Calculation of the Cohomology of Generalized Moment-Angle Complexes

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1 What is a Generalized Moment-Angle Complex?

Let $\mathbb{K}$ be a field, $\mathbb{Z}$ means integers. $[m] = \{1, \ldots, m\}$, $K \subset 2^{[m]}$ is an abstract simplicial complex (in which $\emptyset \in K$), with the facets $\mathcal{F}_K$, which is the set of all the maximal simplexes in K (a simplex is maximal if it is not included in other simplexes). $(\underline{X}, \underline{A}) = \{(X_i, A_i)|i = 1, \ldots, m\}$ means m pairs of topological spaces. Consider K as a category $\mathcal{K}$, whose objects are simplexes, and morphisms are inclusions in K. We can define the functor D from $\mathcal{K}$ to $\mathcal{T}$, the category of topological spaces:

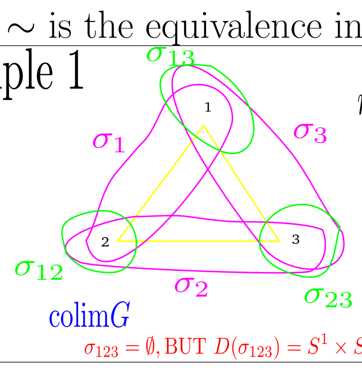
$$\forall \sigma \in \mathcal{K}, \quad D(\sigma) = \prod_{i=1}^{m} Y_i \subset \prod_{i=1}^{m} X_i, \quad Y_j = \begin{cases} X_j & j \in \sigma \\ A_j & j \in [m] \setminus \sigma, \end{cases}$$

and D maps inclusion to inclusion in the obvious way. A generalized moment-angle complex is defined to be

$$\mathcal{Z}_{K,[m]}(\underline{X}, \underline{A}) = \mathrm{colim} D = \sqcup_{\sigma \in \mathcal{K}} D(\sigma)/ \sim,$$

where $\sim$ is the equivalence induced by the inclusions.

Example 1



$\sigma_{123} = \emptyset$, BUT $D(\sigma_{123}) = S^1 \times S^1 \times S^1$

$m = 3$

$\sigma_{i_0 \ldots i_p} = \sigma_{i_0} \cap \cdots \cap \sigma_{i_p}$

$(\underline{X}, \underline{A}) = \{(D_i^2, S_i^1)|i = 1, 2, 3\}$

$\mathcal{F}_K = \{\sigma_1, \sigma_2, \sigma_3\}$

$\sigma_1 = \{1,2\}$ $D(\sigma_1) = D^2 \times D^2 \times S^1$

$\sigma_2 = \{2,3\}$ $D(\sigma_2) = S^1 \times D^2 \times D^2$

$\sigma_3 = \{1,3\}$ $D(\sigma_3) = D^2 \times S^1 \times D^2$

$\mathrm{colim} D = \partial(D^2 \times D^2 \times D^2) = S^5$

Δ is the category with objects $\{\Delta^i\}_{i=0}^{\infty}$ in which Δ^i is the standard geometric simplex of dimension i, and morphisms are linear inclusions.
$G: \mathcal{K} \to \Delta$
is the geometric realization functor, whose geometric realization is $\mathrm{colim} G$.

2 The Main Problem and the Clue

Main Problem:
$$H^*(\mathrm{colim} D; \mathbb{K}) \longleftarrow \lim H^*(D; \mathbb{K})$$
How far away?

By Künneth Formula,
$$\forall \sigma \in \mathcal{K}, H^*(D(\sigma); \mathbb{K}) = \bigotimes_{r \in \sigma} H^*(X_r; \mathbb{K}) \bigotimes_{s \in \overline{\sigma}} H^*(A_s; \mathbb{K}).$$
$$\overline{\sigma} = [m] \setminus \sigma$$

Clue S_*- : singular chain functor
$$S_*(\mathrm{colim} D; \mathbb{K}) \overset{\text{chain equivalence}}{=\!=} \mathrm{colim} S_*(D; \mathbb{K})$$
is not difficult to be satisfied. For example, when A_i is open in X_i, or (X_i, A_i) are CW pairs, $\forall i = 1, \ldots, m$.

The Classical Mayer-Vietoris Argument
Suppose a space is covered by two open subsets:
$$X = U \cup V,$$
then we have a chain equivalence
$$S_*(X) = \mathrm{coker}(\phi), \quad S_*(V) \oplus S_*(U) \xleftarrow{\phi} S_*(U \cap V) \ (+1, -1)$$
and we can use Hom$-$ on them because they are free.

3 A Double Complex Arises

Definition 1. Suppose $\mathfrak{B} = \{B_i\}_i$ is a finite covering by (not necessarily open) subsets of a topological space X, namely $X = \bigcup_i B_i$.

- A singular simplex is $\mathfrak{B}$-small, if there exists an index i, such that its image lies in B_i. $S_*^{\mathfrak{B}}(X; \mathbb{Z})$ means the subgroup of $S_*(X; \mathbb{Z})$ generated by all the $\mathfrak{B}$-small singular simplexes.
- We say that $\mathfrak{B}$ is a double complex cover for X, if the chain inclusion $\mathrm{i}_\sharp : S_*^{\mathfrak{B}}(X; \mathbb{Z}) \to S_*(X; \mathbb{Z})$ induces a chain equivalence.

By a slight modification of the proof in [1], we can prove the following lemma with the fact that every subcomplex (in a CW complex) has an open neighborhood, of which the subcomplex is a deformation retract (a proof could be found in [3]).

Lemma 1. *If X is a CW complex, of which $\mathfrak{B}$ is a cover by subcomplexes, then $\mathfrak{B}$ is a double complex cover for X.*

Immediately, we have (here $G\mathcal{F}_K$ ($D\mathcal{F}_K$, resp.) means the image of $\mathcal{F}_K$ under the functor G (D, resp.))

◦ $G\mathcal{F}_K$ is a double complex cover for colim$G$;

◦ $D\mathcal{F}_K$ is a double complex cover for colim$D$, provided that $(\underline{X}, \underline{A})$ are CW pairs.

The following proposition is the main tool for our computation (details can be found in [1]).

Proposition 1. *Suppose $\mathfrak{B}$ is a (finite) double complex cover for X. Then there is a double complex $C^*(\mathfrak{B}, S^*)$ with differential operator* D *and a product structure, such that there is a ring isomorphism*

$$H_{\mathrm{D}}(C^t(\mathfrak{B}, S^*); \mathbb{Z}) = H^t(X; \mathbb{Z}),$$

where $C^t(\mathfrak{B}, S^*) = \bigoplus_{p+q=t} \bigoplus_{i_0 < \cdots < i_p} S^q(B_{i_0 \ldots i_p}; \mathbb{Z}) = \bigoplus_{p+q=t} S^q(\mathfrak{B}\big|_p; \mathbb{Z})$ $(\mathfrak{B}\big|_p = \coprod_{i_0 < \cdots < i_p} B_{i_0 \ldots i_p})$, $\mathrm{D} = \delta_p + (-1)^p \mathrm{d}$ (d is the usual differential operator in the singular cochain). And we have a spectral sequence associated to the double complex, with

$$E_2^{p,q} = H_\delta^p(\mathfrak{B}; \mathcal{H}^q),$$

where $H_\delta^p(\mathfrak{B}; \mathcal{H}^q)$ means the p-th cohomology of the following chain complex

$$H^q(\mathfrak{B}\big|_0; \mathbb{Z}) \xrightarrow{\delta_0} H^q(\mathfrak{B}\big|_1; \mathbb{Z}) \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{p-1}} H^q(\mathfrak{B}\big|_p; \mathbb{Z}) \xrightarrow{\delta_p} \cdots.$$

More explicitly, choose $c^q \in S^q(\mathfrak{B}\big|_p; \mathbb{Z})$ $(c^q\big|_{i_0 \ldots i_p} \in S^q(B_{i_0 \ldots i_p}; \mathbb{Z})$, the following $S^*\mathrm{i}$ means corresponding inclusions in X after the action of the S^*- functor),

$$(\delta_p c^q)\big|_{i_0 \ldots i_{p+1}} = \sum_{t=0}^{p+1} (-1)^t (S^*\mathrm{i}) c^q\big|_{i_0 \ldots \hat{i_t} \ldots i_{p+1}}.$$

At last, the $E_\infty^{*,*}$-terms converge to $H_{\mathrm{D}}(C^*(\mathfrak{B}, S^*))$ in the following sense (they are all short exact sequences as rings, in which $F_\infty^{*,*}$ are something appearing in the spectral sequence):

$$0 \to F_\infty^{1,t-1} \to H_{\mathrm{D}}(C^t(\mathfrak{B}, S^*)) \to E_\infty^{0,t} \to 0, \qquad 0 \to F_\infty^{2,t-2} \to F_\infty^{1,t-1} \to E_\infty^{1,t-1} \to 0, \ldots$$

and end with $0 \to 0 \to F_\infty^{t,0} \to E_\infty^{t,0} \to 0$.

Corollary 1. *Suppose $D\mathcal{F}_K$ is a double complex cover for colimD. With all the coefficients in $\mathbb{K}$, we have the following isomorphism between $\mathbb{K}$-modules:*

$$H^t(\mathrm{colim} D; \mathbb{K}) = \bigoplus_{p+q=t} E_\infty^{p,q}.$$

As another consequence, we have the following (compare [1, Theorem 15.8])

$$H^*(\mathrm{colim} G; \mathbb{Z}) = H_\delta^*(G\mathcal{F}_K; \mathcal{H}^0) = H_\delta^*(G\mathcal{F}_K; \mathbb{Z}),$$

because in this case the double complex has only one row!

To be a bit more general, for a partition $L|M|N = [m]$, which means $L \cup M \cup N = [m]$ where L, M, N are disjoint subsets (maybe empty), if we define (will be used later)

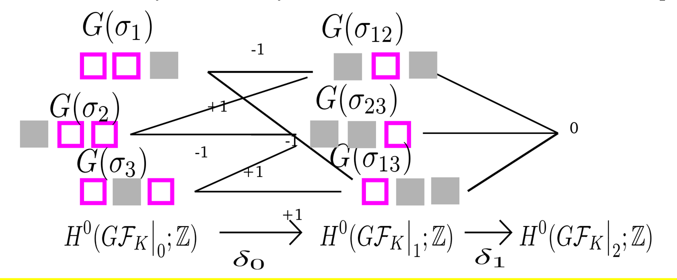
$$\mathcal{F}_K\big|_M^L = \{\sigma \cap M | \sigma \in \mathcal{F}_K, L \subset \sigma\},$$

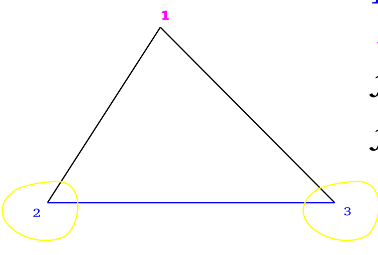
then we have

$$H^*(\mathrm{colim} G\mathcal{F}_K\big|_M^L; \mathbb{Z}) = H_\delta^*(G\mathcal{F}_K\big|_M^L; \mathbb{Z}).$$

The same K as in Example 1

Generally this method to compute cohomology is not as useful as the usual one, because the number of these $\mathbb{Z}$-modules is too large to handle efficiently. (we can try to consider the case of the mobius strip)





$L = \{1\}$ $M = \{2, 3\}$

$\mathcal{F}_K\big|^L = \{\{1, 2\}, \{1, 3\}\}$

$\mathcal{F}_K\big|_M^L = \{\{2\}, \{3\}\}$

$\mathrm{colim} G\mathcal{F}_K\big|_M^L$

By the usual notation, $\mathrm{colim} G\mathcal{F}_K\big|_M^L$ is the star of L in K restricting to its full subcomplex in M.

4 Globalize the "local coefficients" $\mathcal{H}^*$ and An Explicit Formula for E_2-Terms

Generally, we cannot use a "global" notation to unite the "local coefficients". But now the structure of the colim$D$ is so special that we have a canonical way to do this.

Suppose the j-th inclusion $\mathrm{i}_j : A_j \to X_j$, $\mathrm{i}_j^* : H^*(X_j) \to H^*(A_j)$,

$$\bigoplus_{\alpha_j} \mathbb{K}\{\alpha_j\} = \ker(\mathrm{i}_j^*), \quad \bigoplus_{\gamma_j} \mathbb{K}\{\gamma_j\} = \mathrm{coker}(\mathrm{i}_j^*), \quad \bigoplus_{\xi_j} \mathbb{K}\{\xi_j\} = \mathrm{im}(\mathrm{i}_j^*),$$

We fix a basis of cohomology (a vector space over $\mathbb{K}$) for all X_j and A_j, α, γ, ξ runs all the possible (different) elements of this basis. Note that now $H^*(X_j) = \ker(\mathrm{i}_j^*) \oplus \mathrm{im}(\mathrm{i}_j^*)$, $H^*(A_j) = \mathrm{coker}(\mathrm{i}_j^*) \oplus \mathrm{im}(\mathrm{i}_j^*)$. Then we define the Total Cohomology Coefficient Module, which is a free $\mathbb{K}$-module generated by all the possible elements of the basis, by

$$\mathcal{H}^* = \bigoplus_{\substack{\alpha_L, \gamma_M, \xi_N \\ L|M|N=[m]}} \mathbb{K}\{\alpha_L \otimes_{\mathbb{K}} \gamma_M \otimes_{\mathbb{K}} \xi_N\},$$

in which the sum is over all the possible partitions.

The coloring is the free $\mathbb{K}$-module genereted by a particular $\alpha_1 \otimes_{\mathbb{K}} \gamma_{3,4,5,6} \otimes_{\mathbb{K}} \xi_{2,7}$ on , means $X_1 \times X_2 \times A_3 \times A_4 \times A_5 \times A_6 \times X_7$

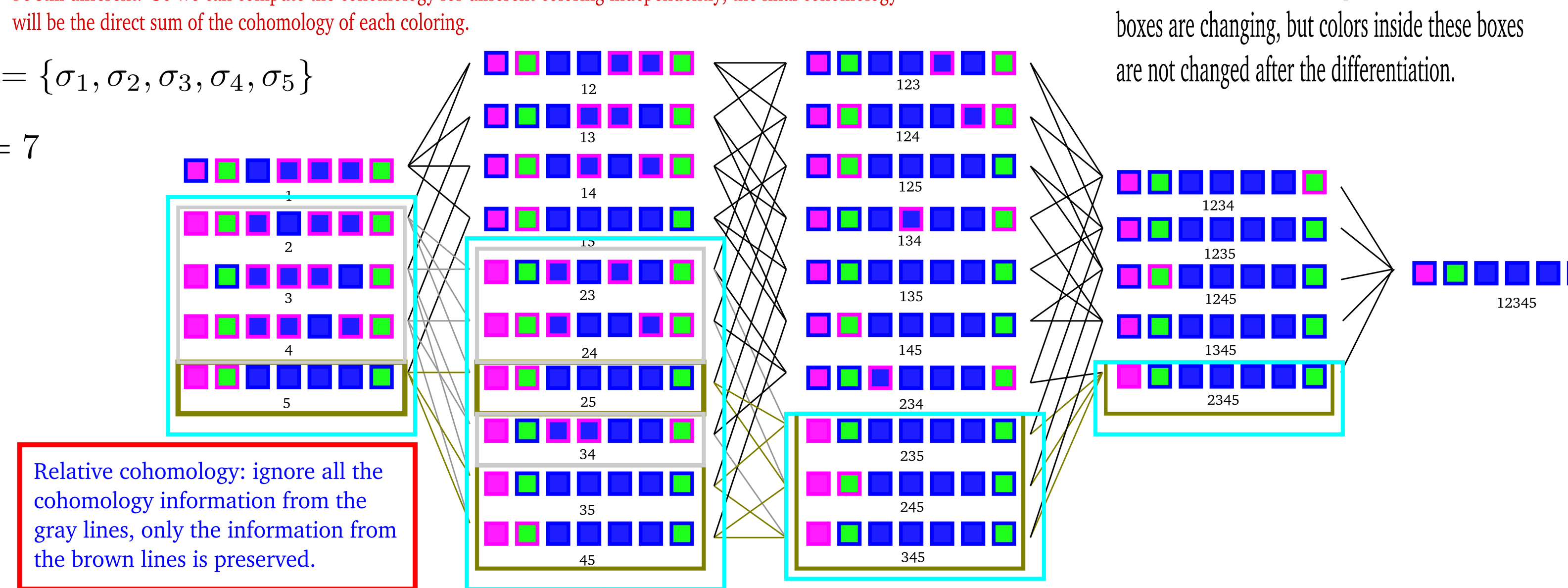
Before the computation, we choose all the non-zero coloring: no and no . The coloring may happen after the differentiation δ, but will never happen in our real computation, because will be fewer after more intersection.

In the general computation, different coloring would not infulence each other: they are zero, or they will be still different. So we can compute the cohomology for different coloring independently, the final cohomology will be the direct sum of the cohomology of each coloring.

The secrets behind the computation of the E_2-terms : boxes are changing, but colors inside these boxes are not changed after the differentiation.

$\mathcal{F}_K = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$

$m = 7$



$$D\mathcal{F}_K\big|_0 \xrightarrow{\delta_0} D\mathcal{F}_K\big|_1 \xrightarrow{\delta_1} D\mathcal{F}_K\big|_2 \xrightarrow{\delta_2} D\mathcal{F}_K\big|_3 \xrightarrow{\delta_3} D\mathcal{F}_K\big|_4$$

This is a big simplex, in which the number of vertexes is equal to the number of the elements in the cover, because now we do not have empty sets from the intersection. Modules in the (and the arrows associated) is something we will really compute in this chain complex.

Generally, for $\alpha_L \otimes_{\mathbb{K}} \gamma_M \otimes_{\mathbb{K}} \xi_N$, we define
to mean the modules with the arrows associated such that no and no ;
to mean the modules with the arrows associated such that no (but may exixts);
to mean the modules with the arrows associated such that no , but do have .

$$H_\delta^p(D\mathcal{F}_K; \mathcal{H}^*) = \bigoplus_{L|M|N=[m], \alpha_L, \gamma_M, \xi_N} H_\delta^p(D\mathcal{F}_K; \mathbb{K}\{\alpha_L \otimes_{\mathbb{K}} \gamma_M \otimes_{\mathbb{K}} \xi_M\})$$

is a simplex with vertexes $\{D(\sigma)|\sigma \in \mathcal{F}_K, L \subset \sigma\}$.

is a subcomplex in , since if exists on some module, then all the modules whose arrows connecting to it from the left in must also have .

$$H_\delta^p(D\mathcal{F}_K; \mathbb{K}\{\alpha_L \otimes_{\mathbb{K}} \gamma_M \otimes_{\mathbb{K}} \xi_M\}) = H_\delta^p(\quad; \mathbb{K})$$

Relative cohomology $\Longrightarrow = H_\delta^p(\quad,\quad; \mathbb{K})$ $\quad \tilde{H}^i(\emptyset) = \begin{cases} \mathbb{K} & i = -1 \\ 0 & i \neq -1 \end{cases}$

By the argument of the long exact sequence $\Longrightarrow = \tilde{H}_\delta^{p-1}(\quad; \mathbb{K}) = \tilde{H}_\delta^{p-1}(G\mathcal{F}_K\big|_M^L; \mathbb{K})$

Theorem 1 (L. Cai, 2011, see [2] for more details)**.** *We have the following formula to compute the E_2-terms $(p, q \geq 0)$:*

$$E_2^{p,q} = H^p(D\mathcal{F}_K; \mathcal{H}^q) = \tilde{H}^{p-1}(\mathrm{colim} G\mathcal{F}_K\big|_M^L; \mathbb{K}).$$

5 $E_2 = E_\infty$: A Miracle, and The Final Result

Theorem 2 (L. Cai, 2011, see [2] for details)**.** *If $D\mathcal{F}_K$ is a double complex cover for* colim$D$*, then we have (with coefficients in $\mathbb{K}$)*

$$E_2^{p,q} = E_\infty^{p,q}.$$

Consequently the final result can be expressed as $(t, p \geq 0)$

$$H^t(\mathrm{colim} D; \mathbb{K}) = \bigoplus_{\substack{L|M|N=[m], \alpha_L, \gamma_M, \xi_N \\ p+|\alpha_L|+|\gamma_M|+|\xi_N|=t}} \tilde{H}^{p-1}(\mathrm{colim} G\mathcal{F}_K\big|_M^L; \mathbb{K}) \otimes_{\mathbb{K}} \mathbb{K}\{\alpha_L \otimes_{\mathbb{K}} \gamma_M \otimes_{\mathbb{K}} \xi_N\},$$

where $|*|$ means the cohomology degree of $*$. We stress that this is an isomorphism between $\mathbb{K}$-modules.

In Example 1, we have $H^1(S^1; \mathbb{K}) = \mathbb{K}\{\gamma\}$, $H^0(S^1; \mathbb{K}) = H^0(D^2; \mathbb{K}) = \mathbb{K}\{\xi\}$. Now we have no α-elements, so colim$G\mathcal{F}_K\big|_M^L$ is the full subcomplex of K restricting in M, and the nontrivial terms from $\tilde{H}^{p-1}(\mathrm{colim} G\mathcal{F}_K\big|_M^L)$ would be when $M = \emptyset$ or $M = \{1, 2, 3\}$. Then we get

$$H^*(S^5; \mathbb{K}) = \tilde{H}^{-1}(\emptyset; \mathbb{K}) \otimes_{\mathbb{K}} \mathbb{K}\{\xi_{1,2,3}\} \bigoplus \tilde{H}^1(S^1; \mathbb{K}) \otimes_{\mathbb{K}} \mathbb{K}\{\gamma_{1,2,3}\}.$$

For more examples and the relation between the previous results, like Tor, please see [2]. Thank you.

References

[1] R. Bott and L. W. Tu. *Differential Forms in Algebraic Topology.* GTM82. Springer, Beijing, 2009.

[2] L. Cai, X. Wang, Z. Lü, and Q. Zheng. A note on double complex and the cohomology of generalized moment angle complexes, 2012 (in preparation).

[3] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, Cambridge, 2002.