Level-zero Lakshmibai-Seshadri paths and the quantum Schubert calculus

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1 Basic notation

G : complex, connected, simply-connected, semisimple Lie group

$B \subset G$: Borel subgroup

$T \subset B$: maximal torus

$W \cong N_G(T)/T$: Weyl group of $\mathfrak{g} = \mathrm{Lie}(G)$;
note that $W \subset \mathrm{GL}(\mathfrak{h}^*)$, where $\mathfrak{h} = \mathrm{Lie}(T)$

$\Delta = \Delta_+ \sqcup \Delta_- \subset \mathfrak{h}^*$: (positive or negative) roots

$\{\alpha_i\}_{i \in I}$: simple roots

$\{h_i\}_{i \in I}$: simple coroots

$w_0 \in W$: the longest element

$B^- := w_0 B w_0 \subset G$: opposite Borel subgroup

$X := G/B$: flag manifold

$X(w) := \overline{BwB/B} \subset X = G/B$:
Schubert variety for $w \in W$;
note that $\dim_{\mathbb{C}} X(w) = \ell(w)$, the length of w

$Y(w) := \overline{B^- wB/B} \subset X = G/B$:

opposite Schubert variety for $w \in W$;

note that $\operatorname{codim}_{\mathbb{C}} Y(w) = \ell(w)$

- $Y(w) = w_0 X(w_0 w)$ for $w \in W$

$\sigma_w := [Y(w)] \in H^{2\ell(w)}(X; \mathbb{Z})$:

cohomology class of $Y(w)$

$\sigma(w_0 w) := [X(w_0 w)] \in H^{2\ell(w)}(X; \mathbb{Z})$:

cohomology class of $X(w_0 w)$;

note that $\sigma_w = \sigma(w_0 w)$ for $w \in W$

Fact

$\{\sigma_w\}_{w \in W}$ form an additive basis for $H^*(X; \mathbb{Z})$ over $\mathbb{Z}$.

Remark

For $w \in W$, $X(w)$ and $Y(w)$ meet transversally at the point $wB/B \in G/B = X$.

Also, $\{\sigma_w\}_{w \in W}$ and $\{\sigma(w)\}_{w \in W}$ are dual bases for $H^*(X; \mathbb{Z})$ under the intersection pairing:

$$\int_X \sigma_w \cdot \sigma(v) = \delta_{v,w} \qquad \text{for } v,\ w \in W$$

2 (small) Quantum cohomology of G/B

$X = G/B$: flag manifold

q_i, $i \in I$: variables

$\mathbb{Z}[q] := \mathbb{Z}[q_i,\, i \in I]$, with $\deg(q_i) = 2n_{\alpha_i}$, where $n_{\alpha_i} := \langle 2\rho,\, h_i \rangle = 2$, with $2\rho := \sum_{\alpha \in \Delta_+} \alpha \in P$

We identify as follows
(including a duality pairing):

$$H^2(X;\mathbb{Z}) \cong P, \qquad H_2(X;\mathbb{Z}) \cong Q^\vee;$$

$P = \bigoplus_{i \in I} \mathbb{Z}\varpi_i$ is the weight lattice,

$Q^\vee := \bigoplus_{i \in I} \mathbb{Z}h_i$ is the coroot lattice.

Also, we identify

$$\begin{aligned} \varpi_i \quad &\longleftrightarrow \quad \sigma_{r_i} \\ h_i \quad &\longleftrightarrow \quad \sigma(r_i), \end{aligned}$$

where $r_i \in W$ is a simple reflection.

- **We have**

$$\boldsymbol{H^2(X;\mathbb{Z}) \ni \sigma_{r_i} = c_1(\mathcal{L}(\varpi_i))},$$

where $\boldsymbol{\mathcal{L}(\varpi_i) := G \times_{P_i} \mathbb{C}(\varpi_i)}$**.**

For $d = \sum_{i \in I} d_i \sigma(r_i) \in H_2(X; \mathbb{Z})$, we set

$$q^d := \prod_{i \in I} q_i^{d_i}$$

$QH^*(X) := H^*(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ (as a $\mathbb{Z}[q]$-module); $\{\sigma_w = \sigma_w \otimes 1\}_{w \in W}$ forms a basis for $QH^*(X)$ over $\mathbb{Z}[q]$.

$$\sigma_u * \sigma_v := \sum_{d \in Q_+^\vee} q^d \sum_{w \in W} N_{u,\,v}^w(d) \sigma_w,$$

where

$d = \sum_{i \in I} d_i \sigma(r_i) \in H_2(X; \mathbb{Z}) = Q^\vee$,

$Q_+^\vee := \sum_{i \in I} \mathbb{Z}_{\geq 0} h_i$, and

$N_{u,\,v}^w(d)$: the 3-point, genus zero Gromov-Witten invariant; this is the number of rational curves $\varphi : \mathbb{P}^1(\mathbb{C}) \to X$ of multidegree d,
i.e., $\varphi_*(\mathbb{P}^1(\mathbb{C})) = d = \sum_{i \in I} d_i \sigma(r_i) \in H_2(X; \mathbb{Z})$,
s.t. $\varphi(p_1) = g_1 Y(u)$, $\varphi(p_2) = g_2 Y(v)$, and
$\varphi(p_3) = g_3 Y(w)$ for three general $g_1, g_2, g_3 \in G$
($p_1, p_2, p_3 \in \mathbb{P}^1(\mathbb{C})$: given, distinct)

- $N_{u,v}^{w}(0) = \int_X \sigma_u \cdot \sigma_v \cdot \sigma_{w_0 w}$

(usual intersection number)

= the coefficient of σ_w in the classical product

$\sigma_u \cdot \sigma_v \in H^*(X; \mathbb{Z})$

• (Degree axiom) The second sum is over all $w \in W$ s.t.

$$\ell(w) = \ell(u) + \ell(v) - \sum_{i \in I} d_i n_{\alpha_i};$$

this follows from the dimension formula

$$\begin{aligned}\dim_{\mathbb{C}} \overline{M}_{0,3}(X, d) &= \dim_{\mathbb{C}} X + \int_d c_1(T_X) \\ &= \ell(w_0) + \langle 2\rho,\ d \rangle \\ &= \ell(w_0) + \sum_{i \in I} d_i \langle 2\rho,\ h_i \rangle \\ &= \ell(w_0) + \sum_{i \in I} d_i n_{\alpha_i},\end{aligned}$$

where $\overline{M}_{0,3}(X, d)$ is the moduli space of stable maps of (multi-)degree d from 3-pointed, genus 0 (projective, connected, reduced, nodal) curves into X.

Note that the space $\overline{M}_{0,3}(X, d)$ can be thought of as a compactification of the set $M_{0,3}(X, d)$ of isomorphism classes of 3-pointed maps of degree d from $\mathbb{P}^1(\mathbb{C})$ into X.

3 Classical Chevalley formula

Fact

For $\alpha_i \in \Pi$ and $w \in W$, we have in $H^*(X;\mathbb{Z})$

$$\boxed{\sigma_{r_i} \cdot \sigma_w = \sum_{\substack{\beta \in \Delta_+ \\ wr_\beta \gtrdot w}} \langle \varpi_i,\, h_\beta \rangle \sigma_{wr_\beta},}$$

where $h_\beta \in Q^\vee$ is the dual root of $\beta \in \Delta_+$, r_β is the associated reflection, and

$$wr_\beta \gtrdot w \quad \Longleftrightarrow \quad \ell(wr_\beta) = \ell(w) + 1.$$

4 Bruhat graph

vertices : elements of W

edges : $w \xrightarrow{\beta} wr_\beta$, $\beta \in \Delta_+$, if $\ell(wr_\beta) = \ell(w) + 1$.

Remark

For u, $v \in W$, we have

$$v = ur_\beta \text{ for } \beta \in \Delta_+$$

if and only if there exists a T-invariant curve C containing uB/B and vB/B in $X = G/B$; note that $[C] = h_\beta \in H_2(X; \mathbb{Z})$.

$\rightsquigarrow$ partial order (Bruhat order) on W, denoted by $>$; we write $wr_\beta \gtrdot w$ if $w \xrightarrow{\beta} wr_\beta$

5 Quantum Chevalley formula

Fact (Fulton and Woodward)

For $\alpha_i \in \Pi$ and $w \in W$, we have in $QH^*(X)$

$$\sigma_{r_i} * \sigma_w = \sum_{\substack{\beta \in \Delta_+ \\ wr_\beta > w}} \langle \varpi_i, h_\beta \rangle \sigma_{wr_\beta} + \sum_{\substack{\beta \in \Delta_+ \\ \ell(wr_\beta) = \ell(w) + 1 - \langle 2\rho, h_\beta \rangle}} q^{h_\beta} \langle \varpi_i, h_\beta \rangle \sigma_{wr_\beta}.$$

Remark For $\beta \in \Delta_+$, we have

$$\langle 2\rho, h_\beta \rangle = \int_{C_\beta} c_1(T_X),$$

where $c_1(T_X) = 2\rho \in H^2(X; \mathbb{Z})$, and

C_β is a unique T-invariant curve in X containing $1B/B$ and $r_\beta B/B$;

note that $[C_\beta] = h_\beta \in H_2(X; \mathbb{Z})$.

In particular, for $\alpha_i \in \Pi$, we have

$$n_{\alpha_i} = \int_{X(r_i)} c_1(T_X) = \langle 2\rho, h_i \rangle = 2.$$

N.B. The equality

$$\ell(wr_\beta) = \ell(w) + 1 - \langle 2\rho,\, h_\beta \rangle$$

above comes from the (co-)dimension computation of a certain closed (reduced, locally irreducible) subscheme $ev_1^{-1}(Y(u)) \cap ev_2^{-1}(X(v)),\ u, v \in W$, of the moduli space $\overline{M}_{0,3}(X, d)$; recall that

$$\begin{aligned}\dim_{\mathbb{C}} \overline{M}_{0,3}(X, d) &= \dim_{\mathbb{C}} X + \int_d c_1(T_X) \\ &= \ell(w_0) + \langle 2\rho,\, d \rangle.\end{aligned}$$

6 Quantum Bruhat graph

vertices : elements of W

edges : $w \xrightarrow{\beta} wr_\beta$, $\beta \in \Delta_+$, if

(1) $wr_\beta \gtrdot w$, or

(2) $\ell(wr_\beta) = \ell(w) + 1 - \langle 2\rho, h_\beta \rangle$

Remark

This does NOT induce a poset structure on W, contrary to the usual Bruhat graph.

7 Affine Lie algebras

$\widehat{\mathfrak{g}} := (\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}) \oplus \mathbb{C}c \oplus \mathbb{C}d$:

untwisted affine Lie algebra/$\mathbb{C}$

$\widehat{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$: Cartan subalgebra

$\widehat{\Pi} := \Pi \cup \{\alpha_0\} \subset (\widehat{\mathfrak{h}})^*$: simple roots

$\widehat{\Pi}^\vee := \Pi^\vee \cup \{h_0\} \subset \widehat{\mathfrak{h}}$: simple coroots

$\widehat{\Delta} := \widehat{\Delta}_+ \sqcup \widehat{\Delta}_- \subset (\widehat{\mathfrak{h}})^*$:

(positive or negative) roots

$\widehat{W} \subset GL((\widehat{\mathfrak{h}})^*)$: (affine) Weyl group

$\widehat{\Delta}_{\mathrm{re}} := \widehat{W}\widehat{\Pi}$: real roots

$\widehat{I} := I \cup \{0\}$

$c = \sum_{j \in \widehat{I}} a_j^\vee h_j$: canonical central element

$\delta = \sum_{j \in \widehat{I}} a_j \alpha_j$: null root

Λ_i, $i \in \widehat{I}$: fundamental weights

$\varpi_i := \Lambda_i - a_i^\vee \Lambda_0$, $i \in I$:

level-zero fundamental weights

$\widehat{\boldsymbol{P}} := \left(\sum_{i \in \widehat{I}} \mathbb{Z}\boldsymbol{\Lambda}_i\right) \oplus \mathbb{Z}\boldsymbol{\delta} \subset (\widehat{\mathfrak{h}})^*$ **: weight lattice**

$\widehat{\boldsymbol{P}}^0 := \left\{\boldsymbol{\lambda} \in \widehat{\boldsymbol{P}} \mid \langle \boldsymbol{\lambda}, \boldsymbol{c} \rangle = 0\right\}$ **:**

set of level-zero weights

$$\mathrm{cl} : \mathbb{R} \otimes_{\mathbb{Z}} \widehat{\boldsymbol{P}}^0 \twoheadrightarrow (\mathbb{R} \otimes_{\mathbb{Z}} \widehat{\boldsymbol{P}}^0)/\mathbb{R}\boldsymbol{\delta} = \bigoplus_{i \in I} \mathbb{R}\,\mathrm{cl}(\boldsymbol{\varpi}_i) \cong \mathfrak{h}_{\mathbb{R}}^* = \bigoplus_{i \in I} \mathbb{R}\boldsymbol{\alpha}_i$$

$\mathrm{cl} : \widehat{\boldsymbol{W}} = \boldsymbol{W} \ltimes \boldsymbol{Q}^{\vee} \to \boldsymbol{W}$, $\boldsymbol{w}t_{\gamma} \mapsto \boldsymbol{w}$,

which is a homomorphism;

note that for $\boldsymbol{\lambda} \in \mathbb{R} \otimes_{\mathbb{Z}} \widehat{\boldsymbol{P}}^0$,

$$\boldsymbol{w}t_{\gamma}\boldsymbol{\lambda} = \boldsymbol{w}\boldsymbol{\lambda} - \langle \boldsymbol{\lambda}, \gamma \rangle \boldsymbol{\delta},$$

where $\boldsymbol{w} \in \boldsymbol{W}$ **and** $\gamma \in \boldsymbol{Q}^{\vee}$.

8 Lakshmibai-Seshadri paths

$\lambda \in P = \bigoplus_{i \in I} \mathbb{Z}\varpi_i$: (integral) weight

Assume that $\lambda \in P$ is regular and dominant.

For $u, v \in W$ s.t. $u > v$ in the Bruhat order, and $a \in \mathbb{Q}$ s.t. $0 < a < 1$,

an a-chain for (u, v) is a sequence

$$u = u_0 \gtrdot u_1 \gtrdot \cdots \gtrdot u_{n-1} \gtrdot u_n = v$$

of elements of W s.t.

$$a\langle \lambda, h_{\beta_k} \rangle \in \mathbb{Z} \quad \text{for } 1 \le k \le n,$$

where $u_{k-1} \gtrdot u_k = u_{k-1} r_{\beta_k}$ for $\beta_k \in \Delta_+$.

A Lakshmibai-Seshadri (LS) path of shape $\lambda \in P$ (λ : regular and dominant) is a pair of sequences

$$(v_1 > v_2 > \cdots > v_s \,; \quad 0 = a_0 < a_1 < \cdots < a_s = 1),$$

with $v_i \in W$ and $a_i \in \mathbb{Q}$, s.t.

$^{\exists}a_k$-chain for (v_k, v_{k+1}) for all $1 \leq k \leq s-1$.

LS(λ) : set of LS paths of shape λ

9 Level-zero LS paths

$\lambda \in \widehat{P}^0$: level-zero weight

$\widehat{W} \subset \mathrm{GL}((\widehat{\mathfrak{h}})^*)$: affine Weyl group

For $\mu, \nu \in \widehat{W}\lambda$, we write $\mu > \nu$ if there exist a sequence

$$\mu = \mu_0, \mu_1, \ldots, \mu_{n-1}, \mu_n = \nu$$

of elements of $\widehat{W}\lambda$ and a sequence

$$\beta_1, \ldots, \beta_n \in \widehat{\Delta}_{\mathrm{re}} \cap \widehat{\Delta}_+$$

s.t. for all $1 \leq k \leq n$,

$$\mu_k = r_{\beta_k}\mu_{k-1}, \quad \langle \mu_{k-1}, h_{\beta_k} \rangle < 0.$$

$\rightsquigarrow$ partial order on $\widehat{W}\lambda \subset (\widehat{\mathfrak{h}})^*$

For $\mu, \nu \in \widehat{W}\lambda$ s.t. $\mu > \nu$ in the (partial) order above, we define:

$\mathrm{dist}(\mu, \nu)$:= the maximal length n of all possible sequences above for (μ, ν);
we write $\nu \to \mu$ if $\mu > \nu$ and $\mathrm{dist}(\mu, \nu) = 1$.

For $\mu > \nu \in \widehat{W}\lambda$ and $a \in \mathbb{Q}$ with $0 < a < 1$, an a-chain for (μ, ν) is a sequence

$$\mu = \mu_0 \leftarrow \mu_1 \leftarrow \cdots \leftarrow \mu_n = \nu$$

of elements of $\widehat{W}\lambda$ s.t.

$$a\langle \mu_{k-1}, h_{\beta_k} \rangle \in \mathbb{Z}_{<0} \quad \text{for } 1 \leq k \leq n,$$

where $\mu_k = r_{\beta_k}\mu_{k-1}$ for $\beta_k \in \widehat{\Delta}_{\mathrm{re}} \cap \widehat{\Delta}_+$.

A (level-zero) LS path of shape $\lambda \in \widehat{P}^0$ is a pair of sequences

$$(\nu_1 > \nu_2 > \cdots > \nu_s\,;$$
$$0 = a_0 < a_1 < \cdots < a_s = 1),$$

with $\nu_i \in \widehat{W}\lambda$ and $a_i \in \mathbb{Q}$, s.t.
${}^{\exists}a_k$-chain for (ν_k, ν_{k+1}) for all $1 \leq k \leq s-1$.

$\widehat{\mathrm{LS}}(\lambda)$: set of level-zero LS paths of shape $\lambda \in \widehat{P}^0$

Assume (for simplicity) that $\lambda \in \widehat{P}^0$ is level-zero dominant and regular, i.e.,

$$\langle \lambda, h_i \rangle > 0 \quad \text{for all } i \in I.$$

Theorem 1.

For $\mu, \nu \in \widehat{W}\lambda$, we have

$$\nu \to \mu$$

if and only if

$$W\,\mathrm{cl}(\lambda) \ni \mathrm{cl}(\nu) \to \mathrm{cl}(\mu) \in W\,\mathrm{cl}(\lambda)$$

is an edge in the quantum Bruhat graph.

Here we identify $\mathfrak{h}^*_{\mathbb{R}} \supset W\,\mathrm{cl}(\lambda) \leftrightarrow W$ since the stabilizer $W_{\mathrm{cl}(\lambda)}$ is trivial.

10 Quantum LS paths

Assume (for simplicity) that $\lambda \in \widehat{P}^0$ is level-zero dominant and regular, i.e.,

$$\langle \lambda, h_i \rangle > 0 \quad \text{for all } i \in I.$$

For $x, y \in W$ and $a \in \mathbb{Q}$ with $0 < a < 1$,
a directed a-path from y to x is a directed path

$$x = y_0 \xleftarrow{\beta_1} y_1 \xleftarrow{\beta_2} y_2 \xleftarrow{\beta_3} \cdots \xleftarrow{\beta_n} y_n = y$$

in the quantum Bruhat graph, with $\beta_k \in \Delta_+$, s.t.

$$a\langle \mathrm{cl}(\lambda), h_{\beta_k} \rangle \in \mathbb{Z} \quad \text{for all } 1 \le k \le n.$$

A quantum LS path of shape $\boldsymbol{\lambda} \in \widehat{P}^0$ (level-zero dominant and regular) is a pair of sequences

$$(x_1, x_2, \ldots, x_s\,;\, 0 = a_0 < a_1 < \cdots < a_s = 1),$$

with $x_i \in W$ and $a \in \mathbb{Q}$, s.t.
$^\exists$directed a_k-path from x_{k+1} to x_k
for all $1 \leq k \leq s - 1$.

QLS($\boldsymbol{\lambda}$) : set of quantum LS paths of shape $\boldsymbol{\lambda}$

For

$$\pi = (\nu_1 > \nu_2 > \cdots > \nu_s\,; \quad 0 = a_0 < a_1 < \cdots < a_s = 1) \in \widehat{\mathrm{LS}}(\lambda),$$

we set

$$\mathrm{cl}(\pi) := (\mathrm{cl}(\nu_1),\, \mathrm{cl}(\nu_2),\, \ldots,\, \mathrm{cl}(\nu_s)\,; \quad 0 = a_0 < a_1 < \cdots < a_s = 1)$$

From Theorem 1, it follows that

$$\widehat{\mathrm{LS}}(\lambda)_{\mathrm{cl}} := \left\{ \mathrm{cl}(\pi) \mid \pi \in \widehat{\mathrm{LS}}(\lambda) \right\} \subset \mathrm{QLS}(\lambda).$$

Here we identify

$$W\,\mathrm{cl}(\lambda) \leftrightarrow W$$

since the stabilizer $W_{\mathrm{cl}(\lambda)}$ is trivial.

In fact, we can prove:

Theorem 2.

$$\widehat{\mathrm{LS}}(\lambda)_{\mathrm{cl}} = \mathrm{QLS}(\lambda).$$

11 Representations

Let $\lambda \in P$ be a dominant integral weight

$V(\lambda)$: the irreducible, finite-dimensional, highest weight $U_q(\mathfrak{g})$-module over $\mathbb{C}(q)$ of highest weight λ

Fact (Littelmann, Joseph, Kashiwara)

The crystal basis of $V(\lambda)$ is isomorphic to the crystal $\mathrm{LS}(\lambda)$ of all LS paths of shape λ.
In particular,

$$\mathrm{ch}\, V(\lambda) = \sum_{\pi \in \mathrm{LS}(\lambda)} e^{\pi(1)},$$

where

$$\pi(1) = \sum_{k=1}^{s} (a_k - a_{k-1}) w_k \lambda$$

$$= w_s \lambda + \sum_{k=1}^{s-1} a_k (w_k \lambda - w_{k+1} \lambda) \in P$$

Let $\boldsymbol{\lambda} = \sum_{i \in I} m_i \varpi_i \in \widehat{P}^0$, with $m_i \in \mathbb{Z}_{\geq 0}$ for $i \in I$.

Theorem 3.

We have an isomorphism of $U_q'(\widehat{\mathfrak{g}})$-crystals:

$$\widehat{\mathrm{LS}}(\lambda)_{\mathrm{cl}} \cong \bigotimes_{i \in I} \left(\widehat{\mathrm{LS}}(\varpi_i)_{\mathrm{cl}}\right)^{\otimes m_i}$$

Theorem 4.

For each $i \in I$, $\widehat{\mathrm{LS}}(\varpi_i)_{\mathrm{cl}}$ is isomorphic to the crystal basis of the level-zero fundamental representation $W(\varpi_i)$ of $U_q'(\widehat{\mathfrak{g}})$, which was introduced by Kashiwara. Here, $W(\varpi_i)$ is a finite-dimensional, irreducible $U_q'(\widehat{\mathfrak{g}})$-module over $\mathbb{C}(q)$.

$U_q'(\widehat{\mathfrak{g}}) = U_q([\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}])$,
where $[\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}] = (\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}) \oplus \mathbb{C}c$

Note that

$$\text{finite-dim. level-zero rep. of } U_q'(\widehat{\mathfrak{g}}) \text{ (of type I)}$$

$$= \text{finite-dim. rep. of } U_q(L(\mathfrak{g})),$$

where $L(\mathfrak{g}) := \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$;
the irreducible ones are

"ℓ"-highest weight representations,

which are parametrized by Drinfeld polynomials (instead of usual highest weights).
Here, "ℓ"-highest weight rep. are highest weight rep. w.r.t. the triangular decomposition:

$$L(\mathfrak{g}) = L(\mathfrak{n}_-) \oplus L(\mathfrak{h}) \oplus L(\mathfrak{n}_+),$$

where $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ is the usual triangular decomposition.

12 On grading

Let

$$\eta = (x_1, x_2, \ldots, x_s\,; \\ 0 = a_0 < a_1 < \cdots < a_s = 1) \in \mathrm{QLS}(\lambda).$$

For each $1 \le k \le s-1$, we can take a shortest directed path d_k from x_{k+1} to x_k, which is automatically a directed a_k-path:

$$\mathrm{d}_k : x_k = y_0 \xleftarrow{\beta_1} y_1 \xleftarrow{\beta_2} y_2 \xleftarrow{\beta_3} \cdots \xleftarrow{\beta_n} y_n = x_{k+1}.$$

We set

$$\mathrm{wt}(\mathrm{d}_k) := \sum_{\substack{1 \le k \le n \\ \ell(y_{k-1}) = \ell(y_k) + 1 - \langle 2\rho, h_{\beta_k} \rangle}} h_{\beta_k} \in Q^\vee;$$

this depends only on x_k and x_{k+1}, and is denoted by $d_{\min}(x_{k+1}, x_k)$ (due to Postnikov).

Theorem 5.

Let $\boldsymbol{\lambda} = \sum_{i \in I} m_i \varpi_i \in \widehat{P}^0$, with $m_i \in \mathbb{Z}_{\geq 0}$ for $i \in I$. For each $\eta \in \mathrm{QLS}(\boldsymbol{\lambda}) = \widehat{\mathrm{LS}}(\boldsymbol{\lambda})_{\mathrm{cl}}$ of the form:

$$\eta = (x_1, x_2, \ldots, x_s; \quad 0 = a_0 < a_1 < \cdots < a_s = 1) \in \mathrm{QLS}(\boldsymbol{\lambda}),$$

we have

$$\mathrm{Deg}_{\boldsymbol{\lambda}}(\eta) = -\sum_{k=1}^{s-1} (1 - a_k) \langle \boldsymbol{\lambda}, d_{\min}(x_{k+1}, x_k) \rangle.$$

Here, $\mathrm{Deg}_{\boldsymbol{\lambda}}(\eta) \in \mathbb{Z}_{\leq 0}$ for $\eta \in \widehat{\mathrm{LS}}(\boldsymbol{\lambda})_{\mathrm{cl}}$ was originally defined to be the nagative $-K$ of the coefficient K of the null root δ in the expression

$$\pi(1) = \boldsymbol{\lambda} - \beta + K\delta, \ \beta \in Q_+, K \in \mathbb{Z}_{\geq 0},$$

where $\pi \in \widehat{\mathrm{LS}}_0(\boldsymbol{\lambda})$ is a unique level-zero LS path s.t. $\mathrm{cl}(\pi) = \eta$ and the "initial direction" ν_1 of π is in $\boldsymbol{\lambda} - Q_+$.

Remark

$\widehat{\mathfrak{g}}$: affine Lie algebra of type $A_N^{(1)}$.

For each $\mu \in \sum_{i \in I} \mathbb{Z}_{\geq 0} \, \mathrm{cl}(\varpi_i)$, we have

$$K_{\mu^t, \lambda^+}(q) = \sum_{\substack{\eta \in \widehat{\mathrm{LS}}(\lambda)_{\mathrm{cl}} \\ e_j \eta = 0 \ (j \in I) \\ \eta(1) = \mu}} q^{-\mathrm{Deg}_\lambda(\eta)}.$$

Here, the right-hand side is

a "weighted" branching multiplicity of

the tensor product $U_q'(\widehat{\mathfrak{g}})$-module $\bigotimes_{i \in I} W(\varpi_i)$

w.r.t. the canonical subalgebra $U_q(\mathfrak{g})$, and

$K_{\mu^t, \lambda^+}(q)$ is

the Kostka-Foulkes polynomial associated to

the partitions μ^t and λ^+, with $|\mu| = |\lambda^+|$.

Fact (Postnikov)

Fix u, $v \in W$. For each $w \in W$, the coefficient of σ_v in $\sigma_u * \sigma_w \in QH^*(X)$ is divisible by $q^{d_{\min}(u, v)}$. Also, $^\exists w \in W$ s.t. the coefficient of σ_v in $\sigma_u * \sigma_w$ equals $q^{d_{\min}(u, v)}$ times a nonzero integer.