Schubert calculus via root datum and sliding laws

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Based on joint work with:

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Overview

G/P= a generalized flag variety; Schubert classes $\{\sigma_\lambda\}$.

Problem: Give nonnegative combinatorial rules for

$$\sigma_\lambda \cdot \sigma_\mu = \sum_\nu C_{\lambda,\mu}^{\nu}(G/P)\sigma_\nu \in H^\star(G/P)$$

(and generalizations to other cohomology theories).

Some frameworks for this problem:

- Schubert polynomials (as analogues of Schur polynomials);
- Quadratic algebras and Dunkl operators;
- Degeneration (checkers, Mondrian tableaux, puzzles);

Root-theoretic Young diagrams

Roots: $\Phi = \Phi^+ \cup \Phi^-$

Simple roots associated to P: $\Delta_P = \{\beta(P)_1, \ldots, \beta(P)_k\}$

Schubert varieties $X_\lambda \subset G/P \leftrightarrow \lambda W_P \in W/W_P$

$w(\lambda) =$ minimal length coset representative of λW_P

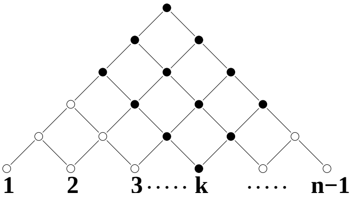
Inversions of $w(\lambda) \subseteq$
$\Lambda_{G/P} = \{\alpha \in \Phi^+ : \alpha$ is a linear combination of $\beta(P)_i$'s$\}$

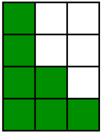
Definition: Call these subsets "λ" of $\Lambda_{G/P}$ **(root-theoretic) Young diagrams**

Naïve theme/question: What efficacy is there in using root-theoretic Young diagrams to study our problem?

Root theoretic Young diagrams (cont.)

Example (Grassmannians): $G/P = \mathrm{Gr}_4(\mathbb{C}^7)$





This situation is especially graphical:

- $\Lambda_{G/P}$ is a planar poset
- λ is a lower order ideal
- Bruhat order is just containment of shapes

A sliding law

[Schützenberger '77]'s sliding law in algebraic combinatorics:

$$\begin{array}{|c|c|}\hline \bullet & a \\ \hline b & \\ \cline{1-1}\end{array} \mapsto \begin{cases} \begin{array}{|c|c|}\hline a & \bullet \\ \hline b & \\ \cline{1-1}\end{array} & \text{if } a < b \\[2em] \begin{array}{|c|c|}\hline b & a \\ \hline \bullet & \\ \cline{1-1}\end{array} & \text{if } b < a \end{cases}$$

Merely one consequence is:

Theorem: [Schützenberger '77]
$C_{\lambda,\mu}^{\nu}(\mathrm{Gr}_k(\mathbb{C}^n)) = \# T \in \mathtt{SYT}(\nu/\lambda)$ that rectify to $T_\mu = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \cline{1-2}\end{array}$.

Perhaps more importantly, jdt unifies (or gets used in) a variety of important tableau algorithms.

Plan of analysis

Theorem: [Thomas-Y. '09] Let G/P be a cominuscule space. Then Schützenberger's rule holds *mutatis mutandis*, replacing standard tableaux with linear extensions of root theoretic Young tableaux, and jeu de taquin by an extension of [R. Proctor '04].

Some tests of the theme we try to address:

(1) Can we replace $H^\star$ by H_T, K, K_T?

(2) Can we go beyond the (co)minuscule setting?

K-theory of Grassmannians

K-theoretic structure constants: $[\mathcal{O}_{X_\lambda}]\cdot[\mathcal{O}_{X_\mu}] = \sum_\nu K_{\lambda,\mu}^{\nu}[\mathcal{O}_{X_\nu}]$.

Additional K-jdt rule: $\begin{array}{|c|c|}\hline \bullet & a \\ \hline a & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \mapsto \begin{array}{|c|c|}\hline a & \bullet \\ \hline \bullet & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$

Not mysterious: $C_{(1),(1)}^{(2,1)}(\mathrm{Gr}_1(\mathbb{C}^2)) = -1$ since $\begin{array}{|c|c|}\hline \bullet & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \mapsto \begin{array}{|c|c|}\hline 1 & \bullet \\ \hline \bullet & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$

Theorem: [Thomas-Y. '09] Suppose $T = \begin{array}{c|c|c|}\cline{2-3} & 1 & 2 \\ \hline \multicolumn{1}{|c|}{1} & 3 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}$ is an increasing tableaux of shape ν/λ. If it K-rectifies to $T_\mu = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ under some order, it rectifies to T_μ under any order.

Theorem: [Thomas-Y. '09] $(-1)^{|\lambda|+|\mu|+|\nu|}K_{\lambda,\mu}^{\nu}(\mathrm{Gr}_k(\mathbb{C}^n)) =$ # increasing tableaux of shape ν/λ that K-rectify to T_μ.

K-theory of Grassmannians (cont)

Some Schubert calculus applications:

(1) (Quiver formulas): The Hecke insertion algorithm of [Buch-Kresch-Shimozono-Tamvakis-Y., '08] has jdt version, just as Robinson-Schensted does, classically.

(2) (Direct sum map):
$\mathrm{Gr}_{k_1}(\mathbb{C}^{n_1}) \times \mathrm{Gr}_{k_2}(\mathbb{C}^{n_2}) \mapsto \mathrm{Gr}_{k_1+k_2}(\mathbb{C}^{n_1+n_2})$:(V,W)$\mapsto V \oplus W$
pulls back:

$$[\mathcal{O}_{X_\nu}] \mapsto \sum_{\lambda,\mu} \hat{K}^{\nu}_{\lambda,\mu}[\mathcal{O}_{X_\lambda}] \otimes [\mathcal{O}_{X_\mu}]$$

New formulas [Thomas-Y., '10] with *full* confluence.

(3) (Extensions to $OG(n, 2n+1)$): [Buch-Ravikumar '10] combined with [Clifford-Thomas-Y., '10];

(4) (All minuscule): conj. [Thomas-Y., '09]; fixed in E_7 and much more [Buch-Samuels, '12+].

K-theory of Grassmannians (cont)

Some *non*-Schubert calculus applications

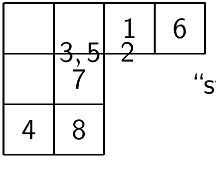
(1) (Longest strictly increasing subsequence problem in random words): [Thomas-Y.,'11] extends the problem of [Ulam '50] and analysis of [Schensted '61].

(2) (Cyclic sieving phenomenon): [Pechenik, '12+] uses `Kjdt` to give a new instance of this phenomenon of [Reiner-Stanton-White, '04].

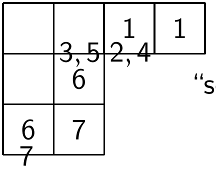
T-equivariant cohomology of Grassmannians

Equiv. structure coeffs: $\sigma_\lambda^T \cdot \sigma_\mu^T = \sum_\nu E_{\lambda,\mu}^{\nu} \sigma_\nu^T \in H_T(\mathrm{Gr}_k(\mathbb{C}^n))$ where $H_T(\mathrm{Gr}_k(\mathbb{C}^n))$ is a module over $H_T(pt) = \mathbb{Z}[t_1, \ldots, t_n]$

Idea: Introduce **edge labeled tableaux** $\{1, 2, \ldots, \ell\}$:



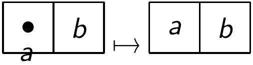
"standard"



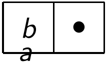
"semistandard"

(cf. [Biedenharn-Louck '89],[Macdonald '92],[Goulden-Greene '94])

"Standard" Ejdt:



(if $a < b$) and



(if $a > b$)

T-equivariant cohomology of Grassmannians (cont)

Not mysterious: $E^{(1)}_{(1),(1)}(Gr_1(\mathbb{C}^2)) = (t_1 - t_2)\sigma^T_{(1)}$ because

$\boxed{\bullet}_{1} \mapsto \boxed{1}$

Assign boxes of $\Lambda_{G/P} = k \times (n-k)$ weights $t_i - t_{i+1}$ via

3	4	5
2	3	4
1	2	3

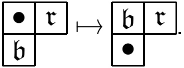
Theorem: [Thomas-Y., '08-'12] $E^{\nu}_{\lambda,\mu}(\mathrm{Gr}_k(\mathbb{C}^n)) = \sum_T \mathtt{wt}(T)$ where T rectifies to T_μ under *column order*. The $\mathtt{wt}(T)$ is a product of weights for edge labels.

This rule is positive in the sense of [Graham '01].

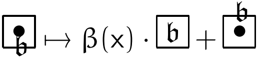
Problem: Develop a form of `Ejdt` that is more "flexible".

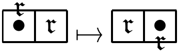
T-equivariant cohomology of Grassmannians (cont)

Our solution: [Thomas-Y., '12] Use the semistandard edge labeled tableaux and the following rules:

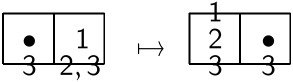
(I) "vertical swap": $\mathfrak{b} \leq \mathfrak{r}$ *(or there is no* $\mathfrak{r}$*)*: 

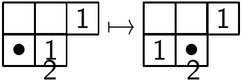
(II) "expansion swap": $\mathfrak{b} \leq \mathfrak{r}$ *and* $\mathfrak{b}$ *an edge label of* x:

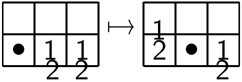


(III) "resuscitation swap": $\mathfrak{b} > \mathfrak{r}$ *(or no* $\mathfrak{b}$*)*: 

(IV) "horizontal swap": (by examples)







Equivariant jeu de taquin and rectification

Fact: [Thomas-Y., '12] `Eqjdt` is a well-defined algorithm: if T is semistandard and lattice, `Eqjdt`(T) is a *formal sum* of semistandard and lattice tableaux.

Let $S_\mu = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}$ be the **highest weight tableau**.

Theorem: [Thomas-Y., '12] Let T be a lattice semistandard tableau of content μ. Then:

(I) `Eqrect`(T) is μ-highest weight for any choice of rectification.

(II) The coefficient $[S_\mu]$`Eqrect`(T) is invariant for these choices.

(III) $[S_\mu]$`Eqrect`(T) can be computed directly from T.

Theorem: [Thomas-Y., '12] $E^{\nu}_{\lambda,\mu} = \sum_T [S_\mu]\mathtt{Eqrect}(T)$, where the sum is over lattice semistandard tableaux of shape ν/λ and having content μ.

Consequences; equivariant K-theory of Grassmannians

These results lead to proofs of the "standard" rule for $E_{\lambda,\mu}^{\nu}$.

The standard rule suggests conjectural generalizations, e.g.:

Equivariant K-theory of Grassmannians: The rule merges the increasing tableau rule for K-theory and the "standard" equivariant rule. Thus we use **increasing tableau with edge labels**. However, we also allow box labels to be marked with $\star$. But, *if i and $i+1$ appear in a row, only $i+1$ can be $\star$-marked*.
Conjecture: [Thomas-Y., '08-'12]

$$(EK)_{\lambda,\mu}^{\nu}(\mathrm{Gr}_k(\mathbb{C}^n)) = \sum_T \mathtt{sgn}(T) \cdot \mathtt{wt}_K(T)$$

where the sum is over all $T \in \mathtt{EqINC}(\nu/\lambda, |\mu|)$ such that
$\mathtt{KErect}(T) = T_\mu$ and
$\mathtt{sgn}(T) = (-1)^{\#\star\text{'s in } T \,+\, \#\text{edge labels in } T \,+\, |\nu|-|\lambda|-|\mu|}$

Equivariant K-theory of Grassmannians (cont)

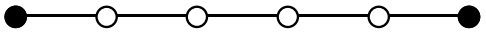
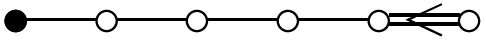
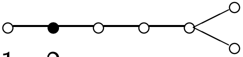
- Can simply turn off features to get our earlier rules.
- Alternative to conjecture of A. Knutson-R. Vakil from 2004.
- Recently, [Knutson '10] uses puzzles to solve a different equivariant K-theory problem.
- Easily seen to be positive in the sense of [Anderson-Griffeth-Miller '09]
- Hope semistandard version (work in progress) will lead to proof.

Adjoint Schubert calculus (joint project in progress with D. Searles)

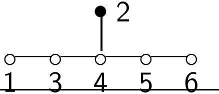
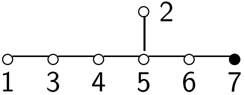
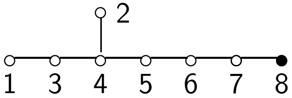
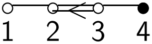
Adjoint varieties are the "next simplest" after the (co)minuscule varieties.

- G/P is **adjoint** if P is associated to an adjoint weight ω (means ω equals the highest weight of Φ^+)
- These are classified (see next two slides)
- ω is **coadjoint** if it is adjoint for the dual root system

Classification of adjoint varieties (classical types)

Root system	*Dynkin Diagram*	*Nomenclature*
A_n	 1 2 ⋯ k ⋯ n	$\mathrm{Flags}(1, n-1; \mathbb{C}^n)$
B_n	 1 2 ⋯ ⋯ n	$OG(2, 2n+1)$
$C_n, n \geq 3$	 1 2 ⋯ ⋯ n	$\mathbb{P}^{2n-1}$
$D_n, n \geq 4$	 n 1 2 ⋯ ⋯ $n-1$	$OG(2, 2n)$

Classification of adjoint varieties (exceptional types)

Root system	*Dynkin Diagram*	*Nomenclature*
E_6		E_6/P_2
E_7		E_7/P_7
E_8		E_8/P_8
F_4		F_4/P_4
G_2		G_2/P_2

Adjoint combinatorics

The adjoint spaces are interesting for our theme since *none* of the following minuscule properties hold in general:

- $\Lambda_{G/P}$ is a planar poset
- λ is a lower order ideal
- Bruhat order is just containment of shapes

But these are "almost true".

One combinatorial commonality: the **adjoint node**, i.e., the highest node of $\Lambda_{G/P}$ (i.e., ω).

Example: $\Lambda_{\mathrm{Flags}(1,n-1,\mathbb{C}^{n-1})}$ within the ambient poset Ω_{GL_n}.

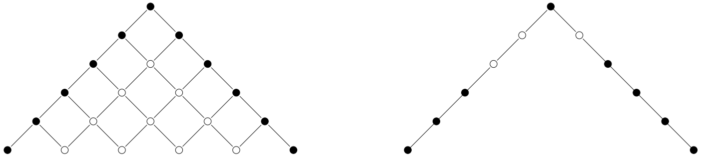


Figure: $\Lambda_{Fl_{1,n-1;n}}$, Ω_{GL_n} and a shape (for $n = 6$)

Adjoint combinatorics (cont)

Example:

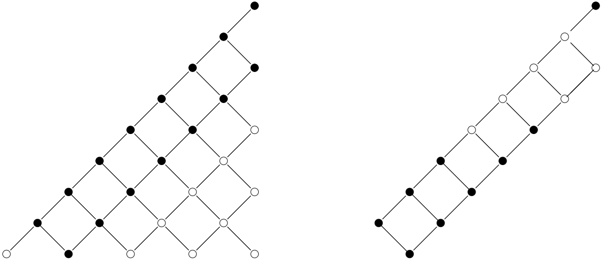


Figure: $\Lambda_{OG(2,2n+1)}$, $\Omega_{SO_{2n+1}}$ and a shape (for $n = 5$)

The short roots of $\Lambda_{OG(2,2n+1)}$ consist of the middle pair of nonadjoint nodes.

Adjoint combinatorics (cont)

Example:

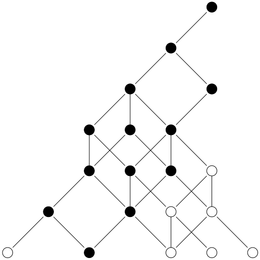
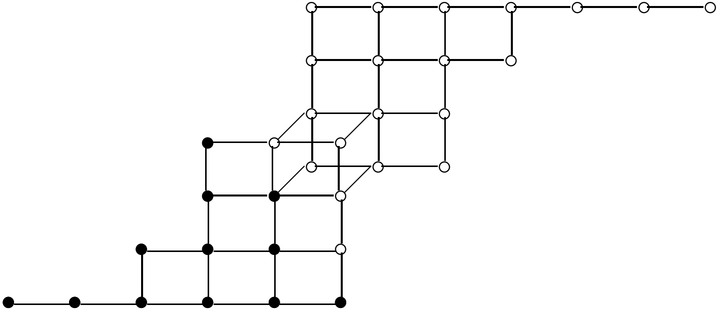


Figure: $\Lambda_{OG(2,2n)}$, $\Omega_{SO_{2n}}(\mathbb{C})$ and a shape (for $n = 5$)

$G/P = OG(2, 2n)$ is the simplest one where $\Lambda_{G/P}$ is non-planar. In fact, only this non-planarity can appear in the adjoint varieties.

Adjoint combinatorics (cont)

Example: A shape in Λ_{E_7/P_7}:



The adjoint node (not used in this case) is the rightmost node.

Adjoint combinatorics (cont)

Index the Schubert classes by $\overline{\lambda} = (\lambda, \bullet)$ or $\overline{\lambda} = (\lambda, \circ)$ depending on whether the adjoint node is used or not.

Fact: If G/P is adjoint then:

(i) $|\Lambda_{G/P}|$ is odd.

(ii) If $\overline{\lambda} = (\lambda, \circ)$ then $|\lambda| < \frac{1}{2}|\Lambda_{G/P}|$.

(iii) If $\overline{\lambda} = (\lambda, \bullet)$ then $|\lambda| > \frac{1}{2}|\Lambda_{G/P}|$

(iv) λ is a lower order ideal in the poset $\Lambda_{G/P} \setminus \{\text{adjoint node}\}$

(v) $(\lambda, \circ) \prec (\mu, \circ)$ and $(\lambda, \bullet) \prec (\mu, \bullet)$ if and only if $\lambda \subseteq \mu$

Main results for (co)adjoint G/P's

Theorem (summary): [Searles-Y., '12+] For classical (co)adjoint G/P's we have explicit nonnegative product rules. They imply:

- In type A_{n-1}; $Fl_{1,n-1;n}$: $C_{\overline{\lambda},\overline{\mu}}^{\overline{\nu}}(Fl_{1,n-1;n}) \in \{0,1\}$

$$C_{(\lambda,\circ),(\mu,\bullet)}^{(\lambda+\mu,\bullet)}(Fl_{1,n-1;n}), C_{(\lambda,\bullet),(\mu,\circ)}^{(\lambda+\mu,\bullet)}(Fl_{1,n-1;n}), C_{(\lambda,\circ),(\mu,\circ)}^{(\lambda+\mu,\circ)}(Fl_{1,n-1;n}),$$
$$C_{(\lambda,\circ),(\mu,\circ)}^{((\lambda+\mu)^{\star},\bullet)}(Fl_{1,n-1;n}), C_{(\lambda,\circ),(\mu,\circ)}^{((\lambda+\mu)_{\star},\bullet)}(Fl_{1,n-1;n}) = 1.$$

- In type C_n; $LG(2,2n)$:

I.
$$C_{(\lambda,\circ),(\mu,\circ)}^{(\nu,\circ)}(LG(2,2n)), C_{(\lambda,\bullet),(\mu,\circ)}^{(\nu,\bullet)}(LG(2,2n)), C_{(\lambda,\circ),(\mu,\bullet)}^{(\nu,\bullet)}(LG(2,2n))$$
$$= C_{\lambda,\mu}^{\nu}(\mathrm{Gr}_2(\mathbb{C}^{2n-1}));$$

II.
$$C_{(\lambda,\circ),(\mu,\circ)}^{(\nu,\bullet)}(LG(2,2n)) \;=\; C_{\lambda,\mu}^{\nu^{\star}}(\mathrm{Gr}_2(\mathbb{C}^{2n-1})) + C_{\lambda,\mu}^{\nu_{\star}}(\mathrm{Gr}_2(\mathbb{C}^{2n-1})).$$

Main results for (co)adjoint G/P's

Theorem (summary, continued): [Searles-Y., '12]

- In type B_n; $OG(2,2n+1)$:

$$C_{\overline{\lambda},\overline{\mu}}^{\overline{\nu}}(OG(2,2n+1)) = 2^{\mathrm{short}(\overline{\nu})-(\mathrm{short}(\overline{\lambda})+\mathrm{short}(\overline{\mu}))} C_{\overline{\lambda},\overline{\mu}}^{\overline{\nu}}(LG(2,2n)),$$

where $\mathrm{short}(\overline{\lambda})$ is the number of short roots of $\overline{\lambda}$, etc.

- In type D_n; $OG(2,2n)$: if $(\overline{\lambda},\overline{\mu},\overline{\nu})$ is of "main type" then

I.
$$C_{(\lambda,\circ),(\mu,\circ)}^{(\nu,\circ)}(OG(2,2n)),\ C_{(\lambda,\bullet),(\mu,\circ)}^{(\nu,\bullet)}(OG(2,2n)),\ C_{(\lambda,\circ),(\mu,\bullet)}^{(\nu,\bullet)}(OG(2,2n)) = 2^{\mathrm{fakeshort}(\pi(\nu))-(\mathrm{fakeshort}(\pi(\lambda))+\mathrm{fakeshort}(\pi(\mu)))} C_{\pi(\lambda),\pi(\mu)}^{\pi(\nu)}(Gr_2(\mathbb{C}^{2n-2}))$$

II.
$$C_{(\lambda,\circ),(\mu,\circ)}^{(\nu,\bullet)}(OG(2,2n)) = 2^{\mathrm{fakeshort}(\pi(\nu))-(\mathrm{fakeshort}(\pi(\lambda))+\mathrm{fakeshort}(\pi(\mu)))} \times \left(C_{\pi(\lambda),\pi(\mu)}^{\pi(\nu)^{\star}}(\mathrm{Gr}_2(\mathbb{C}^{2n-2})) + C_{\pi(\lambda),\pi(\mu)}^{\pi(\nu)_{\star}}(\mathrm{Gr}_2(\mathbb{C}^{2n-2}))\right).$$

Remarks on the theorem

- Earlier work of [Chaput-Perrin '12] (generalizing [Thomas-Y., 09]) gives a jeu de taquin rule when $|\lambda|, |\mu|, |\nu| < \frac{1}{2}|\Lambda_{G/P}|$.
- Our product rules build on the LR rule for $\mathrm{Gr}_2(\mathbb{C}^{2n})$ (easy) to one for $LG(2, 2n)$ (demands an "adjoint node sliding operation") to one for $OG(2, 2n)$ which projects to the $LG(2, 2n)$ case (sort of) and demands a number of "disambiguation rules".
- Ideally, one wants a root-system uniform rule. However, the rules we have show some similarities with each other and the (co)minuscule rule of [Thomas-Y., '09].

Remarks on the theorem (cont)

- We have some control of the structure constants. For example:

Adjoint								
A_{n-1}	B_n	C_n	D_n	G_2	F_4	E_6	E_7	E_8
1	8	1	8	3	8	7	33	975

Coadjoint			
B_n	C_n	G_2	F_4
2	2	2	12

Table: Maximum values of $C_{\lambda,\mu}^{\nu}(G/P)$

(In fact, we know exactly what values can occur in each case.)

Conclusions and summary

We have examined the use of root datum and sliding to study Schubert calculus of G/P in various cohomology theories.

- (Grassmannians): sliding methods extending classical jeu de taquin give us a fairly complete understanding.
- (Minuscules): these methods show promise. For K-theory, they led to explicit (now proved) rules.
- (Beyond minuscules): we discussed the adjoint varieties as a natural step.