Upper bounds for the Gromov width of Grassmannian manifolds

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Abstract

An upper bound for the Gromov width of generalized Grassmannian manifolds with respect to the Kirillov-Kostant-Souriau symplectic form is found by computing certain Gromov-Witten invariants. The approach presented here is closely related to the one used by Gromov in its celebrated Non-Squeezing theorem.

The Gromov width of symplectic manifolds

Let M be a smooth manifold. A **symplectic form** on M is a differential 2-form ω which is closed and nondegenerate. The pair (M, ω) is called a **symplectic manifold**.

Theorem (Darboux's theorem). *Let (M, ω) be a symplectic manifold. For any point of the manifold there are local coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ defined on a neighborhood U of the point such that*

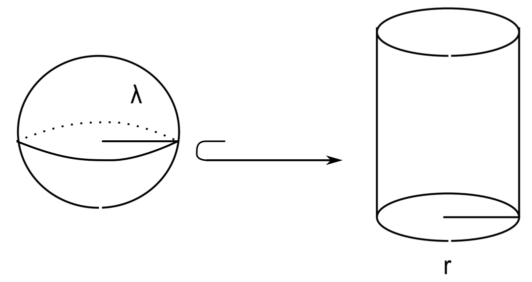
$$\omega|_U = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

A fundamental question in symplectic topology is: how far can U be extendeded symplectically? A way of measuring the "symplectic" size of a manifold is through the **Gromov width** of a symplectic manifold:

$$\mathrm{Gwidth}(M, \omega) := \sup\{\pi r^2 : \exists \text{ a symplectic embedding } B_{2n}(r) \hookrightarrow M\}$$

Motivation:

Theorem (Gromov's Non-squeezing Theorem). *If $\rho : B_{2n}(r) \hookrightarrow B_2(\lambda) \times \mathbb{R}^{2n-2}$ is a symplectic embedding, then $r \leq \lambda$. Thus, $\mathrm{Gwidth}(B^2(\lambda) \times \mathbb{R}^{2n-2}) = \pi\lambda^2$.*



Pseudoholomorphic theory

Let J be a compatible almost-complex structure of (M, ω), i.e., $J : TM \to TM$, $J^2 = -I$, and the formula

$$g(u, v) := \omega(u, Jv)$$

defines a Riemannian metric. A map $\mu : \mathbb{CP}^1 \to M$ is J**-holomorphic** if

$$J \circ d\mu = d\mu \circ j_{st}$$

where j_{st} denotes the standard almost complex structure of $\mathbb{CP}^1$. Let $A \in H_2(M, \mathbb{Z})$. **The moduli space of genus zero simple J-holomorphic curves of degree** A is

$$\mathcal{M}_{A,J} = \{\mu : \mathbb{CP}^1 \to M : J \circ d\mu = d\mu \circ j_{st}, \mu_*[\mathbb{CP}^1] = A, \mu \text{ simple}\}.$$

A J-holomorphic curve $\mu : \mathbb{CP}^1 \to M$ is simple if it can not be written as $\nu \circ f$ where $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ is of the form $f : z \to z^k, k \in \mathbb{Z}$.

- For "generic" J, $\mathcal{M}_{A,J}$ is a smooth oriented manifold of dimension equal to $\dim M + 2c_1(A)$.
- The moduli space $\mathcal{M}_{A,J}$ is not always compact, because of the "bubbling phenomenon".
- We say that a homology class $A \in H_2(M, \mathbb{Z})$ is ω**-indecomposable** if there do not exist homology classes $A_1, A_2 \in H_2(M, \mathbb{Z})$ such that $A = A_1 + A_2$ and $0 < \omega(A_1), \omega(A_2)$. If $A \in H_2(M, \mathbb{Z})$ is ω-indecomposable, the moduli space $\mathcal{M}_{A,J}$ is compact and thus we can associate to it a fundamental class $[\mathcal{M}_{A,J}]$.

Let $A \in H_2(M, \mathbb{Z})$. **The moduli space of genus zero J-holomorphic curves of degree** A **with** k**-marked points** is $\mathcal{M}_{A,J,k} = \mathcal{M}_{A,J} \times_{PSL(2,\mathbb{C})} (\mathbb{CP}^1)^k$.

We have an evaluation map

$$ev_J : \mathcal{M}_{A,J,k} \to M^k.$$

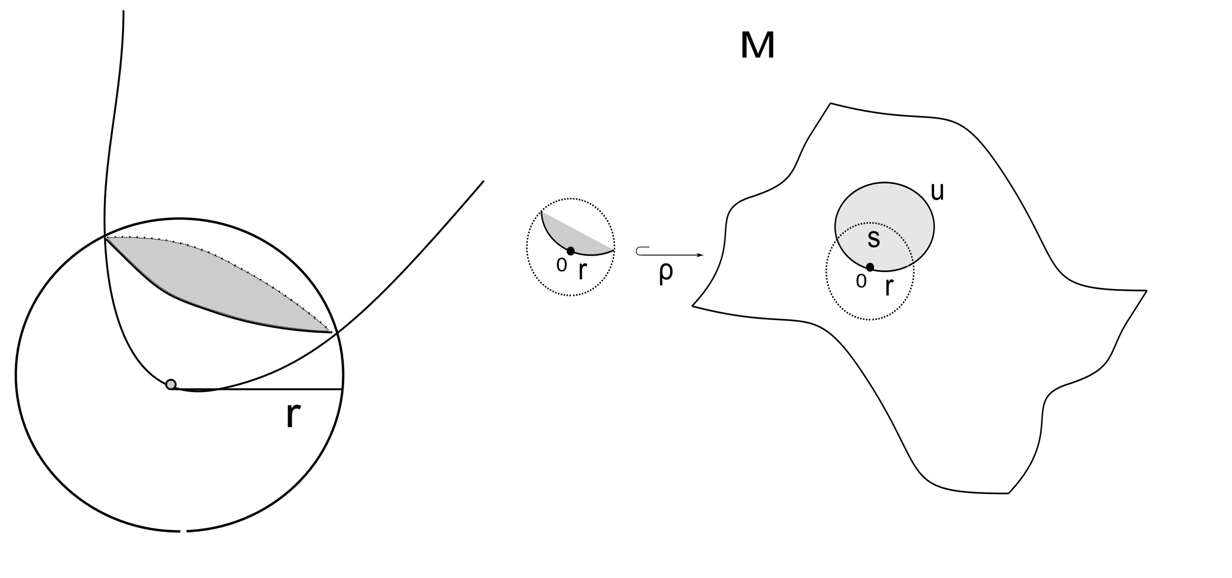
Theorem. *Let (M, ω) be a $2n$-dimensional compact symplectic manifold and $A \in H_2(M, \mathbb{Z}) \backslash \{0\}$ a second homology class. Suppose that for a dense subset of smooth ω-compatible almost complex structures J, the evaluation map*

$$ev_J : \mathcal{M}_{A,J,1} \to M$$

is onto. Then for any symplectic embedding $\rho : B_{2n}(r) \to M$, we have $\pi r^2 \leq \omega(A)$. In particular

$$\mathrm{Gwidth}(M, \omega) \leq \omega(A).$$

Idea of the proof:



The area of a holomorphic curve passing through the origin and bounded by a ball of radius r is bounded from below by πr^2

Let J be an almost complex structure compatible with a symplectic embedding ρ. If u is a J-holomorphic curve passing through the origin, we have that $\pi r^2 \leq$ area s ≤ area u $=\omega(A)$

Proving that the evaluation map, for a generic almost complex structure J, is onto is not an easy task. However, one approach is to prove that certain Gromov-Witten invariant with one of the constrains being a point is different from zero:

Find $X_1, \cdots, X_r \subset M$ such that

$$GW^J_{A,r+1}([pt], [X_1], \cdots, [X_r]) \neq 0$$

where the Gromov-Witten invariant $GW^J_{A,r+1}([pt], [X_1], \cdots, [X_r])$ counts the number of J-holomorphic curves in the class A passing through $pt, X_1, \cdots, X_r$.

- Defining the Gromov Witten invariant in the symplectic category requires big machinary such as virtual fundamental classes and polyfold theory. However under simple conditions the Gromov-Witten invariant can be defined more easily. That is the case when the homology class $A \in H_2(M, \mathbb{Z})$ is ω-indecomposable.

Generalized partial flag manifolds

Let (G, B, N, S) be a Tits system where G is a connected semisimple complex Lie group, B is a Borel subgroup of G, T a maximal torus of G, N the normalizer of T and S a set of simple reflections of the Weyl group $W := N/T$.

Any subgroup P satisfying $B \subset P \subset G$ is called a standard parabolic subgroup of G. Any parabolic subgroup is of the form P_Y where $Y \subset S$ and $P \subset G$ is the group generated by B and Y.

A **generalized partial flag manifold** is a quotient of the form G/P_Y. We say that G/P_Y is a **Grassmannian** manifold if P_Y is a maximal parabolic subgroup. In that case, there exists $\alpha \in S$ such that $Y = S \backslash \{\alpha\}$.

Generalized partial flag manifolds can be identified with coadjoint orbits of compact Lie groups. In other words, let $K, \mathfrak{t}$ be compact forms of G and T respectively, with G acting on $\mathfrak{t}$ by the coadjoint action. Then there exists $\lambda \in \mathfrak{t}^*$ such that $\mathcal{O}_\lambda = K \cdot \lambda \cong G/P_Y$. Coadjoint orbits $\mathcal{O}_\lambda$ are endowed with the Kostant-Kirillov-Souriau symplectic form denoted by ω^λ_{KKS}. Under these identifications, the Kostan-Kirillov-Souriau symplectic form is compatible with the standard complex structure J_s of G/P_Y thus defining a Kähler structure.

The (co)homology of a generalized partial flag manifold can be computed by means of its CW-decomposition given by the Bruhat decomposition. We have that

$$G/P_Y = \bigsqcup_{w \in W'_Y} BwP_Y/P_Y$$

where W'_Y are the elements of minimal length in the cosets W/W_Y and $W_Y \subset W$ is the group generated by $Y \subset W$. For $w \in W'_Y$, we call $C_w = BwP_Y/P_Y$ a Bruhat cell. It is isomorphic to an affine space $\mathbb{C}^{l(w)}$, and its closure $X_w = \overline{C_w}$ is a Schubert variety.

Theorem. *The set of fundamental classes $[X_w]$ of the Schubert varieties X_w, $w \in W'_Y$, are a basis of $H^*(G/P_Y, \mathbb{Z})$.*

Theorem (Duality theorem). *Let w_0 and w_0^Y denote the largest element of W and W_Y respectively. Let B^- be the Borel subgroup of G containing T and opposite to B. For $w \in W'_Y$, let $X^w := B^- wP_Y/P_Y$. Then $[X^w] = [X_{w_0 w w_0^Y}]$ and*

$$X_w \pitchfork X^w = \{wP_Y/P_Y\}.$$

Moduli space of lines for Grassmannians

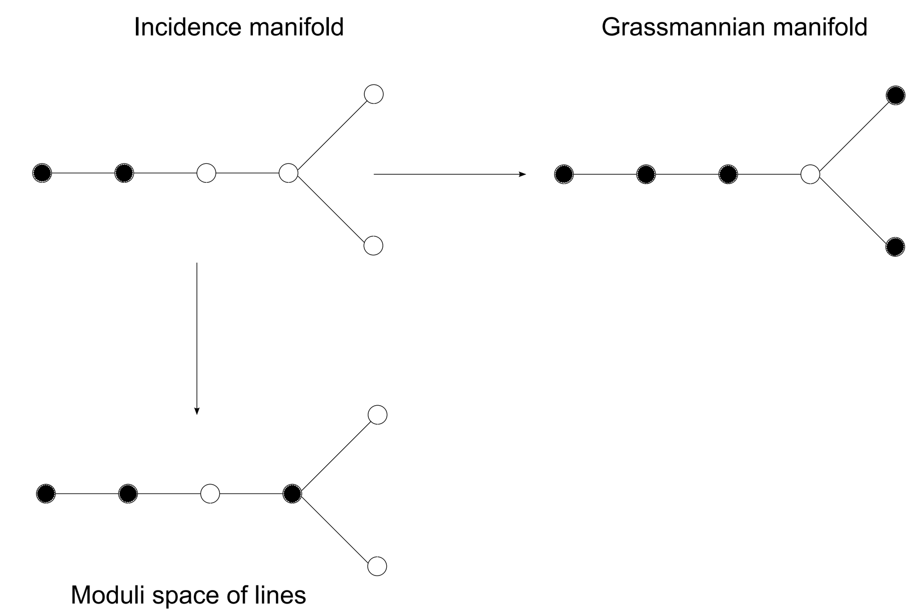
Let $G/P_{S\backslash\{\alpha\}}$ be a Grassmannian manifold where $\alpha \in S$ is a simple reflection. The second homology group $H_2(G/P_{S\backslash\{\alpha\}}, \mathbb{Z})$ is cyclic. Let $A \in H_2(G/P_{S\backslash\{\alpha\}}, \mathbb{Z})$ be the standard generator.

An element of $\mathcal{M}_{A,J_s,0}(G/P_{S\backslash\{\alpha\}}, \omega^\lambda_{KKS})$ is called a line of the Grassmannian manifold $G/P_{S\backslash\{\alpha\}}$. The group G acts on $\mathcal{M}_{A,J_s,0}(G/P_{S\backslash\{\alpha\}}, \omega^\lambda_{KKS})$ and $\mathcal{M}_{A,J_s,1}(G/P_{S\backslash\{\alpha\}}, \omega^\lambda_{KKS})$ transitively if α is a simple reflection coming from a "long" simple root. We have that

$$\mathcal{M}_{A,J_s,0}(G/P_{S\backslash\{\alpha\}}, \omega_{KKS}) \cong G/P_{S\backslash N(\alpha)},$$
$$\mathcal{M}_{A,J_s,1}(G/P_{S\backslash\{\alpha\}}, \omega_{KKS}) \cong G/P_{S\backslash(N(\alpha)\bigsqcup\{\alpha\})}$$

where $N(\alpha)$ denotes the set of simple reflections which are adjacent in the corresponding Dynkin diagram to α. We have the following commutative diagrams:

$$\begin{array}{ccc} \mathcal{M}_{A,J_s,1}(G/P_{S\backslash\{\alpha\}}) \cong G/P_{S\backslash(N(\alpha)\bigsqcup\{\alpha\})} & \xrightarrow{ev} & G/P_{S\backslash\{\alpha\}} \\ \downarrow f & & \\ \mathcal{M}_{A,J_s,0}(G/P_{S\backslash\{\alpha\}}) \cong G/P_{S\backslash N(\alpha)} & & \end{array}$$



Here f denotes the forgetful map.

If $X \subset G/P_{S\backslash\{\alpha\}}$, then $\hat{X} := f(ev^{-1}(X)) \subset G/P_{S\backslash N(\alpha)}$ corresponds to the set of all lines in $G/P_{S\backslash\{\alpha\}}$ incident to X.

In order to avoid ambiguities, we denote the Schubert varieties in $G/P_{S\backslash N(\alpha)}$ as Z_w for $w \in W'_{S\backslash N(\alpha)}$ and the Schubert varieties in $G/P_{S\backslash\{\alpha\}}$ by X_w for $w \in W'_{S\backslash\{\alpha\}}$.

Theorem. *Let $e \in G/P_{S\backslash\{\alpha\}}$ be the class of the identity in the quotient $G/P_{S\backslash\{\alpha\}}$. There exists $w \in W'_{S\backslash\{\alpha\}}$ and $\hat{w} \in W'_{S\backslash N(\alpha)}$ such that $\hat{X}_w = Z_{\hat{w}}$ and $\hat{e} = Z^{\hat{w}}$. In particular*

$$GW^{J_s}_{A,2}([e], [X_w]) = 1$$

Corollary.

$$\mathrm{Gwidth}(G/P_Y, \omega^\lambda_{KKS}) \leq \omega^\lambda_{KKS}(A) = \frac{2(\alpha, \lambda)}{(\alpha, \alpha)}.$$

Example 1. *For the Grassmannian manifold $G(k, n) = \{V^k \subset \mathbb{C}^n\}$ the moduli space of lines is given by the flag manifold*

$$Fl(k-1, k+1; n) = \{V^{k-1} \subset V^{k+1} \subset \mathbb{C}^n\}.$$

For the Grassmannian manifold $G(k, n)$, we can prove that under generic assumptions there is just one line passing through one arbitrary point and $X = \{V^k \in G(k, n) : \mathbb{C} \subset V^k \subset C^{n-1}\} \cong G(k-1, n-2)$. This is illustrated in the next figure for $k = 2, n = 4$.