K-homology of the space of loops on a symplectic group

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1 Motivation of our work

1.1 Hopf algebras $H_*(\Omega_0 SO)$ and $H_*(\Omega Sp)$

- $SU = SU(\infty)$, $SO = SO(\infty)$, $Sp = Sp(\infty)$
- $SU(n) \overset{r}{\hookrightarrow} SO(2n) \overset{c}{\hookrightarrow} SU(2n)$
- $SU(n) \overset{q}{\hookrightarrow} Sp(n) \overset{c}{\hookrightarrow} SU(2n)$

Letting $n \longrightarrow \infty$, we have

- $SU \xrightarrow{r} SO \xrightarrow{c} SU$
- $SU \xrightarrow{q} Sp \xrightarrow{c} SU$
- $\Omega(c \circ r) : \Omega SU \xrightarrow{\Omega r} \Omega_0 SO \xrightarrow{\Omega c} \Omega SU$
- $\Omega(c \circ q) : \Omega SU \xrightarrow{\Omega q} \Omega Sp \xrightarrow{\Omega c} \Omega SU$

These maps induce the following homomorphisms:

$$\Omega(c \circ r)_* : H_*(\Omega SU) \xrightarrow{(\Omega r)_*} H_*(\Omega_0 SO) \xrightarrow{(\Omega c)_*} H_*(\Omega SU),$$

$$\Omega(c \circ q)_* : H_*(\Omega SU) \xrightarrow{(\Omega q)_*} H_*(\Omega Sp) \xrightarrow{(\Omega c)_*} H_*(\Omega SU).$$

Fact 1.1

- $(\Omega r)_* : H_*(\Omega SU) \twoheadrightarrow H_*(\Omega_0 SO)$ is surjective.
- $(\Omega c)_* : H_*(\Omega_0 SO) \hookrightarrow H_*(\Omega SU)$ is injective.
- $(\Omega c)_* : H_*(\Omega Sp) \hookrightarrow H_*(\Omega SU)$ is a split monomorphism.

We wish to describe $H_*(\Omega_0 SO)$ and $H_*(\Omega Sp)$ as Hopf sub-algebras of $H_*(\Omega SU)$.

The "Bott map"

$$g_n : Gr(n, \mathbb{C}^{2n}) \cong U(2n)/U(n) \times U(n) \longrightarrow \Omega SU(2n)$$

induces a homotopy equivalence of H-spaces ($n \longrightarrow \infty$):

$$g_\infty : BU \xrightarrow{\sim} \Omega SU \quad \text{(Bott periodicity theorem).}$$

The map g_∞ induces isomorphisms of Hopf algebras:

$$g_\infty^* : H^*(\Omega SU) \xrightarrow{\sim} H^*(BU),$$

$$g_{\infty *} : H_*(BU) \xrightarrow{\sim} H_*(\Omega SU).$$

The structure of the cohomology ring $H^*(BU)$ and the homology ring (*Pontrjagin ring*) $H_*(BU)$ is well known.

Theorem 1.2

1.
$$H^*(BU) \cong \mathbb{Z}[c_1, c_2, \ldots, c_n, \ldots],$$
where c_n $(n = 1, 2, \ldots)$ are the universal Chern classes. The coalgebra structure is given by
$$\phi(c_n) = \sum_{i+j=n} c_i \otimes c_j \quad (c_0 := 1).$$

2.
$$H_*(BU) \cong \mathbb{Z}[\beta_1, \beta_2, \ldots, \beta_n, \ldots],$$
where β_n $(n = 1, 2, \ldots)$ are induced from $\mathbb{C}P^\infty \simeq BU(1) \longrightarrow BU$. The coalgebra structure is given by
$$\phi(\beta_n) = \sum_{i+j=n} \beta_i \otimes \beta_j \quad (\beta_0 := 1).$$

1.2 Work of Kono-Kozima (1978)

Kono-Kozima considered the homomorphism

$$\begin{aligned}
\Omega(c\circ q)_* : H_*(\Omega SU) &\longrightarrow H_*(\Omega SU),\\
\Omega(c\circ q)_* : H_*(\Omega SU)[[x]] &\longrightarrow H_*(\Omega SU)[[x]],\\
\beta(x) := \sum_{i\geq 0}\beta_i x^i &\longmapsto \Omega(c\circ q)_*(\beta(x))\\
&\qquad := \sum_{i\geq 0}\Omega(c\circ q)_*(\beta_i)x^i.
\end{aligned}$$

Theorem 1.3 (Kono-Kozima)

$$\begin{aligned}
\Omega(c\circ q)_*(\beta(x)) &= \beta(x)/\beta(-x),\\
\Omega(c\circ q)_*(\beta_n) &= \{1+(-1)^{n+1}\}\beta_n + 2r_n \quad (n=1,2,\ldots)\\
&\qquad (\exists r_n \in \tilde{H}_*(BU)^2).
\end{aligned}$$

Example 1.4

$$\Omega(c \circ q)_*(\beta_1) = 2\beta_1,$$
$$\Omega(c \circ q)_*(\beta_2) = 2\beta_1^2,$$
$$\Omega(c \circ q)_*(\beta_3) = 2(\beta_3 - \beta_2\beta_1 + \beta_1^3),$$
$$\Omega(c \circ q)_*(\beta_4) = 2(2\beta_3\beta_1 - 2\beta_2\beta_1^2 + \beta_1^4),$$
$$\Omega(c \circ q)_*(\beta_5) = 2(\beta_5 - \beta_3\beta_2 - \beta_4\beta_1 + \beta_2^2\beta_1 + 3\beta_3\beta_1^2 - 3\beta_2\beta_1^3 + \beta_1^5).$$

Since the polynomials in the right hand side are always divisible by 2, they defined the elements z_n $(n = 1, 2, \ldots)$ as

$$z_n := \frac{1}{2}(\Omega q)_*(\beta_n) \quad (n = 1, 2, \ldots),$$
$$(\Omega q)_*(\beta(x)) = 1 + 2xz(x), \quad z(x) := \textstyle\sum_{j \geq 0} z_{j+1}x^j.$$

Theorem 1.5 (Kono-Kozima)

- ring structure:

$$H_*(\Omega Sp) \cong \mathbb{Z}[z_1, z_3, \ldots, z_{2n-1}, \ldots]$$

- coalgebra structure:

$$\phi(z_n) = z_n \otimes 1 + 1 \otimes z_n + 2 \sum_{i+j=n,\, i\geq 1,\, j\geq 1} z_i \otimes z_j$$

- The elements z_{2n} $(n = 1, 2, \ldots)$ are defined recursively by the formula:

$$z_{2n} + \sum_{i+j=2n,\, i\geq 1,\, j\geq 1} (-1)^i z_i z_j = 0.$$

1.3 Symmetric functions

1.3.1 Ring of symmetric functions

- Λ: ring of symmetric functions with integer coefficients
- e_i $(i = 0, 1, 2, \ldots)$: i-th elementary symmetric functions
- h_i $(i = 0, 1, 2, \ldots)$: i-th complete symmetric functions
- generating functions:

$$E(t) = \sum_{i \geq 0} e_i t^i = \prod_{i \geq 1} (1 + y_i t), \quad H(t) = \sum_{i \geq 0} h_i t^i = \prod_{i \geq 1} \frac{1}{1 - y_i t}$$

- relation:

$$H(t)E(-t) = 1$$

- s_λ: Schur function corresponding to a partition $\lambda \in \mathcal{P}$, the set all partitions.

Fact 1.6

- Λ is a polynomial ring over the integers $\mathbb{Z}$ generated by e_i $(i = 1, 2, \ldots)$ (or h_i $(i = 1, 2, \ldots)$):
$$\Lambda = \mathbb{Z}[e_1, e_2, \ldots] = \mathbb{Z}[h_1, h_2, \ldots].$$
- Λ becomes a commutative and co-commutative Hopf algebra over $\mathbb{Z}$ with the coproduct
$$\phi(e_k) = \sum_{i+j=k} e_i \otimes e_j, \quad \phi(h_k) = \sum_{i+j=k} h_i \otimes h_j.$$
- Λ and its graded dual $\Lambda^* = \mathrm{Hom}\,(\Lambda, \mathbb{Z})$ are isomorphic as Hopf algebras under the "Hall inner product" $\langle \cdot\, , \cdot \rangle : \Lambda \times \Lambda \longrightarrow \mathbb{Z}$.
- $\{s_\lambda\}_{\lambda \in \mathcal{P}}$ is a free $\mathbb{Z}$-basis for Λ: $\Lambda = \bigoplus_{\lambda \in \mathcal{P}} \mathbb{Z} s_\lambda$.

1.3.2 Schur P- and Q-functions

- Q_k $(k = 0, 1, 2, \ldots)$ are defined as the coefficients of t^k in the following generating function ($Q_0 := 1$):

$$Q(t) = \sum_{k \geq 0} Q_k t^k := H(t)E(t) = \frac{H(t)}{H(-t)} = \prod_{i \geq 1} \frac{1 + y_i t}{1 - y_i t}.$$

- The identity $Q(t)Q(-t) = 1$ implies the following relations:

$$Q_i^2 + 2 \sum_{j=1}^{i} (-1)^j Q_{i+j} Q_{i-j} = 0 \quad (i \geq 1).$$

- We define Γ as the subalgebra of Λ generated by Q_i's:

$$\Gamma = \frac{\mathbb{Z}[Q_1, Q_2, \ldots, Q_i, \ldots]}{(Q_i^2 + 2 \sum_{j=1}^{i} (-1)^j Q_{i+j} Q_{i-j} \ (i \geq 1))}.$$

- Q_λ: Schur Q-function corresponding to a strict partition $\lambda \in \mathcal{SP}$, the set of all strict partitions.
- P_k $(k = 0, 1, 2, \ldots)$ are defined by the formula:

$$P_k := \frac{1}{2} Q_k = \frac{1}{2} \sum_{i+j=k} h_i e_j, \quad P_0 := 0.$$

- P_i's satisfy the relations:

$$P_i^2 + 2 \sum_{j=1}^{i-1} (-1)^j P_{i+j} P_{i-j} + (-1)^i P_{2i} = 0 \quad (i \geq 1).$$

- We define Γ' as the subalgebra of Λ generated by P_i's:

$$\begin{aligned} \Gamma' &= \frac{\mathbb{Z}[P_1, P_2, \ldots, P_i, \ldots]}{(P_i^2 + 2\sum_{j=1}^{i-1}(-1)^j P_{i+j} P_{i-j} + (-1)^i P_{2i} \ (i \geq 1))} \\ &= \mathbb{Z}[P_1, P_3, \ldots, P_{2n-1}, \ldots]. \end{aligned}$$

- $P_\lambda = 2^{-\ell(\lambda)} Q_\lambda$: Schur P-functions corresponding to a strict partition $\lambda \in \mathcal{SP}$.

Fact 1.7

- The subalgebras Γ and Γ' have natural Hopf algebra structures with the coproduct given by

$$\begin{aligned} \phi(Q_k) &= \sum_{i+j=k} Q_i \otimes Q_j, \\ \phi(P_k) &= P_k \otimes 1 + 1 \otimes P_k + 2 \sum_{i+j=k,\, i\geq 1,\, j\geq 1} P_i \otimes P_j. \end{aligned}$$

- Γ and Γ' are mutually dual Hopf algebras: $\Gamma^* \cong \Gamma', \quad (\Gamma')^* \cong \Gamma$.
- $\{Q_\lambda\}_{\lambda\in\mathcal{SP}}$ (resp. $\{P_\lambda\}_{\lambda\in\mathcal{SP}}$) forms a free $\mathbb{Z}$-basis for Γ (resp. Γ'): $\Gamma = \bigoplus_{\lambda\in\mathcal{SP}} \mathbb{Z}Q_\lambda$ (resp. $\Gamma' = \bigoplus_{\lambda\in\mathcal{SP}} \mathbb{Z}P_\lambda$).

1.3.3 Identification with symmetric functions

- We make the following identification of Hopf algebras:

$$\begin{aligned} H^*(\Omega SU) \cong H^*(BU) = \mathbb{Z}[c_1, c_2, \ldots] &\xrightarrow{\sim} \Lambda = \mathbb{Z}[e_1, e_2, \ldots], \\ c_i &\longmapsto e_i, \\ H_*(\Omega SU) \cong H_*(BU) = \mathbb{Z}[\beta_1, \beta_2, \ldots] &\xrightarrow{\sim} \Lambda = \mathbb{Z}[h_1, h_2, \ldots], \\ \beta_i &\longmapsto h_i. \end{aligned}$$

- Under this identification, we have

$$\begin{array}{ccc} H_*(\Omega SU) & \longleftrightarrow & \Lambda \\ \beta(x) = \sum_{i \geq 0} \beta_i x^i & \longleftrightarrow & H(x) = \sum_{i \geq 0} h_i x^i \\ \Omega(c \circ q)_* : H_*(\Omega SU) \to H_*(\Omega SU) & \longleftrightarrow & \Phi : \Lambda \longrightarrow \Lambda \\ \Omega(c \circ q)_*(\beta(x)) = \beta(x)/\beta(-x) & \longleftrightarrow & \Phi(H(x)) = H(x)/H(-x) \end{array}$$

Since $\Phi(H(x)) = H(x)/H(-x) = H(x)E(x) = Q(x)$, we have

$$\Phi(h_n) = Q_n = 2P_n \ (n = 1, 2, \ldots)$$

Under the identification $H_*(\Omega SU) \cong \Lambda$, the element $z_n \in H_{2n}(\Omega Sp)$ defined by Kono-Kozima is nothing but the Schur P-function P_n.

Observation 1.8

- There is a natural identification of Hopf algebras:

$$H_*(\Omega Sp) = \mathbb{Z}[z_1, z_3, z_5, \ldots] \quad \xrightarrow{\sim} \quad \Gamma' = \mathbb{Z}[P_1, P_3, P_5, \ldots].$$

- The homomorphism $(\Omega q)_* : H_*(\Omega SU) \longrightarrow H_*(\Omega Sp)$ corresponds to the homomorphism $\Lambda \longrightarrow \Gamma'$, $h_i \longmapsto 2P_i$.
- The homomorphism $(\Omega c)_* : H_*(\Omega Sp) \hookrightarrow H_*(\Omega SU)$ corresponds to the natural inclusion $\Gamma' \hookrightarrow \Lambda$.

Similarly, putting $\tau_n := (\Omega r)_*(\beta_n)$ $(n = 1, 2, \ldots)$, we have

Observation 1.9 There is a natural identification of Hopf algebras:

$$\begin{aligned} H_*(\Omega_0 SO) &= \frac{\mathbb{Z}[\tau_1, \tau_2, \ldots, \tau_n, \ldots]}{(\tau_n^2 + 2\sum_{j=1}^{n}(-1)^j \tau_{n+j}\tau_{n-j} \ (n \geq 1))} \\ &\xrightarrow{\sim} \Gamma = \frac{\mathbb{Z}[Q_1, Q_2, \ldots, Q_n, \ldots]}{(Q_n^2 + 2\sum_{j=1}^{n}(-1)^j Q_{n+j}Q_{n-j} \ (n \geq 1))}. \end{aligned}$$

- The homomorphism $(\Omega r)_* : H_*(\Omega SU) \twoheadrightarrow H_*(\Omega_0 SO)$ corresponds to the surjection $\Lambda \twoheadrightarrow \Gamma$, $h_i \longmapsto Q_i$.
- The homomorphism $(\Omega c)_* : H_*(\Omega_0 SO) \hookrightarrow H_*(\Omega SU)$ corresponds to the natural inclusion $\Gamma \hookrightarrow \Lambda$.

2 E-homology Hopf algebras of $\Omega_0 SO$ and ΩSp

2.1 Hopf algebras $E_*(\Omega_0 SO)$ and $E_*(\Omega Sp)$

- $E^*(\quad)$: "complex oriented" generalized cohomology theory with the orientation class $x^E \in E^2(\mathbb{C}P^\infty)$
- $E_*(\quad)$: corresponding homology theory
- $E_* := E_*(\mathrm{pt}) = E^{-*}$: coefficient ring
- $\mu_E(x, y) = x + y + \sum_{i \geq 1, j \geq 1} a^E_{ij} x^i y^j$ ($a^E_{ij} \in E_{2(i+j-1)}$): commutative formal group law associated with $E^*(\quad)$
- $[-1]_E(x) = \iota_E(x) = \overline{x} = -x + \sum_{j \geq 2} a'_j x^j$: formal inverse, i.e., $\mu_E(x, [-1]_E(x)) \equiv 0$
- $[1]_E(x) := x$, $[n]_E(x) := \mu_E([n-1]_E(x), x)$ ($n > 1$): n-series

The maps

$$\Omega(c \circ r) : \Omega SU \xrightarrow{\Omega r} \Omega_0 SO \xrightarrow{\Omega c} \Omega SU,$$

$$\Omega(c \circ q) : \Omega SU \xrightarrow{\Omega q} \Omega Sp \xrightarrow{\Omega c} \Omega SU$$

induce the following homomorphisms:

$$\Omega(c \circ r)_* : E_*(\Omega SU) \xrightarrow{(\Omega r)_*} E_*(\Omega_0 SO) \xrightarrow{(\Omega c)_*} E_*(\Omega SU),$$

$$\Omega(c \circ q)_* : E_*(\Omega SU) \xrightarrow{(\Omega q)_*} E_*(\Omega Sp) \xrightarrow{(\Omega c)_*} E_*(\Omega SU).$$

Fact 2.1

- $(\Omega r)_* : E_*(\Omega SU) \twoheadrightarrow E_*(\Omega_0 SO)$ is surjective.
- $(\Omega c)_* : E_*(\Omega_0 SO) \hookrightarrow E_*(\Omega SU)$ is injective.
- $(\Omega c)_* : E_*(\Omega Sp) \hookrightarrow E_*(\Omega SU)$ is a split monomorphism.

Theorem 2.2 (Adams)

1.
$$E^*(\Omega SU) \cong E^*(BU) \cong E^*[[c_1^E, c_2^E, \ldots, c_n^E, \ldots]],$$
where c_n^E $(n = 1, 2, \ldots)$ are the E-theory Chern-Conner-Floyd classes. The coalgebra structure is given by
$$\phi(c_n^E) = \sum_{i+j=n} c_i^E \otimes c_j^E \quad (c_0^E := 1).$$

2.
$$E_*(\Omega SU) \cong E_*(BU) \cong E_*[\beta_1^E, \beta_2^E, \ldots, \beta_n^E, \ldots],$$
where β_n^E $(n = 1, 2, \ldots)$ are induced from $\mathbb{C}P^\infty \simeq BU(1) \longrightarrow BU$. The coalgebra structure is given by
$$\phi(\beta_n^E) = \sum_{i+j=n} \beta_i^E \otimes \beta_j^E \quad (\beta_0^E := 1).$$

Theorem 2.3

$$\begin{aligned}\Omega(c\circ r)_*(\beta^E(x)) &= \beta^E(x)/\beta^E([-1]_E(x)),\\ \Omega(c\circ q)_*(\beta^E(x)) &= \beta^E(x)/\beta^E([-1]_E(x)).\end{aligned}$$

Putting

$$\tau_n^E := (\Omega r)_*(\beta_n^E)\ (n=0,1,2,\ldots),\quad \tau^E(x) := \sum_{i\geq 0}\tau_i^E x^i,$$

we have the following relation:

$$\tau^E(x)\tau^E([-1]_E(x)) = 1.$$

Proposition 2.4

- ring structure:

$$E_*(\Omega_0 SO) = E_*[\tau_1^E,\tau_2^E,\ldots,\tau_n^E,\ldots]/(\tau^E(x)\tau^E([-1]_E(x))=1)$$

- coalgebra structure:

$$\phi(\tau_n^E) = \sum_{i+j=n} \tau_i^E \otimes \tau_j^E \quad (\tau_0^E = 1)$$

We define the elements η_n^E $(n = 0, 1, 2, \ldots)$ in $E_*(\Omega Sp)$ so as to satisfy the relation

$$(\Omega q)_*(\beta^E(x)) = 1 + [2]_E(x)\eta^E(x),\ \eta^E(x) := \sum_{j\geq 0} \eta_{j+1}^E x^j,\ \eta_0^E := 0.$$

For example, we have

$$\begin{aligned}
(\Omega q)_*(\beta_1^E) &= 2\eta_1^E, \\
(\Omega q)_*(\beta_2^E) &= 2\eta_2^E + \alpha_2^E \eta_1^E, \\
(\Omega q)_*(\beta_3^E) &= 2\eta_3^E + \alpha_2^E \eta_2^E + \alpha_3^E \eta_1^E,
\end{aligned}$$

where we put $[2]_E(x) = \mu_E(x,x) = \sum_{k\geq 1} \alpha_k^E x^k = 2x + \sum_{k\geq 2} \alpha_k^E x^k$.

We have the following relation:

$$(1+[2]_E(x)\eta^E(x))(1+[-2]_E(x)\eta^E([-1]_E(x)))=1,$$

where $[-2]_E(x)=[2]_E([-1]_E(x))$.

From this, we can eliminate η_{2n} $(n=1,2,\ldots)$, and we obtain

Proposition 2.5 (F. Clarke (1981))

- ring structure:

$$E_*(\Omega Sp)=E_*[\eta_1^E,\eta_3^E,\ldots,\eta_{2n-1}^E,\ldots]$$

- coalgebra structure:

$$\phi(\eta_l^E)=\eta_l^E\otimes 1+1\otimes\eta_l^E+\sum_{i+j+k=l}\alpha_{k+1}^E\eta_i^E\otimes\eta_j^E,$$

  where $[2]_E(x)=\mu_E(x,x)=\sum_{k\geq 1}\alpha_k^E x^k$.

2.2 E-homology Schur P- and Q-functions

Definition 2.6 We define q_k^E $(k = 0, 1, 2, \ldots)$ as the coefficients of t^k in the following generating function ($q_0^E := 1$):

$$q^E(t) = \sum_{k \geq 0} q_k^E t^k := \frac{H(t)}{H([-1]_E(t))} = \frac{H(t)}{H(\bar{t})} = \prod_{i \geq 1} \frac{1 - y_i \bar{t}}{1 - y_i t}.$$

The following relation holds:

$$q^E(t) q^E(\bar{t}) = 1.$$

Definition 2.7 We define Γ^E as the subalgebra of $\Lambda^E := E_* \otimes_{\mathbb{Z}} \Lambda$ generated by q_i^E's:

$$\Gamma^E := \frac{E_*[q_1^E, q_2^E, \ldots, q_i^E, \ldots]}{(q^E(t) q^E(\bar{t}) = 1)} \hookrightarrow \Lambda^E.$$

Definition 2.8 We define p_k^E $(k = 0, 1, 2, \ldots)$ so as to satisfy the relation

$$\underline{q^E(t) = 1 + [2]_E(t)p^E(t), \quad p^E(t) := \sum_{j\geq 0} p_{j+1}^E t^j \quad p_0^E := 0.}$$

For example, we have

$$\begin{aligned} q_1^E &= 2p_1^E, \\ q_2^E &= 2p_2^E + \alpha_2^E p_1^E, \\ q_3^E &= 2p_3^E + \alpha_2^E p_2^E + \alpha_3^E p_1^E, \end{aligned}$$

where we put $[2]_E(t) = \mu_E(t,t) = \sum_{k\geq 1} \alpha_k^E t^k$.

The following relation holds:

$$(1 + [2]_E(t)p^E(t))(1 + [-2]_E(t)p^E([-1]_E(t))) = 1.$$

From this, we can eliminate p^E_{2n} $(n = 1, 2, \ldots)$.

Definition 2.9 We define Γ'^E as the subalgebra of Λ^E generated by p^E_i's:

$$\begin{aligned}\Gamma'^E &= \frac{E_*[p^E_1, p^E_2, \ldots, p^E_i, \ldots]}{(1 + [2]_E(t)p^E(t))(1 + [-2]_E(t)p^E([-1]_E(t))) = 1)} \\ &= E_*[p^E_1, p^E_3, \ldots, p^E_{2n-1}, \ldots] \hookrightarrow \Lambda^E.\end{aligned}$$

Proposition 2.10

1. We make the following identification of Hopf algebras:

$$\begin{aligned}E_*(\Omega SU) \cong E_*(BU) = E_*[\beta^E_1, \beta^E_2, \ldots] &\xrightarrow{\sim} \Lambda^E = E_*[h_1, h_2, \ldots], \\ \beta^E_i &\longmapsto h_i.\end{aligned}$$

2. There is a natural identification of Hopf algebras:

$$E_*(\Omega_0 SO) = \frac{E_*[\tau_1^E, \tau_2^E, \ldots, \tau_n^E, \ldots]}{(\tau^E(t)\tau^E(\bar{t}) = 1)} \xrightarrow{\sim} \Gamma^E = \frac{E_*[q_1^E, q_2^E, \ldots, q_n^E, \ldots]}{(q^E(t)q^E(\bar{t}) = 1)}$$

- The homomorphism $(\Omega r)_* : E_*(\Omega SU) \twoheadrightarrow E_*(\Omega_0 SO)$ corresponds to the surjection $\Lambda^E \twoheadrightarrow \Gamma^E,\ h_i \longmapsto q_i^E$.
- The homomorphism $(\Omega c)_* : E_*(\Omega_0 SO) \hookrightarrow E_*(\Omega SU)$ corresponds to the natural inclusion $\Gamma^E \hookrightarrow \Lambda^E$.

$$E_*(\Omega Sp) = E_*[\eta_1^E, \eta_3^E, \eta_5^E, \ldots] \xrightarrow{\sim} \Gamma'^E = E_*[p_1^E, p_3^E, p_5^E, \ldots]$$

- The homomorphism $(\Omega q)_* : E_*(\Omega SU) \longrightarrow E_*(\Omega Sp)$ corresponds to the homomorphism $\Lambda^E \longrightarrow \Gamma'^E,\ H(t) \longmapsto 1 + [2]_E(t)p^E(t)$.

- The homomorphism $(\Omega c)_* : E_*(\Omega Sp) \hookrightarrow E_*(\Omega SU)$ corresponds to the natural inclusion ${\Gamma'}^E \hookrightarrow \Lambda^E$.

2.3 E-(co)homology factorial Schur P- and Q-functions

Ikeda-Naruse constructed the *K-theoretic factorial P- and Q-functions*.

- $x \oplus y = x + y + \beta xy, \quad x \ominus y = \dfrac{x-y}{1+\beta y}$
- $b = (b_1, b_2, \ldots)$: a set of parameters
- $[x|b]^k := (x \oplus b_1)(x \oplus b_2) \cdots (x \oplus b_k)$
- $[[x|b]]^k := (x \oplus x)[x|b]^{k-1}$
- $\mathcal{SP}_n$: the set of all strict partitions λ with $\ell(\lambda) = r \leq n$
- $[x|b]^\lambda := \prod_{j=1}^r [x_j|b]^{\lambda_j}$ and $[[x|b]]^\lambda := \prod_{j=1}^r [[x_j|b]]^{\lambda_j}$

Definition 2.11 (Ikeda-Naruse (2012))

1. For a strict partition $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathcal{SP}_n$, define

$$GP_\lambda(x_1, \ldots, x_n|b) := \frac{1}{(n-r)!} \sum_{w \in \mathcal{S}_n} w \left[[x|b]^\lambda \prod_{i=1}^{r} \prod_{j=i+1}^{n} \frac{x_i \oplus x_j}{x_i \ominus x_j} \right],$$

$$GQ_\lambda(x_1, \ldots, x_n|b) := \frac{1}{(n-r)!} \sum_{w \in \mathcal{S}_n} w \left[[[x|b]]^\lambda \prod_{i=1}^{r} \prod_{j=i+1}^{n} \frac{x_i \oplus x_j}{x_i \ominus x_j} \right].$$

2.

$$\begin{aligned} GP_\lambda(x_1, \ldots, x_n) &:= GP_\lambda(x_1, \ldots, x_n|0) \in G\Gamma_n, \\ GQ_\lambda(x_1, \ldots, x_n) &:= GQ_\lambda(x_1, \ldots, x_n|0) \in G\Gamma_{n,+}. \end{aligned}$$

3. For a strict partition $\lambda \in \mathcal{SP} = \cup_{n \geq 1} \mathcal{SP}_n$, define

$$\begin{aligned} GP_\lambda(x) &:= \varprojlim GP_\lambda(x_1, \dots, x_n) \in G\Gamma = \varprojlim G\Gamma_n, \\ GQ_\lambda(x) &:= \varprojlim GQ_\lambda(x_1, \dots, x_n) \in G\Gamma_+ = \varprojlim G\Gamma_{n,+}. \end{aligned}$$

We generalize the K-theoretic factorial P- and Q-functions to the generalized cohomology theory and define "*E-cohomology factorial (equivariant) Schur P- and Q-functions*".

- $\mu(x,y) = \mu_E(x,y)$: formal group law correspoding to E-theory
- $x +_\mu y := \mu(x,y)$: formal group sum
- $\bar{x}$: formal inverse of x, i.e. $\bar{x} +_\mu x = 0$
- $b = (b_1, b_2, ...)$: a set of parameters
- $[x|b]^k := \prod_{i=1}^{k}(x +_\mu b_i)$
- $[[x|b]]^k := (x +_\mu x)[x|b]^{k-1}$
- $[x|b]^\lambda = \prod_{j=1}^{r}[x_j|b]^{\lambda_j}$ and $[[x|b]]^\lambda = \prod_{j=1}^{r}[[x_j|b]]^{\lambda_j}$ for $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathcal{SP}_n$.

Definition 2.12 (E-cohomology factorial Schur P- and Q-functions)

$$P^E_\lambda(x_1,\dots,x_n|b) \quad := \frac{1}{(n-r)!}\sum_{w\in\mathcal{S}_n} w\left[[x|b]^\lambda \prod_{i=1}^{r}\prod_{j=i+1}^{n}\frac{x_i +_\mu x_j}{x_i +_\mu \bar{x}_j}\right],$$

$$Q^E_\lambda(x_1,\dots,x_n|b) \quad := \frac{1}{(n-r)!}\sum_{w\in\mathcal{S}_n} w\left[[[x|b]]^\lambda \prod_{i=1}^{r}\prod_{j=i+1}^{n}\frac{x_i +_\mu x_j}{x_i +_\mu \bar{x}_j}\right].$$

Using a Cauchy type kernel $\Delta = \prod_{i,j\geq 1}\frac{1-\bar{x}_i y_j}{1-x_i y_j}$, we define the "*$E$-homology factorial (equivariant) Schur P- and Q-functions*" $p^E_\lambda(y|b)$ and $q^E_\lambda(y|b)$:

Definition 2.13 (E-homology factorial Schur P- and Q-functions) 1.

$$\prod_{i,j\geq 1} \frac{1-\bar{x}_i y_j}{1-x_i y_j} = \sum_{\lambda\in\mathcal{SP}} Q^E_\lambda(x|b)p^E_\lambda(y|b),$$

$$\prod_{i,j\geq 1} \frac{1-\bar{x}_i y_j}{1-x_i y_j} = \sum_{\lambda\in\mathcal{SP}} P^E_\lambda(x|b)q^E_\lambda(y|b).$$

2. $p^E_\lambda(y) := p^E_\lambda(y|0)$ and $q^E_\lambda(y) := q^E_\lambda(y|0)$.

When $\lambda = (k)$ $(k = 1, 2, \ldots)$ (one row), $p^E_{(k)}(y)$ (resp. $q^E_{(k)}(y)$) coincides with p^E_k (resp. q^E_k) introduced in §2.2.

Fact 2.14 $\{q^E_\lambda(y)\}_{\lambda\in\mathcal{SP}}$ (resp. $\{p^E_\lambda(y)\}_{\lambda\in\mathcal{SP}}$ forms a free E_*-basis for Γ^E (resp. Γ'^E): $\Gamma^E = \bigoplus_{\lambda\in\mathcal{SP}} E_* q^E_\lambda(y)$ (resp. $\Gamma'^E = \bigoplus_{\lambda\in\mathcal{SP}} E_* p^E_\lambda(y)$).

3 K-homology Schur P- and Q-functions

We focus on the case of K-homology theory. We use the $\mathbb{Z}/2\mathbb{Z}$-graded "representable K-homology theory" $K_*(\quad) = K_0(\quad) \oplus K_1(\quad)$.

- $\mu_K(x, y) = x + y - xy$
- $[-1]_K(x) = \overline{x} = -\dfrac{x}{1-x}$
- $[2]_K(x) = 2x - x^2$ ($\alpha_1^K = 2$, $\alpha_2^K = -1$, $\alpha_i^K = 0$ $(i \geq 3)$)
- $K_* = K_*(\mathrm{pt}) = K_0(\mathrm{pt}) = \mathbb{Z}$
- In §2.2, we defined the "*ring of K-homology Schur P- and Q-functions*":

$$K_0(\Omega_0 SO) \cong \Gamma^K = \frac{\mathbb{Z}[q_1^K, q_2^K, \dots, q_n^K, \dots]}{(q^K(x)q^K([-1]_K(x)) = 1)} = \bigoplus_{\lambda \in \mathcal{SP}} \mathbb{Z} q_\lambda^K(y),$$

$$K_0(\Omega Sp) \cong \Gamma'^K = \mathbb{Z}[p_1^K, p_3^K, \dots, p_{2n-1}^K, \dots] = \bigoplus_{\lambda \in \mathcal{SP}} \mathbb{Z} p_\lambda^K(y).$$

Motivated by the construction of the *dual stable Grothendieck polynomials* $\{g_\lambda\}_{\lambda \in \mathcal{P}}$ due to Lam-Pylyavskyy, we define their "type C analogues".

Definition 3.1

1. For a strict partition $\lambda = (\lambda_1 > \cdots > \lambda_r > 0)$, we define $\mathrm{Tab}(\lambda)$ as the set of tableaux of shape λ in alphabet $1' < 1 < 2' < 2 < \cdots$ with condition that each rows and columns are weakly increasing.
   Let $\mathrm{Tab}'(\lambda)$ be the subset of $\mathrm{Tab}(\lambda)$ with the property that for each row, the first box contains a primed number,
   in other words, all the entries in the diagonal boxes are primed).

2. Define $gp_\lambda(y) = gp_\lambda(y_1, y_2, \ldots)$, $gq_\lambda(y) = gq_\lambda(y_1, y_2, \ldots)$ by

$$\begin{aligned} gp_\lambda(y) &:= \sum_{T \in \mathrm{Tab}'(\lambda)} y^T, \\ gq_\lambda(y) &:= \sum_{T \in \mathrm{Tab}(\lambda)} y^T, \end{aligned}$$

where $y^T := \prod_{i \in T} y_i^{c(i)} \prod_{i' \in T} x_i^{r(i')}$ with

- $c(i)$: the number of columns containing i
- $r(i')$: the number of rows containing i'

Example 3.2

$T =$

$1'$	$1'$	2	3
	$2'$	3	3
		3	

$y^T = y_1 y_2^2 y_3^2$

Conjecture 3.3 $gp_\lambda(y) = p_\lambda^K(y)$ and $gq_\lambda(y) = q_\lambda^K(y)$ ($\lambda \in \mathcal{SP}$).

Remark 3.4

1. The conjecture above is shown to be true for one row case i.e., $\lambda = (k)$ $(k = 1, 2, \ldots)$.
2. Since $p_\lambda^K(y)$ and $q_\lambda^K(y)$ are symmetric functions by definition, the fact that $gp_\lambda(y)$ and $gq_\lambda(y)$ are symmetric functions follows from the above conjecture.

There is a formula related to the stable dual Grothendieck polynomials $g_\lambda(y)$:

Proposition 3.5 (hook sum formula)

$$gp_k(y) = \sum_{a=1}^{k} g_{a1^{k-a}}(y)$$

Conjecture 3.6

$$gp_\rho(y) = g_\rho(y) \text{ for } \rho = (k, k-1, \ldots, 2, 1),\ k \leq n.$$

Remark 3.7 For a geometric reason, $gp_\lambda(y)$ and $gq_\lambda(y)$ should be positive linear combinations of the dual stable Grothendieck polynomials $g_\mu(y)$ (hence Schur polynomials $s_\mu(y)$).

Example 3.8

$$
\begin{aligned}
gp_1 &= h_1, \\
gp_2 &= h_1^2 + h_1, \\
gp_3 &= (h_1^3 - h_2h_1 + h_3) + (2h_1^2 - h_2) + h_1, \\
gp_4 &= (h_1^4 - 2h_2h_1^2 + 2h_3h_1) + (3h_1^3 - 4h_2h_1 + 2h_3) + (3h_1^2 - 2h_2) \\
&\quad +h_1, \\
gp_5 &= (h_1^5 - 3h_2h_1^3 + 3h_3h_1^2 + h_2^2h_1 - h_4h_1 - h_3h_2 + h_5) \\
&\quad +(4h_1^4 - 9h_2h_1^2 + 7h_3h_1 + h_2^2 - 2h_4) + (6h_1^3 - 9h_2h_1 + 4h_3) \\
&\quad +(4h_1^2 - 3h_2) + h_1, \\
gp_6 &= (h_1^6 - 4h_2h_1^4 + 4h_3h_1^3 + 3h_2^2h_1^2 - 2h_4h_1^2 - 4h_3h_2h_1 + 2h_5h_1 + h_3^2) \\
&\quad +(5h_1^5 - 16h_2h_1^3 + 3h_5 + 14h_3h_1^2 + 7h_2^2h_1 - 7h_4h_1 - 5h_3h_2) \\
&\quad +(10h_1^4 - 24h_2h_1^2 + 17h_3h_1 + 4h_2^2 - 6h_4) + (10h_1^3 - 16h_2h_1 + 7h_3) \\
&\quad +(5h_1^2 - 4h_2) + h_1,
\end{aligned}
$$

$$
\begin{aligned}
gp_{21} &= (h_2h_1 - h_3) + h_2, \\
gp_{31} &= (h_2h_1^2 - h_3h_1) + (2h_2h_1 - h_3) + h_2, \\
gp_{41} &= (h_2h_1^3 - h_3h_1^2 - h_2^2h_1 + h_4h_1 + h_3h_2 - h_5) \\
&\quad +(3h_2h_1^2 - 3h_3h_1 - h_2^2 + 2h_4) + (3h_2h_1 - 2h_3) + h_2, \\
gp_{51} &= (h_2h_1^4 - h_3h_1^3 - 2h_2^2h_1^2 + h_4h_1^2 + 3h_3h_2h_1 - h_5h_1 - h_3^2) \\
&\quad +(4h_2h_1^3 - 4h_3h_1^2 - 5h_2^2h_1 + 4h_4h_1 + 4h_3h_2 - 2h_5) \\
&\quad +(6h_2h_1^2 - 6h_3h_1 - 3h_2^2 + 4h_4) + (4h_2h_1 - 3h_3) + h_2, \\
gp_{32} &= (h_2^2h_1 - h_4h_1 - h_3h_2 + h_5) + (h_2h_1^2 + h_2^2 - 2h_4) + (2h_2h_1) \\
&\quad +h_2, \\
gp_{42} &= (h_2^2h_1^2 - 2h_3h_2h_1 + h_3^2) + (h_2h_1^3 - h_3h_1^2 + 2h_2^2h_1 - 2h_3h_2) \\
&\quad +(3h_2h_1^2 - 2h_3h_1 + h_2^2) + (3h_2h_1 - h_3) + h_2, \\
gp_{321} &= (h_3h_2h_1 - h_4h_1^2 + h_5h_1 - h_3^2) + (h_3h_1^2 - 3h_4h_1 + h_3h_2 + h_5) \\
&\quad +(2h_3h_1 - 2h_4) + h_3
\end{aligned}
$$