Positivity in Schubert calculus

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If Z is a subscheme in G/P and $[Z] = \sum \alpha_w X_w$, then $\alpha_w \geq 0$.

Standard tools to show positivity

If a connected algebraic group G acts on a variety X, the corresponding action on cohomology is trivial, so $[g \cdot V] = [V]$ for a subvariety V and an element g in G. If G acts transitively on X, one can use it to make $g \cdot V$ meet a given variety W transversally.

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Duality: Consider the cohomology ring $H^*(G/P, \mathbf{Z})$ with Schubert classes X_w. For any w there exists only one w' such that $X_w \cdot X_{w'} \neq 0$ and $\dim X_w + \dim X_{w'} = \dim G/P$.

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Vector bundles and positivity

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Griffiths, Kleiman, Bloch-Gieseker investigated positive polynomials.

Theorem. *(Fulton-Lazarsfeld) A polynomial*

$$P = \alpha_1 s_{\lambda_1} + \ldots + \alpha_k s_{\lambda_k}$$

is positive iff $\alpha_i \geq 0$ *for any* i *and* $\sum \alpha_i > 0$.

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- Hard-Lefschetz Theorem for these groups, proved by Gabber.

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where $f_k : M \to \mathcal{J}^k(M, N)$ is the k-jet extension of f.

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We specialize. Let $c = \operatorname{codim} C$. We take a projective variety X of dimension c. We then choose E to be trivial bundle and F an ample bundle of the corresponding ranks.

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We specialize. Let $c = \operatorname{codim} C$. We take a projective variety X of dimension c. We then choose E to be trivial bundle and F an ample bundle of the corresponding ranks.
$\Sigma(E,F)$ is a cone in $\mathcal{J}(E,F)$, and we obtain the class $z(\Sigma(E,F), \mathcal{J}(E,F))$.

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We conclude, by Fulton-Lazarsfeld, that all the coefficients α_λ are nonnegative with at least one strictly positive, so $\mathcal{T}^\Sigma \neq 0$.

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A Lagrange singularity class is any closed pure dimensional algebraic subset of $\mathcal{J}^k(V)$ which is invariant w.r.t. the action of H.

$\widetilde{Q}$-polynomials

Given a vector bundle E, we set $\widetilde{Q}_i(E) = c_i(E)$ and for $i \geq j$,

$$\widetilde{Q}_{i,j}(E) = \widetilde{Q}_i(E)\widetilde{Q}_j(E) + 2\sum_{p=1}^{j}(-1)^p\widetilde{Q}_{i+p}(E)\widetilde{Q}_{j-p}(E)\,.$$

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$[\Omega_I(V_\bullet)] = \Omega_I$. We have $\Omega_I = \widetilde{Q}_I(R^*)$, where R is the tautological subbundle on $LG(V)$ (PP, 1986).

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Theorem. *(MM+PP+AW, 2007) For any Lagrange singularity class Σ, the Thom polynomial $\mathcal{T}^\Sigma$ is a nonnegative combination of $\widetilde{Q}$-functions.*

Let $i : G = LG(V) \hookrightarrow \mathcal{J}$ be the inclusion. We look at the coefficients α_I of the expression $i^*[\Sigma] = \sum \alpha_I \, \widetilde{Q}_I(R^*)$.

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By deformation to the normal cone, we have in $A_* G$ the equality

$$i^*[\Sigma] = j^*[C] \, .$$

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Proposition. *Let $\pi : E \to X$ be a globally generated bundle on a proper homogeneous variety X. Let C be a cone in E, and let Z be a nonnegative algebraic cycle in X of the complementary dimension. Then the intersection $[C] \cdot [Z]$ is nonnegative.*

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Additionally, we assume that Σ is stable with respect to enlarging the dimension of W.

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Let $\tau : LG(V,\omega) \to X$ denote the Lagrange Grassmann bundle parametrizing Lagrangian linear submanifolds in V_x, $x \in X$.

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$\pi : \mathcal{J}^k(W, \xi) \to LG(V, \omega)$.
The space $\mathcal{J}^k(W, \xi)$ fibers over X. It is equal to the pull-back:

$$\mathcal{J}^k(W, \xi) = \tau^* \left(\bigoplus_{i=3}^{k+1} \mathrm{Sym}^i(W^*) \otimes \xi \right) .$$

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The element $[\Sigma(W, \xi)]$ of $H^*(\mathcal{J}^k(W, \xi), \mathbf{Z})$, is called the *Legendrian Thom polynomial* of Σ, and denoted by $\mathcal{T}^{\Sigma}$.

Let $\xi, \alpha_1, \alpha_2, \ldots, \alpha_n$ be vector spaces of dimension one and let

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$$F_h^+ := \bigoplus_{i=1}^{h} \alpha_i \,, \qquad F_h^- := \bigoplus_{i=1}^{h} \alpha_{n-i+1}^* \otimes \xi \,, \qquad (h = 1, 2, \ldots, n)$$

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Consider two Borel groups $B^\pm \subset Sp(V,\omega)$, preserving the flags $F_\bullet^\pm$. The orbits of $B^\pm$ in $LG(V,\omega)$ form two "opposite" cell decompositions $\{\Omega_I(F_\bullet^\pm, \xi)\}$ of $LG(V,\omega)$, $I \subset \rho$ strict.

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$$Z_{I\lambda}^+ := \tau^{-1}(\sigma_\lambda) \cap \Omega_I(F_\bullet^+, \xi) \, .$$

Theorem. *Fix* $I \subset \rho$ *and* λ*. Suppose that the vector bundle* $\mathcal{J}$ *is globally generated. Then, in* $\mathcal{J}$*, the intersection of* $\Sigma(W, \xi)$ *with the closure of any* $\pi^{-1}(Z^-_{I\lambda})$ *is represented by a nonnegative cycle.*

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We shall apply the Theorem in the situation when all α_i are equal to the same line bundle α (i.e. $W = \alpha^{\oplus n}$) and $\alpha^{-m} \otimes \xi$ is globally generated for $m \geq 3$.

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We shall apply the Theorem in the situation when all α_i are equal to the same line bundle α (i.e. $W = \alpha^{\oplus n}$) and $\alpha^{-m} \otimes \xi$ is globally generated for $m \geq 3$.
Consider the following three cases: the base is always $X = \mathbf{P}^n$ and

$$\xi_1 = \mathcal{O}(-2), \alpha_1 = \mathcal{O}(-1)$$

$$\xi_2 = \mathcal{O}(1), \alpha_2 = \mathbf{1}$$

$$\xi_3 = \mathcal{O}(-3), \alpha_3 = \mathcal{O}(-1).$$

We obtain symplectic bundles $V_i = \alpha_i^{\oplus n} \oplus (\alpha_i^* \otimes \xi_i)^{\oplus n}$ with twisted symplectic forms ω_i, $i = 1, 2, 3$.

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where $p_i : X \to \mathbf{P}^n$, $i = 1, 2$, are the projections. Let v_i denote $p_i^*(c_1(\mathcal{O}(1)))$ for $i = 1, 2$.

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The space $LG(V, \omega)$ has a cell decomposition $Z^-_{I,a,b}$. The dual basis of cohomology (in the sense of linear algebra) is denoted by

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We have $e_{I,a,b} = e_{I,0,0}\; v_1^a v_2^b$ and $e_{I,0,0} = [\overline{\Omega_I(F^+_\bullet, \xi)}]$.

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Using this theorem, one can obtain a one-parameter family of bases in the ring of Legendrian characteristic classes giving rise to positive expansions of all Legendrian Thom polynomials.

$\mathbf{A_8}$:

$18840\,\widetilde{Q}[61] + 20160\,\widetilde{Q}[7] + 3123\,\widetilde{Q}[421] + 5556\,\widetilde{Q}[43]+$

$15564\,\widetilde{Q}[52]+$

$t(71856\,\widetilde{Q}[6] + 3999\,\widetilde{Q}[321] + 55672\,\widetilde{Q}[51] + 34780\,\widetilde{Q}[42])+$

$t^2(64524\,\widetilde{Q}[41] + 24616\,\widetilde{Q}[32] + 105496\,\widetilde{Q}[5])+$

$t^3(36048\,\widetilde{Q}[31] + 81544\,\widetilde{Q}[4])+$

$t^4(8876\,\widetilde{Q}[21] + 34936\,\widetilde{Q}[3])+$

$t^5 7848\,\widetilde{Q}[2]+$

$t^6 720\,\widetilde{Q}[1]$

$\mathbf{E_8}$:

$$93\,\widetilde{Q}[421] + 108\,\widetilde{Q}[43] + 204\,\widetilde{Q}[52] + 72\,\widetilde{Q}[61]+$$

$$t(99\,\widetilde{Q}[321] + 216\,\widetilde{Q}[51] + 414\,\widetilde{Q}[42])+$$

$$t^2(246\,\widetilde{Q}[41] + 246\,\widetilde{Q}[32])+$$

$$t^3 126\,\widetilde{Q}[31]+$$

$$t^4 24\,\widetilde{Q}[21]$$

Theorem. *(W. Graham) Let* $X = G/B$ *be the flag variety for a complex semisimple group* G *and with maximal torus* $T \subset B$*, and let* $\{\sigma_w \in H_T^* X : w \in W\}$ *be the basis of (*B*-invariant) Schubert classes. Let* α_i *be the simple roots which are negative on* B*. Then in the expansion*

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Inspired by D. Anderson's proof of the theorem of Graham, we show the following result

Theorem. *The intersection of any nonnegative cycle on $LG(V,\omega)$ with any $\overline{Z_{I\lambda}^+}$ is represented by a nonnegative cycle.*

X is homogeneous. For any automorphism of X which is covered by a map of ξ and α_i's, we obtain an automorphism of $LG(V,\omega) \to X$ transforming the fibers to fibers.

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Lemma. ΓB^- *is globally generated.*

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Lemma. *ΓB^- is globally generated.*

Corollary. *The group ΓB^- acts on $LG(V,\omega)$, preserving fibers, and in each fiber its orbits coincide with the strata of the stratification $\{\Omega_J^-\}$.*

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Lemma. *The group H acts transitively on each stratum Ω_J^-: G transports any fiber to any other fiber, and ΓB^- acts transitively inside the fibers.*

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Proof of the theorem Let $Y \subset LG(V,\omega)$ be a subvariety. We can use the Bertini-Kleiman transversality theorem for H acting on Ω_J^-. There exists an open, dense subset $U_{JI\lambda} \subset H$ with the following property: if $h \in U_{JI\lambda}$, then $h \cdot (Y \cap \Omega_J^-)$ meets transversally $\overline{Z_{I\lambda}^+} \cap \Omega_J^-$.

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