# K-Chevalley Rule for Kac-Moody Flag Manifolds

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# Kac-Moody flag manifold

| $G$ | $B$ | $T$ |
|---|---|---|
| Kac-Moody "group" | Borel | max torus |

$X = G/B$ thick Kac-Moody flag manifold

$X_w^\circ = B_- wB/B$ Schubert cell (codim $\ell(w)$)

$X_w = \overline{X_w^\circ} = \bigsqcup_{v \geq w} X_v^\circ$ Schubert variety

$X_{\mathrm{id}}^\circ$ big cell

$$X = \bigsqcup_{w \in W} X_w^\circ$$

Flag ind-variety

$$X_{\mathrm{ind}} = \bigsqcup_{w \in W} BwB/B \subset X$$

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# Exhaustion of $X$ by quasicompact open sets

$S \subset W$ finite Bruhat order ideal
($w \in S$ and $v \leq w \Rightarrow v \in S$)

$$\Omega_S = \bigsqcup_{w \in S} X_w^\circ = \bigcup_{w \in S} wX_{\mathrm{id}}^\circ$$

$$wX_{\mathrm{id}}^\circ \cong X_{\mathrm{id}}^\circ \cong \mathrm{Spec}\,\mathbb{C}[x_1, x_2, \ldots] \qquad \text{if } \dim(X) = \infty.$$

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# $K$-theory ring

$K^*_{B_-}(\Omega_S)$ Grothendieck group of $B_-$-equivariant coherent sheaves on $\Omega_S$

$$K^*_{B_-}(\Omega_S) = \bigoplus_{w \in \Omega_S} K^*_{B_-}(pt) O_{X_w}$$

$$K^*_{B_-}(X) := \lim_{\longleftarrow} K^*_{B_-}(\Omega_S) \cong \prod_{w \in W} K^*_{B_-}(pt) O_{X_w}$$

$$K^*_{B_-}(pt) \cong K^*_T(pt) = R(T) = \mathbb{Z}[e^\lambda \mid \lambda \in \Lambda].$$

$\Lambda$: weight lattice

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# K-GKM ring

$X^T = WB/B \cong W$ $T$-fixed points in $X$

$$\mathrm{res} : K^*_{B_-}(X) \hookrightarrow K^*_{B_-}(X^T) \cong \mathrm{Fun}(W, R(T))$$

$$i_w : \{pt\} \to \{wB/B\} \subset X^T \subset X$$

$$\mathrm{res}(c)(w) = i_w^*(c) \qquad \text{for } c \in K^*_{B_-}(X),\ w \in W.$$

$f : W \to R(T)$ satisfies the K-GKM (Goresky-Kottwitz-Macpherson) condition [Harada,Henriques,Holm] if

$$f(s_\alpha w) - f(w) \in (1 - e^\alpha)R(T) \qquad \text{for all } \alpha, w$$

**Theorem**

*[Kostant and Kumar for $X_{ind}$] [Kashiwara; Lam,Schilling,S. for $X$]*

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# Localization of Schubert classes

Let $\psi^v : W \to R(T)$ be the image of $O_{X_v}$ under $\mathrm{res}$.

Theorem

*[Kostant,Kumar] [Lam, Schilling, S.]*

*The functions $\{\psi^v \mid v \in W\}$ are uniquely determined by:*

- $\psi^v(id) = \delta_{v,id}$ *for* $v \in W$.
- *For* $w \neq id$ *let* $ws_i < w$. *Then*

$$\psi^v(w) = \begin{cases} \psi^v(ws_i) & \text{if } vs_i > v \\ (1 - e^{-w\alpha_i})\psi^{vs_i}(w) + e^{-w\alpha_i}\psi^v(ws_i) & \text{if } vs_i < v. \end{cases}$$

I have a `sage-combinat` program to compute $\psi^v(w)$.

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weight $\lambda \in \Lambda$

Line bundle class $L^\lambda \in K^*_{B_-}(X)$.

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# $K$-Chevalley coefficients

Define $b^w_{v,\lambda} \in R(T)$ by

$$L^\lambda O_v = \sum_w b^w_{v,\lambda} O_w.$$

For $\dim(X) < \infty$ the elements $L^\lambda$ generate $K^*_{B_-}(X)$.

For $\dim(X) = \infty$ they do not.

In affine type they still determine the product in $K^*_{B_-}(X)$ [Kashiwara,S.].

In general ???

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# Main result

## Theorem (Lenart, S.)

*Explicit effective (cancellation-free) formulas for $b_{v,\lambda}^{w}$ for $\lambda$ dominant and for $\lambda$ antidominant.*

Two combinatorial versions:

- Using Lakshmibai-Seshadri (LS) paths (canonical Littelmann paths)
- Using the alcove path model of [Lenart,Postnikov]

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# Idea of proof

- Push-pull (divided difference) operator on Schubert classes leads to recurrence for $b_{v\lambda}^{w}$.
- Show the LS path formula satisfies the recurrence.
- Deduce alcove path formula using a new direct bijection from LS paths to alcove paths.

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# $(\lambda, b)$-Bruhat order

Fix dominant $\lambda$. For $b \in \mathbb{Q}$, define

$$w \lessdot_{\lambda,b} ws_\alpha$$

if

$$w \lessdot ws_\alpha \qquad \text{and} \qquad b\langle \alpha^\vee , \lambda \rangle \in \mathbb{Z}$$

# Lakshmibai-Seshadri paths

An LS path of shape $\lambda$ is a pair of sequences $p = (b_i; \sigma_i)$

$$0 = b_1 < b_2 < \cdots < b_m < b_{m+1} = 1 \qquad b_i \in \mathbb{Q}$$

and

$$\sigma_1 <_{\lambda, b_2} \sigma_2 <_{\lambda, b_3} \cdots <_{\lambda, b_m} \sigma_m \qquad \sigma_i \in W/W_\lambda.$$

This data specifies a piecewise linear path given by walking along the vectors

$$b_m \sigma_m \lambda, \quad , \cdots , \quad b_2 \sigma_2 \lambda, \quad b_1 \sigma_1 \lambda.$$

$\iota(p) = \sigma_m$ is the initial direction of $p$

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# Littelmann's Theorem

$\lambda$ dominant

### Theorem (Littelmann)

*The set of LS paths of shape $\lambda$ affords the crystal graph of the irreducible integrable G-module of highest weight $\lambda$.*

# Deodhar's lift "up"

$vW_\lambda = \sigma \leq \tau$ in $W/W_\lambda$.

**Theorem (Deodhar)**

*There is a unique Bruhat-minimum $w \in W$ such that*

$$v \leq w \qquad \textit{and} \qquad wW_\lambda = \tau.$$

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$vW_\lambda = \sigma \leq \tau$ in $W/W_\lambda$.

## Theorem (Deodhar)

*There is a unique Bruhat-minimum $w \in W$ such that*

$$v \leq w \qquad \textit{and} \qquad wW_\lambda = \tau.$$

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$w = \mathrm{up}(v, \tau)$

# Lifting an LS path

Let $v = v_0 \in W$, $p = (b_i; \sigma_i)$ an LS path with $vW_\lambda = \sigma_0 \leq \sigma_1 = \phi(p)$.

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# $K$-Chevalley rule, LS paths, dominant weight

$$L^\lambda O_v = \sum_w b^w_{v,\lambda} O_w$$

**Theorem**

*[Pittie,Ram for $\dim(X) < \infty$] [Lenart, S.]*
*For $\lambda$ dominant*

$$b^w_{v,\lambda} = \sum_p e^{p(1)}$$

*where $p$ is an LS path of shape $\lambda$, $vW_\lambda \le \phi(p)$ and $\mathrm{up}(v,p) = w$.*

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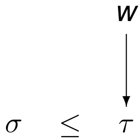

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# $K$-Chevalley rule: LS paths, antidominant weight

**Theorem**

*[Griffeth,Ram for* $\dim(X) < \infty$*] [Lenart, S.] For* $\lambda$ *dominant*

$$b^w_{v,-\lambda} = (-1)^{\ell(w)-\ell(v)} \sum_p e^{-p(1)}$$

*where* $p$ *is an LS path of shape* $\lambda$ *with* $\iota(p) \leq wW_\lambda$ *and* $\mathrm{down}(w,p) = v$*.*

# $\lambda$-hyperplanes

$\lambda$ dominant weight

A $\lambda$-hyperplane is a pair $h = (\alpha^\vee, k)$ with $\alpha^\vee$ a coroot, $k \in \mathbb{Z}$, and

$$0 \leq k < \langle \alpha^\vee, \lambda \rangle.$$

$$H_{\alpha^\vee, k} = \{x \in \Lambda_\mathbb{R} \mid \langle \alpha^\vee, x \rangle = k\}$$

Relative height of $h$: $\mathrm{rht}(h) \in \mathbb{Q}$

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# lex order on $\lambda$-hyperplanes

Fix any total order $1 < 2 < \cdots < r$ on the Dynkin node set $I$.

$$\alpha^\vee = \sum_{i=1}^{r} c_i \alpha_i^\vee \qquad \beta^\vee = \sum_{i=1}^{r} d_i \alpha_i^\vee$$

$$(\alpha^\vee, k) < (\beta^\vee, \ell)$$

means

$$\frac{1}{\langle \alpha^\vee , \lambda \rangle}(k, c_1, c_2, \dots, c_r) <_{\text{lex}} \frac{1}{\langle \beta^\vee , \lambda \rangle}(\ell, d_1, d_2, \dots, d_r).$$

In particular

$$\mathrm{rht}(\alpha^\vee, k) = \frac{k}{\langle \alpha^\vee , \lambda \rangle} \le \frac{\ell}{\langle \beta^\vee , \lambda \rangle} = \mathrm{rht}(\beta^\vee, \ell)$$

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$(h_1, h_2, \ldots, h_q)$ is $[v, w]$-adapted if

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reflection across hyperplane $H_{\alpha^\vee,k}$.

**Theorem (Lenart, S.)**

*For $\lambda$ dominant,*

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# Forgetful bijection from alcove paths to LS paths

Fix $v$. There is a crystal graph isomorphism

$$\{\ [v, w]\text{-adapted } h_1 < h_2 < \cdots < h_q \text{ for some } w \geq v\ \} \longleftrightarrow \{\ \text{LS paths } p \text{ of shape } \lambda \text{ with } \phi(p) \geq v\ \}$$

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- Let $0 < b_2 < b_3 < \cdots < b_m < 1$ be the distinct relative heights (and set $b_1 = 0$).
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# Alcove model feature

Demazure module inside highest weight module

$$U_q(\mathfrak{b}_+) \cdot v_{w\lambda} \subset V(\lambda)$$

Opposite Demazure module:

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In the alcove model the crystal graphs of (opposite) Demazure crystals can be generated using only $\mathbb{Z}$-labeled Bruhat (co)covers and lex order on branches; no signature rule is needed.

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# Ongoing and future work

$G_{\mathrm{af}}$ affine type

Consider LS paths of shape $\lambda$ of level zero, projected to classical weight lattice.

[Lenart, Naito, Sagaki, Schilling, S.]

- Get tensor product of Kirillov-Reshetikhin crystals.
- Combinatorics is controlled by quantum Chevalley rule in $QH^*(G/P)$ where $G$ is finite-dimensional and $P$ is given by the stabilizer of $\lambda$.
- Character is Macdonald specialization $P_\lambda(x; q, 0)$.

[Lenart,Postnikov] Conjecture for Chevalley rule in $QK^*(G/B)$.

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