# Schubert Calculus for Weighted Grassmannian Orbifolds

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**Abstract**

The *weighted Grassmannian* wGr defined by Corti-Reid [1] is a projective variety with at worst orbifold singularities, given by the *Plücker relations* regarded as a *weighted* homogeneous polynomials with certain weights. It carries a natural action of an $(n-1)$-dimensional torus w$R$ and we study its rational equivariant cohomology. We introduce the equivariant *weighted Schubert classes*, and after we show that they form a basis of the cohomology, we give an explicit formula for the structure constants with respect to this Schubert basis. We also find a particular rational basis $\{\mathrm{w}u_1, \cdots, \mathrm{w}u_{n-1}\}$ of Lie(w$R$)$^*$, in which those structure constants are polynomials with non-negative coefficients, up to a permutation on the weights. Furthermore, we find the relation between the factorial Schur functions and our weighted Schubert classes.

## §1.Weighted Grassmannians:

Let $d < n$ be positive integers. Let $\{{n \atop d}\} := \{\lambda = \{\lambda_1, \cdots, \lambda_d\} \subset \{1, \cdots, n\}\}$ where $\lambda_1 < \cdots < \lambda_d$. Let $\mathbb{C}^{\{{n \atop d}\}}$ be the affine space of the *Plücker coordinates* $\{x_\lambda, \lambda \in \{{n \atop d}\}\}$. Let $\mathrm{aPl}^\times := \mathrm{aPl}^\times(d,n)$ be the quasi-affine variety in $\mathbb{C}^{\{{n \atop d}\}} - \{0\}$ given by the famous *Plücker relations*. The $n$-dimensional complex torus $T_\mathbb{C}$ acts on $\mathrm{aPl}^\times$ by

$$\rho : T_\mathbb{C} \to (\mathbb{C}^\times)^{\{{n \atop d}\}}, \quad (t_1, \cdots, t_n) \mapsto (\, t_\lambda := t_{\lambda_1} \cdots t_{\lambda_d}\,)_{\lambda \in \{{n \atop d}\}}.$$

Recall that the usual Grassmannian Gr is the quotient of $\mathrm{aPl}^\times$ by a diagonal torus $D_\mathbb{C}$ in $T_\mathbb{C}$.

Now choose $\quad \vec{w} := (w_1, \cdots, w_n) \in (\mathbb{Z}_{\geq 0})^n \quad$ and $\quad a \in \mathbb{Z}_{\geq 1}$.

We introduce the *weighted diagonal* in $T_\mathbb{C}$: $\mathrm{w}D_\mathbb{C} := \{(t^{dw_1+a}, \cdots, t^{dw_n+a}) \in T_\mathbb{C} \mid t \in \mathbb{C}^\times\}$.

**Definition (Corti-Reid [1])**

The *weighted Grassmannian* is a projective variety with at worst orbifold singularities, defined by

$$\mathrm{wGr} : \mathrm{wGr}(d,n) := \mathrm{aPl}^\times / \mathrm{w}D_\mathbb{C} \quad \subset \mathbb{P}(\mathbb{C}^{\{{n \atop d}\}})_w$$

where $\mathbb{P}(\mathbb{C}^{\{{n \atop d}\}})_w$ is the weighted projective space with the weights $(w_\lambda := a + \sum_{l \in \lambda} w_l)_{\lambda \in \{{n \atop d}\}}$. It carries the residual action of $\mathrm{w}R_\mathbb{C} := T_\mathbb{C}/\mathrm{w}D_\mathbb{C}$. Remark that the ordinary Grassmannian Gr is a special case when $\vec{w} = 0$, $a = 1$. We use $D_\mathbb{C}$ and $R_\mathbb{C}$ for the corresponding tori.

We study a *quasi-cell decomposition* of wGr, i.e. each "cell" is a quotient of a *usual* cell by a finite group action, generalizing the ordinary Schubert cell-decomposition, and it implies

**Theorem 1**

$H^*(\mathrm{wGr})$ is concentrated in even degree and $H^*_{\mathrm{w}R}(\mathrm{wGr})$ is a free $H^*(B\mathrm{w}R)$-module.

Note that the coefficients of cohomology is assumed to be $\mathbb{Q}$ in this poster.

## §2.Weighted Schubert Classes:

Consider the natural maps for the Borel constructions of Gr, $\mathrm{aPl}^\times$ and wGr

$$ER \times_R \mathrm{Gr} \xleftarrow{h} ET \times_T \mathrm{aPl}^\times \xrightarrow{\mathrm{w}h} E\mathrm{w}R \times_{\mathrm{w}R} \mathrm{wGr}$$

**Proposition 2**

$h^* : H^*_R(\mathrm{Gr}) \to H^*_T(\mathrm{aPl}^\times)$ is an isomorphism of rings over $H^*(BR)$.
$\mathrm{w}h^* : H^*_{\mathrm{w}R}(\mathrm{wGr}) \to H^*_T(\mathrm{aPl}^\times)$ is an isomorphism of rings over $H^*(B\mathrm{w}R)$.

**Definition**

Let $\mathrm{a}\Omega_\lambda (\subset \mathrm{aPl}^\times)$ be the preimage of the Schubert variety $\Omega_\lambda (\subset \mathrm{Gr})$ under the quotient map $\mathrm{aPl}^\times \to \mathrm{Gr}$. Define

$$\mathrm{a}\tilde{S}_\lambda := [\mathrm{a}\Omega_\lambda]_T \in H^*_T(\mathrm{aPl}^\times) \quad \text{and} \quad \mathrm{w}\tilde{S}_\lambda := (\mathrm{w}h^*)^{-1}(\mathrm{a}\tilde{S}_\lambda) \in H^*_{\mathrm{w}R}(\mathrm{wGr}).$$

The class $\mathrm{a}\tilde{S}_\lambda$ is related to the *usual* $R$-equivariant Schubert class $\tilde{S}_\lambda$ in $H^*_R(\mathrm{Gr})$, by $\tilde{S}_\lambda = (h^*)^{-1}(\mathrm{a}\tilde{S}_\lambda)$. Furthermore, our weighted Schubert class $\mathrm{w}\tilde{S}_\lambda$ can also be interpreted geometrically as follows.

**Remark**

$H^*_T(\mathrm{aPl}^\times)$ is naturally identified with the w$R$-*equivariant cohomology of weighted Grassmannian orbifold stack* $\widetilde{\mathrm{wGr}} := [\mathrm{aPl}^\times/\mathrm{w}D_\mathbb{C}]$. The isomorphism $\mathrm{w}h^*$ is nothing but the identification to the cohomology of its coarse moduli space wGr. Under this identification, $\mathrm{w}\tilde{S}_\lambda$ should be regarded as the class associated to the w$R$-invariant substack $[\mathrm{a}\Omega_\lambda/\mathrm{w}D_\mathbb{C}]$ which may be called the *weighted Schubert stack*.

## §3. Equivariant Structure Constants:

Working out the *GKM (Goresky-Kottwitz-Macpherson)* description for $\mathrm{w}\tilde{S}_\lambda$, we can prove

**Proposition 3**

$\{\mathrm{w}\tilde{S}_\lambda, \lambda \in \{{n \atop d}\}\}$ is an $H^*(B\mathrm{w}R)$-basis of $H^*_{\mathrm{w}R}(\mathrm{wGr})$.

This allows us to define the equivariant structure constants for $H^*_{\mathrm{w}R}(\mathrm{wGr})$:

**Definition**

For $\lambda, \mu, \nu \in \{{n \atop d}\}$, define $\mathrm{w}\tilde{c}^\nu_{\lambda\mu} \in H^*(B\mathrm{w}R)$ by $\mathrm{w}\tilde{S}_\lambda \cdot \mathrm{w}\tilde{S}_\mu = \sum_\nu \mathrm{w}\tilde{c}^\nu_{\lambda\mu} \mathrm{w}\tilde{S}_\nu$.

**The Equivariant Parameters for w$R$**

Let $\{y_1, \cdots, y_n\}$ be the standard basis of $\mathrm{Lie}(T)^*$ and identify $H^*(BT) = \mathbb{Q}[y_1, \cdots, y_n]$. Let

$$\mathrm{w}u_i := (y_{i+1} - y_i) - \tfrac{(w_{i+1} - w_i)}{w_{\mathrm{id}}} y_{\mathrm{id}} \quad \text{for } i = 1, \cdots, n-1,$$

where $\mathrm{id} \in \{{n \atop d}\}$ is the unique minimum element in the Bruhat order and $y_\lambda := y_{\lambda_1} + \cdots + y_{\lambda_d}$. The quotient map $T \to \mathrm{w}R$ induces the identification

$$H^*(B\mathrm{w}R) \cong \mathbb{Q}[\mathrm{w}u_1, \cdots, \mathrm{w}u_{n-1}] \subset H^*(BT)$$

In particular, $H^*(BR) = \mathbb{Q}[y_2 - y_1, y_3 - y_2, \cdots, y_n - y_{n-1}]$.

## §4. Computing the Structure Constants:

An explicit formula for $\mathrm{w}\tilde{c}^\nu_{\lambda\mu}$ is obtained from the Knutson-Tao's puzzle formula [2], interpreted through the isomorphisms $h^*$ and $\mathrm{w}h^*$, as follows.

• **1st STEP** Under the isomorphism $h^*$, we have

$$\mathrm{a}\tilde{S}_\lambda \mathrm{a}\tilde{S}_\mu = \sum_\nu \tilde{c}^\nu_{\lambda\mu} \mathrm{a}\tilde{S}_\nu \qquad \text{in} \quad H^*_T(\mathrm{aPl}^\times) \tag{1}$$

where $\tilde{c}^\nu_{\lambda\mu} \in H^*(BR)$ is the equivariant structure constants for the ordinary Grassmannian, which is computed in [2] by equivariant puzzles. This expands the product in terms of $\mathrm{a}\tilde{S}_\nu$'s by regarding $H^*_T(\mathrm{aPl}^\times)$ as an $H^*(BR)$-module.

• **2nd STEP** We want to rewrite (1) into an expansion formula of the product in $H^*_T(\mathrm{aPl}^\times)$ as the $H^*(B\mathrm{w}R)$-module. Since $\tilde{c}^\nu_{\lambda\mu}$ is a polynomial of $y_{i+1} - y_i$'s, the key is the following formula essentially equivalent to the *equivariant Pieri rule*:

**Lemma 4**

$$(y_j - y_i) \cdot \mathrm{a}\tilde{S}_\nu = \left( (y_j - y_i) - \frac{w_j - w_i}{w_\nu} y_\nu \right) \cdot \mathrm{a}\tilde{S}_\nu + \sum_{\nu' \to \nu} \frac{w_j - w_i}{w_\nu} \cdot \mathrm{a}\tilde{S}_{\nu'}.$$

Since the coefficients on the RHS are in $H^*(B\mathrm{w}R)$, it converts the $H^*(BR)$-multiplication to the $H^*(B\mathrm{w}R)$-multiplication. We apply this formula to (1) iteratively and obtain the desired expansion:

$$\mathrm{a}\tilde{S}_\lambda \mathrm{a}\tilde{S}_\mu = \sum_\nu \mathrm{w}\tilde{c}^\nu_{\lambda\mu} \mathrm{a}\tilde{S}_\nu.$$

The resulting formula for $\mathrm{w}\tilde{c}^\nu_{\lambda\mu}$ obtained by this procedure is written in our paper.

• **3rd STEP** By the isomorphism $\mathrm{w}h^*$, we conclude that $\mathrm{w}\tilde{c}^\nu_{\lambda\mu}$ is the equivariant structure constant for $H^*_{\mathrm{w}R}(\mathrm{wGr})$:

$$\mathrm{w}\tilde{S}_\lambda \mathrm{w}\tilde{S}_\mu = \sum_\nu \mathrm{w}\tilde{c}^\nu_{\lambda\mu} \mathrm{w}\tilde{S}_\nu.$$

□

The term $(y_j - y_i) - \frac{w_j - w_i}{w_\nu} y_\nu$ in Lemma 5 is written non-negatively in terms of the basis $\mathrm{w}u_i$'s if $w_1 \leq \cdots \leq w_n$. Therefore the equivariant positivity for the formula of $\tilde{c}^\nu_{\lambda\mu}$ implies

**Theorem 5**

$\mathrm{w}\tilde{c}^\nu_{\lambda\mu}$ is a polynomial of $\mathrm{w}u_1, \cdots, \mathrm{w}u_{n-1}$ with non-negative coefficients if $w_1 \leq \cdots \leq w_n$.

## §5. Relation to the Factorial Schur Functions:

Let $x = (x_1, \cdots, x_d)$ and $a = (a_1, \cdots, a_n)$ be sequences of variables. The *factorial Schur function* $s_\lambda(x|a)$ is a polynomilas in $x$ and $b$. A relation to the ordinary equivariant Schubert classes $\tilde{S}_\lambda$ is that the pullback $\tilde{S}_\lambda|_\mu$ to a fixed point indexed by $\mu$ is given by (c.f. [4], [3], [2])

$$\tilde{S}_\lambda|_\mu = s_\lambda(-y(\mu)| - \bar{y}).$$

where $\bar{y} := (y_n, \cdots, y_1)$ and $y(\mu) := (y_{\mu_1}, \cdots, y_{\mu_d})$. We generalized this to $\tilde{S}_\lambda$ by introducing

$$y^\mu := (\, y_1 - (w_1/w_\mu) y_\mu \,, \;\cdots\;,\; y_n - (w_n/w_\mu) y_\mu \,)$$

**Theorem 6**

$$\mathrm{w}\tilde{S}_\lambda|_\mu = s_\lambda(-y^\mu(\mu)| - \bar{y}^\mu).$$

## §6. Example wGr(2, 4): 

The *Bruhat order* in $\{{4 \atop 2}\}$ is given by

$$\mathrm{id} = 34 \leftarrow 24 \leftarrow 23, 14 \leftarrow 13 \leftarrow 12.$$

$\mathrm{aPl}^\times(2,4)$ in $\mathbb{C}^{\{{4 \atop 2}\}} - \{0\}$ is given by the *Plücker* relation: $x_{14}x_{23} - x_{13}x_{24} + x_{12}x_{34} = 0$.

From $\tilde{S}_{23}\tilde{S}_{14} = (y_4 - y_1)\tilde{S}_{13}$ in $H^*_R(\mathrm{Gr})$, we can derive

$$\mathrm{w}\tilde{S}_{23}\mathrm{w}\tilde{S}_{14} = \left( (y_4 - y_1) - \frac{(w_4 - w_1)}{w_{13}} y_{13} \right) \mathrm{w}\tilde{S}_{13} + \left( \frac{w_4 - w_1}{w_{13}} \right) \mathrm{w}\tilde{S}_{12}.$$

## §7. Example wGr(1, n): 

The *Bruhat order* in $\{{n \atop 1}\}$ is given by

$$\{n\} \leftarrow \{n-1\} \leftarrow \cdots \leftarrow \{2\} \leftarrow \{1\}.$$

Since $\mathrm{aGr}(1,n)^\times = \mathbb{C}^n \backslash \{0\}$, we have $\mathrm{wGr}(1,n) = \mathbb{CP}_{b_1, \cdots, b_n}$ where $b_i = w_i + a$ and

$$\mathbb{CP}_{b_1,\cdots,b_n} \supset \mathbb{CP}_{b_1,\cdots,b_{n-1}} \supset \cdots \supset \mathbb{CP}_{b_1,b_2,b_3} \supset \mathbb{CP}_{b_1,b_2} \supset \mathrm{pt} \quad \text{(weighted Schubert varieties).}$$

We find that $H^*_{\mathrm{w}R}(\mathrm{wGr}(1,n)) \cong H^*_T(\mathrm{aGr}(1,n)^\times) \cong \mathbb{Q}[y_1, \cdots, y_n]/(y_1 \cdots y_n)$ and

$$\mathrm{w}\tilde{S}_{\{n\}} = 1, \quad \mathrm{w}\tilde{S}_{\{n-1\}} = y_n, \quad \cdots, \quad \mathrm{w}\tilde{S}_{\{k\}} = y_{k+1} \cdots y_n, \quad \cdots, \quad \mathrm{w}\tilde{S}_{\{1\}} = y_2 \cdots y_n.$$

It follows from our formula for $\mathrm{w}\tilde{c}^\nu_{\lambda\mu}$ that

$$\mathrm{w}\tilde{S}_{\{n-1\}} \cdot \mathrm{w}\tilde{S}_{\{k\}} = \left( y_n - \frac{b_n}{b_k} y_k \right) \mathrm{w}\tilde{S}_{\{k\}} + \frac{b_n}{b_k} \mathrm{w}\tilde{S}_{\{k-1\}}.$$

This is a special case of the following weighted analogue of the equivairant Pieri-rule:

**Lemma 7 (Weighted Equivariant Pieri-rule for wGr)**

$$\mathrm{w}\tilde{S}_{\mathrm{div}} \mathrm{w}\tilde{S}_\lambda = \left( y_{\mathrm{id}} - \frac{w_{\mathrm{id}}}{w_\lambda} y_\lambda \right) \mathrm{w}\tilde{S}_\lambda + \sum_{\lambda' \to \lambda} \frac{w_{\mathrm{id}}}{w_\lambda} \mathrm{w}\tilde{S}_{\lambda'}$$

## References

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