Character sheaves on a symmetric space and Kostka polynomials

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Kostka polynomials $K_{\lambda,\mu}(t)$

$\lambda = (\lambda_1, \dots, \lambda_k)$: **partition of n**

$\lambda_i \in \mathbb{Z}_{\geq 0}, \quad \lambda_1 \geq \cdots \geq \lambda_k \geq 0, \quad \sum_i \lambda_i = n$

$\mathcal{P}_n = \{\text{partitions of } n\}$

$s_\lambda(x) = s_\lambda(x_1, \dots, x_k) \in \mathbb{Z}[x_1, \dots, x_k]$: **Schur function**

$P_\lambda(x; t) = P_\lambda(x_1, \dots, x_k; t) \in \mathbb{Z}[x_1, \dots, x_k; t]$: **Hall-Littlewood function**

For $\lambda, \mu \in \mathcal{P}_n$, $K_{\lambda,\mu}(t)$: **Kostka polynomial** defined by

$$s_\lambda(x) = \sum_{\mu \in \mathcal{P}_n} K_{\lambda,\mu}(t) P_\mu(x; t)$$

$K_{\lambda,\mu}(t) \in \mathbb{Z}[t], \qquad (K_{\lambda,\mu}(t))_{\lambda,\mu \in \mathcal{P}_n}$: transition matrix of two basis $\{s_\lambda(x)\}$, $\{P_\mu(x; t)\}$ of the space of homog. symm. poly. of degree n

Geometric realization of Kostka polynomials

In 1981, Lusztig gave a geometric realization of Kostka polynomials in connection with the closure of nilpotent orbits.

$V = \mathbb{C}^n, \quad G = GL(V)$
$\mathcal{N} = \{x \in \mathrm{End}(V) \mid x : \text{ nilpotent } \}$: nilpotent cone

$$\mathcal{P}_n \simeq \mathcal{N}/G$$

$$\lambda \leftrightarrow G\text{-orbit } \mathcal{O}_\lambda \ni x : \text{ Jordan type } \lambda$$

- **Closure relations :**

$$\overline{\mathcal{O}}_\lambda = \coprod_{\mu \le \lambda} \mathcal{O}_\mu \quad (\overline{\mathcal{O}}_\lambda : \text{ Zariski closure of } \mathcal{O}_\lambda)$$

dominance order on $\mathcal{P}_n$

For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k), \quad \mu = (\mu_1, \mu_2, \dots, \mu_k)$,

$\mu \le \lambda \Leftrightarrow \sum_{i=1}^j \mu_i \le \sum_{i=1}^j \lambda_i$ for each j.

Notation: $n(\lambda) = \sum_{i \geq 1}(i-1)\lambda_i$

Define $\widetilde{K}_{\lambda,\mu}(t) = t^{n(\mu)} K_{\lambda,\mu}(t^{-1})$: **modified Kostka polynomial**

$K = \mathrm{IC}(\overline{\mathcal{O}}_\lambda, \mathbb{C})$: Intersection cohomology complex

$$K : \cdots \longrightarrow K_{i-1} \xrightarrow{d_{i-1}} K_i \xrightarrow{d_i} K_{i+1} \xrightarrow{d_{i+1}} \cdots$$

$K = (K_i)$: bounded complex of $\mathbb{C}$-sheaves on $\overline{\mathcal{O}}_\lambda$

$\mathcal{H}^i K = \mathrm{Ker}\, d_i / \mathrm{Im}\, d_{i-1}$: i-th cohomology sheaf

$\mathcal{H}^i_x K$: the stalk at $x \in \overline{\mathcal{O}}_\lambda$ of $\mathcal{H}^i K$ (finite dim. vecotr space over $\mathbb{C}$)

Known fact : $\mathcal{H}^i K = 0$ for odd i.

Theorem (Lusztig)

For $x \in \mathcal{O}_\mu$,

$$\widetilde{K}_{\lambda,\mu}(t) = t^{n(\lambda)} \sum_{i \geq 0} (\dim_{\mathbb{C}} \mathcal{H}^{2i}_x K) t^i$$

In particular, $K_{\lambda,\mu}[t] \in \mathbb{Z}_{\geq 0}[t]$. (theorem of Lascoux-Schützenberger)

Representation theory of $GL_n(\mathbb{F}_q)$

$\mathbb{F}_q$: finite field of q elements with $\operatorname{ch}\mathbb{F}_q = p$
$\overline{\mathbb{F}}_q$: algebraic closure of $\mathbb{F}_q$

$$G = GL_n(\overline{\mathbb{F}}_q) \supset B = \left\{ \begin{pmatrix} * & \cdots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & * \end{pmatrix} \right\} \supset U = \left\{ \begin{pmatrix} 1 & \cdots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

B : Borel subgroup, U : maximal unipotent subgroup

$F : G \to G, (g_{ij}) \mapsto (g_{ij}^q)$: Frobenius map
$G^F = \{g \in G \mid F(g) = g\} = G(\mathbb{F}_q)$: finite subgroup

$\operatorname{Ind}_{B^F}^{G^F} 1$: the character of G^F obtained by inducing up 1_{B^F}

$$\operatorname{Ind}_{B^F}^{G^F} 1 = \sum_{\lambda \in \mathcal{P}_n} (\deg \chi^\lambda) \rho^\lambda,$$

ρ^λ: irreducible character of G^F corresp. to $\chi^\lambda \in S_n^\wedge \simeq \mathcal{P}_n$.

$G_{\text{uni}} = \{g \in G \mid u\text{: unipotent}\} \subset G, \quad G_{\text{uni}} \simeq \mathcal{N}, u \leftrightarrow u - 1$

- $G_{\text{uni}}/G \simeq \mathcal{P}_n, \quad \mathcal{O}_\lambda \leftrightarrow \lambda$

$\mathcal{O}_\lambda$: F-stable $\Longrightarrow \mathcal{O}_\lambda^F$: single G^F-orbit, $u_\lambda \in \mathcal{O}_\lambda^F$

Theorem (Green)

$$\rho^\lambda(u_\mu) = \widetilde{K}_{\lambda,\mu}(q)$$

Remark : Lusztig's result $\Longrightarrow$ the character values of ρ^λ at **unipotent elements** are described in terms of intersection cohomology complex.

Theory of character sheaves $\Longrightarrow$ describes **all the character values** of **any irreducible characters** in terms of certain simple perverse sheaves.

Enhanced nilpotent cone

$\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)}), \quad \sum_{i=1}^{r} |\lambda^{(i)}| = n$: ***r*-partition of *n***

$\mathcal{P}_{n,r}$: the set of r-partitions of n

S (2004) : for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_{n,r}$, introduced $K_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t) \in \mathbb{Q}(t)$:
Kostka functions associated to complex reflection groups
as the transition matrix between the bases of Schur functions $\{s_{\boldsymbol{\lambda}}(x)\}$ and "Hall-Littlewood functions" $\{P_{\boldsymbol{\mu}}(x;t)\}$.

Achar-Henderson (2008) : geometric realization
of Kostka functions for $r = 2$

$V = \mathbb{C}^n, \quad \mathcal{N}$: nilpotent cone
$\mathcal{N} \times V$: **enhanced nilpotent cone**, action of $G = GL(V)$

Achar-Henderson, Travkin :

$$(\mathcal{N} \times V)/G \simeq \mathcal{P}_{n,2}, \quad \mathcal{O}_{\boldsymbol{\lambda}} \leftrightarrow \boldsymbol{\lambda}$$

$K = \mathrm{IC}(\overline{\mathcal{O}}_{\boldsymbol{\lambda}}, \mathbb{C})$: Intersection cohomology complex

Theorem (Achar-Henderson)

$\mathcal{H}^i K = 0$ for odd i. For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_{n,2}$, and $(x, v) \in \mathcal{O}_{\boldsymbol{\mu}} \subseteq \overline{\mathcal{O}}_{\boldsymbol{\lambda}}$,

$$t^{a(\boldsymbol{\lambda})} \sum_{i \geq 0} (\dim_{\mathbb{C}} \mathcal{H}^{2i}_{(x,v)} K) t^{2i} = \widetilde{K}_{\boldsymbol{\lambda}, \boldsymbol{\mu}}(t),$$

where $a(\boldsymbol{\lambda}) = 2n(\lambda^{(1)}) + 2n(\lambda^{(2)}) + |\lambda^{(2)}|$ for $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$.

$\mathcal{N} \times V \rightsquigarrow G_{\mathrm{uni}} \times V \hookrightarrow G \times V$ (over $\overline{\mathbb{F}}_q$) : action of $G = GL(V)$

Finkelberg-Ginzburg-Travkin (2008) : Theory of character sheaves on $G \times V$ (certain G-equiv. simple perverse sheaves)

$\Longrightarrow$ **"character table"** of $(G \times V)^F$

good basis of the space of G^F-invariant functions on $(G \times V)^F$

Finite symmetric space $GL_{2n}(\mathbb{F}_q)/Sp_{2n}(\mathbb{F}_q)$

$G = GL(V) \simeq GL_{2n}(\overline{\mathbb{F}}_q), \quad V = (\overline{\mathbb{F}}_q)^{2n}, \quad \mathrm{ch}\,\mathbb{F}_q \neq 2$

$\theta : G \to G$, $\theta(g) = J^{-1}({}^t g^{-1})J$: involution, $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$

$K := \{g \in G \mid \theta(g) = g\} \simeq Sp_{2n}(\overline{\mathbb{F}}_q) \quad G/K$: symmetric space over $\overline{\mathbb{F}}_q$

$G^F \simeq GL_{2n}(\mathbb{F}_q) \supset Sp_{2n}(\mathbb{F}_q) \simeq K^F$

G^F acts on $G^F/K^F \rightsquigarrow 1_{K^F}^{G^F}$: induced representation

$H(G^F, K^F) := \mathrm{End}_{G^F}(1_{K^F}^{G^F})$: **Hecke algebra** asoc. to (G^F, K^F)

- $H(G^F, K^F)$: commutative algebra
- $H(G^F, K^F)^\wedge$: natural labeling by $(GL_n^F)^\wedge$
- $K^F \backslash G^F / K^F$: natural labeling by $\{$ conj. classes of $GL_n^F\}$

Theorem (Bannai-Kawanka-Song, 1990)

The character table of $H(G^F, K^F)$ can be obtained from the character table of GL_n^F by replacing $q \mapsto q^2$.

Geometric setting for G/K

$$G^{\iota\theta} = \{g \in G \mid \theta(g) = g^{-1}\} \\ = \{g\theta(g)^{-1} \mid g \in G\},$$

where $\iota : G \to G, g \mapsto g^{-1}$.

The map $G \to G,\ g \mapsto g\theta(g)^{-1}$ gives isom. $G/K \xrightarrow{\sim} G^{\iota\theta}$.

K acts by left mult $\curvearrowright G/K \simeq G^{\iota\theta} \curvearrowleft K$ acts by conjugation.

$$K\backslash G/K \simeq \{K\text{-conjugates of } G^{\iota\theta}\}$$

Henderson : Geometric reconstruction of BKS (not complete)

Lie algebra analogue

$\mathfrak{g} = \mathfrak{gl}_{2n}, \quad \theta : \mathfrak{g} \to \mathfrak{g}$: involution, $\mathfrak{g} = \mathfrak{g}^{\theta} \oplus \mathfrak{g}^{-\theta}$,

$\mathfrak{g}^{\pm\theta} = \{x \in \mathfrak{g} \mid \theta(x) = \pm x\}$, $\quad K$-stable subspace of $\mathfrak{g}$

$\mathfrak{g}^{-\theta}_{\mathrm{nil}} = \mathfrak{g}^{-\theta} \cap \mathcal{N}_{\mathfrak{g}}$: analogue of nilpotent cone $\mathcal{N}$, K-stable subset of $\mathfrak{g}^{-\theta}$

$$\mathfrak{g}^{-\theta}_{\mathrm{nil}}/K \simeq \mathcal{P}_n, \quad \mathcal{O}_\lambda \leftrightarrow \lambda$$

Theorem (Henderson + BKS, 2008)

Let $K = \mathrm{IC}(\overline{\mathcal{O}}_\lambda, \bar{\mathbb{Q}}_l)$, $x \in \mathcal{O}_\mu \subset \overline{\mathcal{O}}_\lambda$. Then $\mathcal{H}^i K = 0$ unless $i \equiv 0 \pmod 4$, and

$$t^{2n(\lambda)} \sum_{i \geq 0} (\dim \mathcal{H}^{4i}_x K) t^{2i} = \widetilde{K}_{\lambda,\mu}(t^2)$$

Exotic symmetric space $GL_{2n}/Sp_{2n} \times V$

(Joint work with K. Sorlin)

Bannai-Kawanaka-Song

Green, Lusztig

Henderson

$\widetilde{K}_{\lambda,\mu}(t)$ $(r = 1)$ $GL_n \longrightarrow GL_{2n}/Sp_{2n}$ $\widetilde{K}_{\lambda,\mu}(t^2)$ $(r = 1)$

Want to show

$\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t^{1/2})$ $(r = 2)$ $GL_n \times V_n \longrightarrow GL_{2n}/Sp_{2n} \times V_{2n}$ $\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$ $(r = 2)$

???

Achar-Henderson

Finkelberg-Ginzburg-Travkin

Kato's exotic nilcone

$G = GL(V) \simeq GL_{2n}(\overline{\mathbb{F}}_q), \quad \dim V = 2n, \quad K = G^{\theta}.$

$\mathcal{X} = G^{\iota\theta} \times V$: K action

Problem

- Find a good class of K-equivariant simple perverse sheaves on $G^{\iota\theta} \times V$, i.e., **"character sheaves"** on $G^{\iota\theta} \times V$
- Find a good basis of K^F-equivariant functions on $(G^{\iota\theta} \times V)^F$, i.e., **"irreducible characters"** of $(G^{\iota\theta} \times V)^F$, and compute their values, i.e., computaion of the **"character table"**

Remark : $\mathcal{X}_{\mathrm{uni}} := G^{\iota\theta}_{\mathrm{uni}} \times V \simeq \mathfrak{g}^{-\theta}_{\mathrm{nil}} \times V$: **Kato's exotic nilcone**

Kato $\mathcal{X}_{\mathrm{uni}}/K \simeq (\mathfrak{g}^{-\theta}_{\mathrm{nil}} \times V)/K \simeq \mathcal{P}_{n,2}, \quad \mathcal{O}_{\boldsymbol{\mu}} \leftrightarrow \boldsymbol{\mu} \in \mathcal{P}_{n,2}$

Natural bijection with GL_n-orbits of enhanced nilcone, compatible with closure relations (**Achar-Henderson**)

Constrcution of character sheaves on $\mathcal{X}$

$T \subset B$ θ-stable maximal torus, θ-stable Borel sdubgroup of G

M_n : maximal isotropic subspace of V stable by B^θ

$\widetilde{\mathcal{X}} = \{(x, v, gB^\theta) \in G^{\iota\theta} \times V \times K/B^\theta \mid g^{-1}xg \in B^{\iota\theta}, g^{-1}x \in M_n\}$
$\pi : \widetilde{\mathcal{X}} \to \mathcal{X}, (x, v, gB^\theta) \mapsto (x, v),$
$\alpha : \widetilde{\mathcal{X}} \to T^{\iota\theta}, (x, v, gB^\theta) \mapsto \overline{g^{-1}xg}, \quad (b \mapsto \overline{b}$: projection $B^{\iota\theta} \to T^{\iota\theta}$)

$$T^{\iota\theta} \xleftarrow{\ \alpha\ } \widetilde{\mathcal{X}} \xrightarrow{\ \pi\ } \mathcal{X}$$

$\mathcal{E}$: tame local system on $T^{\iota\theta} \rightsquigarrow K_{T,\mathcal{E}} = \pi_*\alpha^*\mathcal{E}[\dim \mathcal{X}]$
$K_{T,\mathcal{E}}$: semisimple perverse sheaf on $\mathcal{X}$

Definition $\widehat{\mathcal{X}}$: (**Character sheaves on $\mathcal{X}$**) K-equiv. simple perverse sheaves on $\mathcal{X}$, appearing as a direct summand of various $K_{T,\mathcal{E}}$.

$\mathbb{F}_q$-structures on $\chi_{T,\mathcal{E}}$ and Green functions

$(T, \mathcal{E})$ as before
Assume T : F-stable, (but B: not necessarily F-stable), $\quad F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$.

Obtain canonical isomorphism $\varphi : F^*K_{T,\mathcal{E}} \xrightarrow{\sim} K_{T,\mathcal{E}}$

Define a **characteristic function** $\chi_{K,\varphi}$ of $K = K_{T,\mathcal{E}}$ by

$$\chi_{K,\varphi}(z) = \sum_i (-1)^i \operatorname{Tr}(\varphi, \mathcal{H}^i_z K) \qquad (z \in \mathcal{X}^F)$$

$\chi_{K,\varphi}$: K^F-invariant function on $\mathcal{X}^F$
Put $\chi_{T,\mathcal{E}} = \chi_{K_{T,\mathcal{E}},\varphi}$ for each $(T, \mathcal{E})$.

Proposition-Definition

$\chi_{T,\mathcal{E}}|_{\mathcal{X}^F_{\mathrm{uni}}}$ is independent of the choice of $\mathcal{E}$ on $T^{\iota\theta}$. We define $Q_T : \mathcal{X}^F_{\mathrm{uni}} \to \bar{\mathbb{Q}}_l$ by $Q_T = \chi_{T,\mathcal{E}}|_{\mathcal{X}^F_{\mathrm{uni}}}$, and call it **Green function** on $\mathcal{X}^F_{\mathrm{uni}}$.

Character formula

For $s \in G^{\iota\theta}$, semisimple, $Z_G(s)$: θ-stable, and $Z_G(s) \times V$ has a similar structure as $\mathcal{X} = G^{\iota\theta} \times V$. Then Green function $Q_{T'}^{Z_G(s)}$ (T': θ-stable maximal torus in $Z_G(s)$) can be defined simialr to Q_T

Theorem (Character formula)

Let $s, u \in (G^{\iota\theta})^F$ be such that $su = us$, with s: semisimple, u: unipotent. Assume that $\mathcal{E} = \mathcal{E}_\vartheta$: F-stable tame local system on $T^{\iota\theta}$ with $\vartheta \in (T^{\iota\theta, F})^\wedge$. Then

$$\chi_{T,\mathcal{E}}(su, v) = |Z_K(s)^F|^{-1} \sum_{\substack{x \in K^F \\ x^{-1}sx \in T^{\iota\theta,F}}} Q_{xTx^{-1}}^{Z_G(s)}(u, v)\vartheta(x^{-1}sx)$$

Remark The computation of the function $\chi_{T,\mathcal{E}}$ is reduced to the computation of Green functions $Q_{xTx^{-1}}^{Z_G(s)}$ for various semisimple $s \in G^{\iota\theta}$.

Orthogonality relations

Theorem (Orthogonality relations for $\chi_{T,\mathcal{E}}$)

Assume that T, T' are F-stable, θ-stable maximal tori in G as before. Let $\mathcal{E} = \mathcal{E}_\vartheta, \mathcal{E}' = \mathcal{E}_{\vartheta'}$ be tame local systems on $T^{\iota\theta}$, $T'^{\iota\theta}$ with $\vartheta \in (T^{\iota\theta,F})^\wedge, \vartheta' \in (T'^{\iota\theta,F})^\wedge$. Then

$$
\begin{aligned}
|K^F|^{-1} & \sum_{(x,v)\in\mathcal{X}^F} \chi_{T,\mathcal{E}}(x,v)\chi_{T',\mathcal{E}'}(x,v) \\
&= |T^{\theta,F}|^{-1}|T'^{\theta,F}|^{-1} \sum_{\substack{n\in N_K(T^\theta,T'^\theta)^F \\ t\in T^{\iota\theta,F}}} \vartheta(t)\vartheta'(n^{-1}tn)
\end{aligned}
$$

Theorem(Orthogonality relations for Green functions)

$$
|K^F|^{-1} \sum_{(u,v)\in\mathcal{X}^F_{\mathrm{uni}}} Q_T(u,v)_{T'}(u,v) = \frac{N_K(T^\theta, T'^\theta)^F|}{|T^{\theta,F}||T'^{\theta,F}|}
$$

Springer correspondence

We consider the case $\mathcal{E} = \bar{\mathbb{Q}}_l$: constant sheaf on $T^{\iota\theta}$.
Then $K_{T,\mathcal{E}} = \pi_* \bar{\mathbb{Q}}_l[\dim \mathcal{X}]$.

$M_0 \subset M_1 \subset \cdots \subset M_n$: isotorpic flag stable by B^θ

Define $\mathcal{X}_m = \bigcup_{g \in K} g(B^{\iota\theta} \times M_m)$. Then $\mathcal{X}_m$: closed in $\mathcal{X} = \mathcal{X}_n$,

$$\mathcal{X}_0 \subset \mathcal{X}_1 \subset \cdots \subset \mathcal{X}_n = \mathcal{X}.$$

$W_n = N_K(T^\theta)/T^\theta$: Weyl group of type C_n, $W_n^\wedge \simeq \mathcal{P}_{n,2}$

Proposition 1

$\pi_* \bar{\mathbb{Q}}_l[\dim \mathcal{X}]$: a semisimple perverse sheaf with W_n-action, is decomposed as

$$\pi_* \bar{\mathbb{Q}}_l[\dim \mathcal{X}] \simeq \bigoplus_{\boldsymbol{\mu} \in \mathcal{P}_{n,2}} V_{\boldsymbol{\mu}} \otimes \mathrm{IC}(\mathcal{X}_{m(\boldsymbol{\mu})}, \mathcal{L}_{\boldsymbol{\mu}})[\dim \mathcal{X}_{m(\boldsymbol{\mu})}],$$

$V_{\boldsymbol{\mu}}$: standard irred. W_n-module, $m(\boldsymbol{\mu}) = |\mu^{(1)}|$ for $\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)})$,
$\mathcal{L}_{\boldsymbol{\mu}}$: local system on a smooth open subset of $\mathcal{X}_{m(\boldsymbol{\mu})}$.

Theorem (Springer correspondence)

For each $\boldsymbol{\mu} \in \mathcal{P}_{n,2}$,

$$\mathrm{IC}(\mathcal{X}_{m(\boldsymbol{\mu})}, \mathcal{L}_{\boldsymbol{\mu}})|_{\mathcal{X}_{\mathrm{uni}}} \simeq IC(\overline{\mathcal{O}}_{\boldsymbol{\mu}}, \bar{\mathbb{Q}}_l) \qquad \text{(up to shift).}$$

Hence $V_{\boldsymbol{\mu}} \mapsto \mathcal{O}_{\boldsymbol{\mu}}$ gives a bijective correspondence $W_n^{\wedge} \simeq \mathcal{X}_{\mathrm{uni}}/K$.

Remark

1. Springer correspondence was first proved by Kato for the exotic nilcone by using Ginzburg theroy on affine Hecke algebras.
2. The proof of the theorem is divided into two steps. In the first step, we show the existence of the bijection $W_n^{\wedge} \xrightarrow{\sim} \mathcal{X}_{\mathrm{uni}}/K$. In the second step, we determine this map explcitly, by using an analogy of the restriction theorem due to Lusztig.

Green functions and Springer correspondence

We denote by T_w F-stable, θ-stable maximal torus of G corresp. to $w \in W_n \subset S_{2n}$.

For $A_{\boldsymbol{\mu}} = \mathrm{IC}(\overline{\mathcal{O}}_{\boldsymbol{\mu}}, \bar{\mathbb{Q}}_l)[\dim \mathcal{O}_{\boldsymbol{\mu}}]$, we have a unique isomorphism $\varphi_{\boldsymbol{\mu}} : F^*A_{\boldsymbol{\mu}} \xrightarrow{\sim} A_{\boldsymbol{\mu}}$ induced from $\varphi : F^*K_{T_1,\bar{\mathbb{Q}}_l} \xrightarrow{\sim} K_{T_1,\bar{\mathbb{Q}}_l}$

By using the Springer correspondence, we have

$$Q_{T_w} = (-1)^{\dim \mathcal{X} - \dim \mathcal{X}_{\mathrm{uni}}} \sum_{\boldsymbol{\mu} \in \mathcal{P}_{n,2}} \chi^{\boldsymbol{\mu}}(w) \chi_{A_{\boldsymbol{\mu}}, \varphi_{\boldsymbol{\mu}}},$$

where $\chi^{\boldsymbol{\mu}}$ is the irreducible characters of W_n corresp. to $V_{\boldsymbol{\mu}}$.

Define, for each $\boldsymbol{\lambda} \in \mathcal{P}_{n,2}$,

$$Q_{\boldsymbol{\lambda}} = |W_n|^{-1} \sum_{w \in W_n} \chi^{\boldsymbol{\lambda}}(w) Q_{T_w}$$

Then by using the orthogonality relations for Green functions, we have

Proposition 2

For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_{n,2}$,

$$|K^F|^{-1} \sum_{(u,v)\in\mathcal{X}_{\mathrm{uni}}^F} Q_{\boldsymbol{\lambda}}(u,v)Q_{\boldsymbol{\mu}}(u,v) = |W_n|^{-1} \sum_{w\in W_n} |T_w^{\theta,F}|^{-1}\chi^{\boldsymbol{\lambda}}(w)\chi^{\boldsymbol{\mu}}(w).$$

Remark By definition, we have

$$Q_{\boldsymbol{\lambda}} = (-1)^{\dim\mathcal{X}-\dim\mathcal{X}_{\mathrm{uni}}}\chi_{A_{\boldsymbol{\lambda}},\varphi_{\boldsymbol{\lambda}}}$$

for $A_{\boldsymbol{\lambda}} = \mathrm{IC}(\overline{\mathcal{O}}_{\boldsymbol{\lambda}}, \bar{\mathbb{Q}}_l)[\dim\mathcal{O}_{\boldsymbol{\lambda}}]$.

Characterization of Kostka polynomials

For any character χ of W_n, we define

$$R(\chi) = \frac{\prod_{i=1}^{n}(t^{2i}-1)}{|W_n|}\sum_{w\in W_n}\frac{\varepsilon(w)\chi(w)}{\det_{V_0}(t-w)},$$

where ε : sign character of W_n, and V_0 : reflection module of W_n.

$R(\chi) =$ graded multiplicity of χ in the coinvariant algerba $R(W_n)$.

Define a matrix $\Omega = (\omega_{\boldsymbol{\lambda},\boldsymbol{\mu}})_{\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathcal{P}_{n,2}}$ by

$$\omega_{\boldsymbol{\lambda},\boldsymbol{\mu}} = t^N R(\chi^{\boldsymbol{\lambda}}\otimes\chi^{\boldsymbol{\mu}}\otimes\varepsilon).$$

Define a partial order $\boldsymbol{\lambda}\le\boldsymbol{\mu}$ on $\mathcal{P}_{n,2}$ by the condition for any j;

$$\sum_{i=1}^{j}(\lambda_i^{(1)}+\lambda_i^{(2)}) \le \sum_{i=1}^{j}(\mu_i^{(1)}+\mu_i^{(2)})$$

$$\sum_{i=0}^{j}(\lambda_i^{(1)}+\lambda_i^{(2)})+\lambda_{j+1}^{(1)} \le \sum_{0=1}^{j}(\mu_i^{(1)}+\mu_i^{(2)})+\mu_{j+1}^{(1)}.$$

Theorem (S)

There exists a unique matrices $P = (p_{\boldsymbol{\lambda},\boldsymbol{\mu}}), \varLambda = (\xi_{\boldsymbol{\lambda},\boldsymbol{\mu}})$ over $\mathbb{Q}[t]$ satisfying the equation

$$P \varLambda\, {}^t P = \Omega$$

subject to the condition that $\varLambda$ is a diagonal matrix and

$$p_{\boldsymbol{\lambda},\boldsymbol{\mu}} = \begin{cases} 0 & \text{unless } \boldsymbol{\mu} \le \boldsymbol{\lambda}, \\ t^{a(\boldsymbol{\lambda})} & \text{if } \boldsymbol{\mu} = \boldsymbol{\lambda}. \end{cases}$$

Then the entry $p_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ coincides with $\widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t)$.

Remark. Under this setup, we have

$$\omega_{\boldsymbol{\lambda},\boldsymbol{\mu}}(q) = |K^F||W_n|^{-1} \sum_{w \in W_n} |T_w^{\theta,F}|^{-1} \chi^{\boldsymbol{\lambda}}(w)\chi^{\boldsymbol{\mu}}(w).$$

Conjecture of Achar-Henderson

Main Theorem

Let $\mathcal{O}_{\boldsymbol{\lambda}}$ be the orbit in $\mathcal{X}_{\mathrm{uni}}$ corresp. to $\boldsymbol{\lambda} \in \mathcal{P}_{n,2}$, and put $K = \mathrm{IC}(\overline{\mathcal{O}}_{\boldsymbol{\lambda}}, \bar{\mathbb{Q}}_l)$. Then for $(x, v) \in \mathcal{O}_{\boldsymbol{\mu}} \subset \overline{\mathcal{O}}_{\boldsymbol{\lambda}}$, we have $\mathcal{H}^i K = 0$ unless $i \equiv 0 \pmod 4$, and

$$t^{a(\boldsymbol{\lambda})} \sum_{i \geq 0} (\dim \mathcal{H}^{4i}_{(x,v)} K) t^{2i} = \widetilde{K}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(t).$$

Remarks.

1. The theorem was first proved (for the exotic nilcone over $\mathbb{C}$) by Kato by a different meothod.
2. By a similar argument, we can (re)prove a result of Henderson concerning the orbits in $G^{\iota\theta} \simeq \mathfrak{g}^{-\theta}$, without appealing the result from BKS.

K^F-invariant functions on $(G^{\iota\theta} \times V)^F$

Let $\widehat{\mathcal{X}}$ be the set of character sheaves on $\mathcal{X} = G^{\iota\theta} \times V$.

Put $\widehat{\mathcal{X}}^F = \{A \in \widehat{\mathcal{X}} \mid F^*A \simeq A\}$.

For each $A \in \widehat{\mathcal{X}}^F$, fix an isomorphism $\varphi_A : F^*A \xrightarrow{\sim} A$, and consider the characteristic function χ_{A,φ_A}.

Let $C_q(\mathcal{X})$ be the $\bar{\mathbb{Q}}_l$-space of K^F-invariant functions on $\mathcal{X}^F$.

Theorem

1. There exists an algorithm of computing χ_{A,φ_A} for each $A \in \widehat{\mathcal{X}}^F$.
2. The set $\{\chi_{A,\varphi_A} \mid A \in \widehat{\mathcal{X}}^F\}$ gives a basis of $C_q(\mathcal{X})$.