Tableaux and Eulerian properties of the symmetric group

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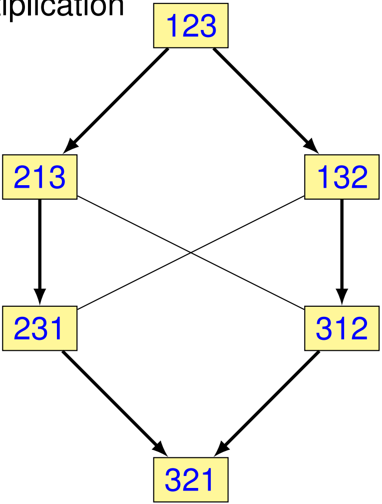
Ehresmann defined an order (called Bruhat order) on the symmetric group, which plays a fundamental role in geometry, algebra and representation theory:
two permutations are consecutive if they have consecutive length and differ by multiplication by a transposition (on the right or left).
thick arrows = permutohedron



The EB-order possesses many symmetry properties.
For example, in every interval, there are as many permutations of even length than of odd length.
This type of properties is accounted by the notion of Eulerian structure : given a graded poset X, its incidence matrix E (i.e. $E[x,y] = 1 \Leftrightarrow x \leq y$) is graded.

$$E = E_0 + E_1 + E_2 + \dots$$

The poset is Eulerian if $E^{-1} = E_0 - E_1 + E_2 - E_3 + \dots$.

The same notion applies to a directed graph with a rank.

Young tableaux may be viewed as chains of permutations w.r. to the EB-order.

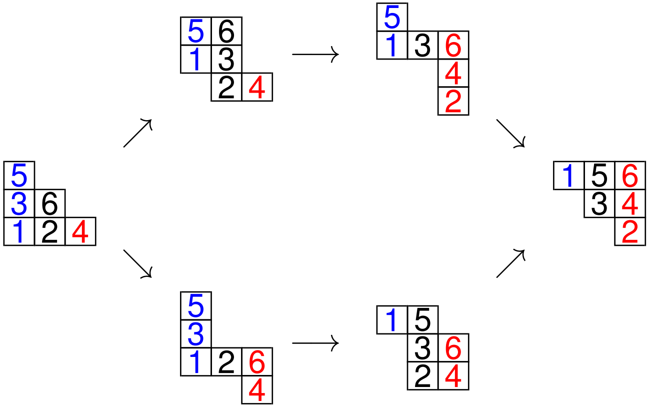
$$\begin{array}{cccc} 2 & 2 & 1 & 1 \\ 4 & 3 & 3 & 2 \\ 3 & 4 & 4 & 4 \\ 1 & 1 & 2 & 3 \end{array} \Leftrightarrow 1342 < 1432 < 2431 < 3421$$

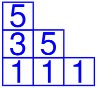
tableau chain

first permutation= left key = 1342 ,
last permutation=right key=3421

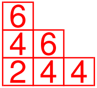
But keys may be obtained by the jeu de taquin . . .

Keys by jeu de taquin on consecutive columns :





= left key

rightkey =


Instead of permutations, one records also shapes, and one can use one of the following equivalent ways of denoting the same object :

key = set of columns embedded into each other
$\Leftrightarrow$ *tableau congruent to some word of type* $\ldots 3^{v_3} 2^{v_2} 1^{v_1}$
$\Leftrightarrow$ *monomial* $x^v = x_1^{v_1} x_2^{v_2} \ldots$
$\Leftrightarrow$ *weight* $v = [v_1, v_2, \ldots]$.

For the above example, the right key is

$$\mathcal{C}(t) = \left\{ \begin{array}{|c|}\hline 4 \\ \hline\end{array}\,,\ \begin{array}{|c|}\hline 6 \\ \hline 4 \\ \hline\end{array}\,,\ \begin{array}{|c|}\hline 6 \\ \hline 4 \\ \hline 2 \\ \hline\end{array} \right\} \Leftrightarrow \begin{array}{|c|c|c|}\hline 6 & & \\ \hline 4 & 6 & \\ \hline 2 & 4 & 4 \\ \hline\end{array} \equiv \begin{array}{|c|c|c|}\hline 6 & 6 & \\ \hline 4 & 4 & 4 \\ \hline & & 2 \\ \hline\end{array}$$

$$\Leftrightarrow x_2 x_4^3 x_6^2 \Leftrightarrow [0, 1, 0, 3, 0, 2]$$

Keys are very useful to describe Demazure characters, but they can also be used for other problems in the theory of tableaux. For example, the shape of a product $t_1 t_2$ of two tableaux is equal to the sum of the shapes of t_1 and t_2 if and only if

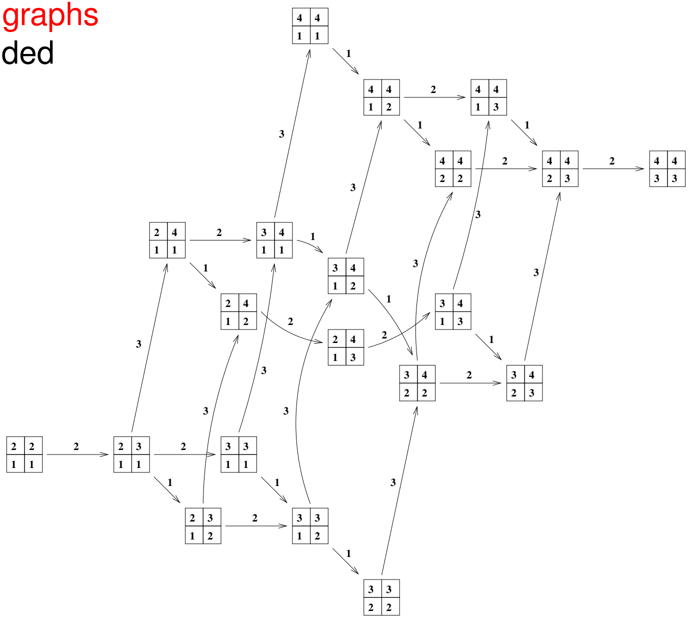
$$\text{right key}(t_1) \;\leq\; \text{left key}(t_2) \quad \text{(componentwise)}$$

Thus, the square of a tableau has twice the shape of the tableau iff the tableau is a key. The square of the preceding tableau, which is not a key, is

$$\begin{array}{|c|cc}\hline 5 & & \\ \hline 3 & \multicolumn{1}{c|}{6} & \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \quad \begin{array}{|c|cc}\hline 5 & & \\ \hline 3 & \multicolumn{1}{c|}{6} & \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \;=\; \begin{array}{|c|c|ccc}\hline 5 & 6 & & & \\ \hline 3 & 4 & & & \\ \hline 2 & 3 & \multicolumn{1}{c|}{5} & & \\ \hline 1 & 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{6} \\ \hline \end{array}$$

whose shape is $[2, 2, 3, 5] \neq 2 \times [1, 2, 3]$

Taquin is not enough
Crystal graphs
are needed



Crystal graphs decompose the set of tableaux of the same shape into *i*-strings:
Example of a 3-string (one pairing 43)

$$\begin{array}{|c|c|}\hline 3&4\\\hline\end{array}\qquad \to \qquad \begin{array}{|c|c|}\hline 3&4\\\hline\end{array}\qquad \to \qquad \begin{array}{|c|c|}\hline 3&4\\\hline\end{array}\qquad \to \qquad \begin{array}{|c|c|}\hline 4&4\\\hline\end{array}$$

$$\begin{array}{lllllll}
34 & & 34 & & 34 & & 44 \\
223 & \to & 223 & \to & 223 & \to & 223 \\
11233 & & 11234 & & 11244 & & 11244
\end{array}$$

In fact, after paring and eliminating the letters $\neq 3, 4$, a 3-string reduces to

$$333 \to 334 \to 344 \to 444$$

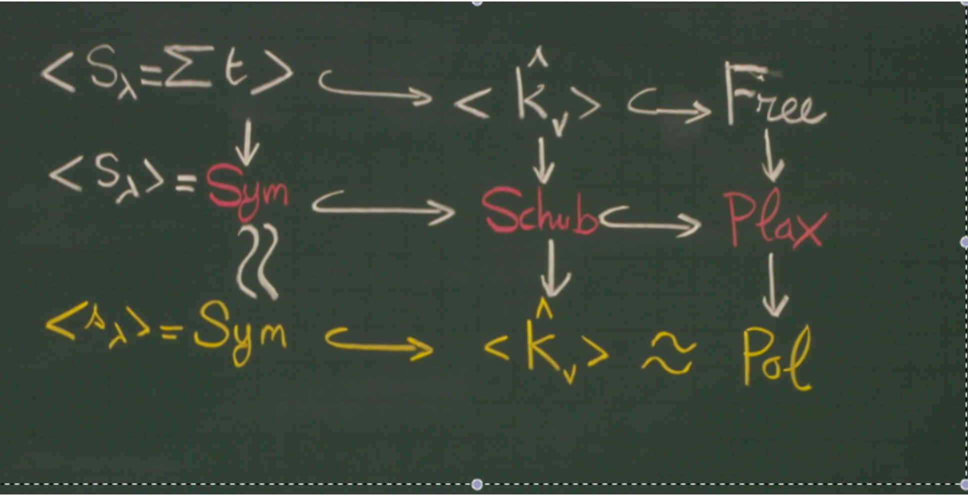
Action of s_i = symmetry with respect to the middle of the *i*-string
Action of π_i on the head of the string = sum of all the elements of the string

One can also use $\widehat{\pi}_i = \pi_i - 1 = \pi_i$ - identity.
It sends the head of a i-string to the sum of all the other elements of the string.

$$\begin{array}{l} 3\,4 \\ 2\,2\,3 \\ 1\,1\,2\,3\,3 \end{array} \; \widehat{\pi}_3 = \begin{array}{l} 3\,4 \\ 2\,2\,3 \\ 1\,1\,2\,3\,4 \end{array} + \begin{array}{l} 3\,4 \\ 2\,2\,3 \\ 1\,1\,2\,4\,4 \end{array} + \begin{array}{l} 4\,4 \\ 2\,2\,3 \\ 1\,1\,2\,4\,4 \end{array}$$

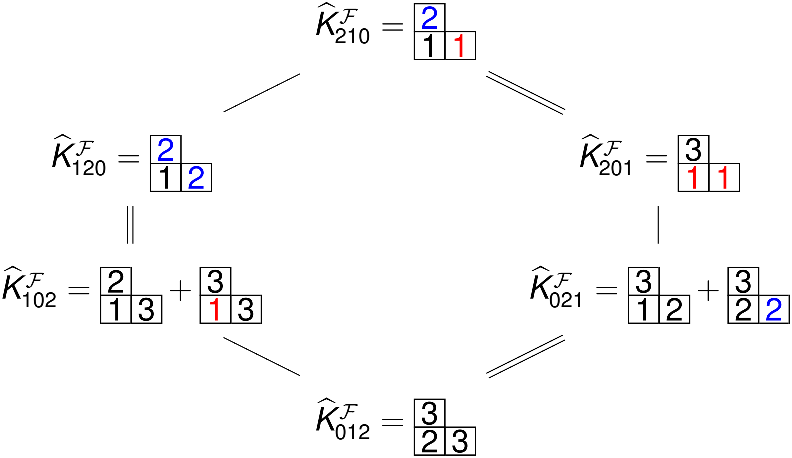
These operators lift similar operators on $\mathfrak{Pol}$:

$$s_i \text{ transpose } x_i, x_{i+1},$$
$$\pi_i \text{ is the operator } f \to (x_i f - x_{i+1} f^{s_i})(x_i - x_{i+1})^{-1}$$
$$\widehat{\pi}_i = \pi_i - 1 \text{ is the operator } f \to (f - f^{s_i})(x_i/x_{i+1} - 1)^{-1}$$



<S_λ=Σ t> ↪ <k̂_v> ↪ Free
<S_λ> = Sym ↪ Schub ↪ Plax
<s_λ> = Sym ↪ <k̂_v> ≈ Pol

Tableaux generated by $\widehat{\pi}_1$, $\widehat{\pi}_2$, starting from a Yamanouchi tableau:



Given a partition λ, one defines $K_\lambda^{\mathcal{F}} = \widehat{K}_\lambda^{\mathcal{F}} = \ldots 2^{\lambda_2} 1^{\lambda_1}$, and the other key polynomials by recursion. For i, v such that $v_i > v_{i+1}$, then

$$K_{vs_i}^{\mathcal{F}} = K_v^{\mathcal{F}} \pi_i \quad \& \quad \widehat{K}_{vs_i}^{\mathcal{F}} = \widehat{K}_v^{\mathcal{F}} \widehat{\pi}_i$$

Facts. $\widehat{K}_v^{\mathcal{F}}$ is the sum of all tableaux with right key v.

$$K_v^{\mathcal{F}} = \sum_{u \le v} \widehat{K}_u^{\mathcal{F}}$$

The tableaux in $K_v^{\mathcal{F}}$ index a basis of a Demazure module.

The construction of the bases of Demazure modules is not symmetrical in the left and right keys. To recover symmetry, one defines operators on $\mathfrak{Free} \otimes \mathfrak{Free}$.

$$\begin{aligned} \mathbb{U}_i &= \widehat{\pi}_i \otimes s_i + 1 \otimes \widehat{\pi}_i \\ &= \pi_i \otimes s_i + 1 \otimes \theta_i \,, \end{aligned}$$

with $\theta_i = \widehat{\pi}_i - s_i = \pi_i - 1 - s_i$.

Caution! These operators do not satisfy the braid relations.

Since one uses $1, \pi_i, \widehat{\pi}_i$ on the left factor, this factor remains in the linear span of the $K_v^{\mathcal{F}}$. One can project this factor on $\mathfrak{Pol}$ without loss of information.

On the right factor, one uses also the s_i, which preserve tableaux, but not the linear span of the $K_v^{\mathcal{F}}$. Thus, one takes V_λ, the linear span of all tableaux of shape λ as second space.

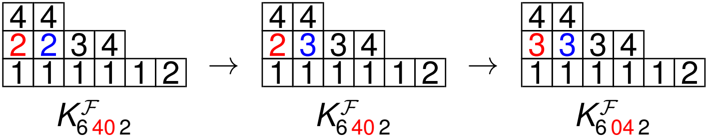
Proposition. *Given a strict partition $\lambda \in \mathbb{N}^n$, a reduced decomposition $s_i \dots s_j$, and $v = s_i \dots s_j$, then*

$$K_\lambda^{\mathcal{F}} \otimes \widehat{K}_\lambda^{\mathcal{F}} \, \mathbb{U}_i \cdots \mathbb{U}_j = \sum_u \sum_t K_u^{\mathcal{F}} \otimes t$$

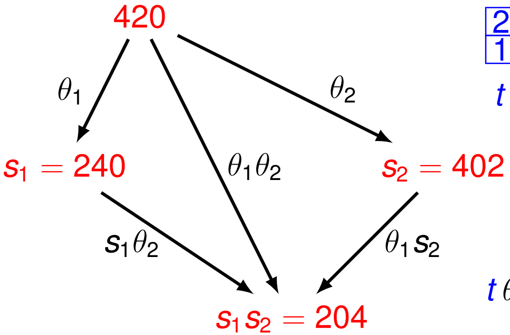
sum over all weights $u \in \mathbb{N}^n$ which are a permutation of λ, all tableaux t which have left key u and right key v.

Proof. By induction on the length of the reduced decomposition. Adding one factor $\Cup_j$ and looking at the action on j-strings, or heads of j-strings is easy, as well as the modification of the left key. Reasoning is reduced to words in the two letters $j, j+1$.

Here is an example of a 2-string, writing the variable letters in color. The letter in red determines the left key among $K^{\mathcal{F}}_{6402}, K^{\mathcal{F}}_{6042}$.



Considering the component on the right only, given a strict partition λ and a reduced decomposition $s_i \dots s_j$, one obtains the Tableauhedron with top $\lambda s_i \dots s_j$ by the expansion of $(s_i{+}\theta_i{+}1)\dots(s_j{+}\theta_j{+}1)$.



expansion of

$\begin{array}{llll} 2&2&& \\ 1&1&1&1 \end{array}\;(s_1{+}\theta_1{+}1)(s_2{+}\theta_2{+}s_2)$

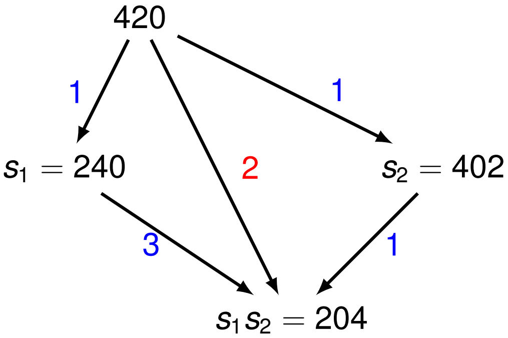
$t\,\theta_1 = \begin{array}{llll} 2&2&& \\ 1&1&1&2 \end{array}$

$t\,\theta_2 = \begin{array}{llll} 2&3&& \\ 1&1&1&1 \end{array}$

$t\,\theta_1 s_2 = \begin{array}{llll} 3&3&& \\ 1&1&1&3 \end{array}$

$t\,\theta_1\theta_2 = \begin{array}{llll} 2&2&& \\ 1&1&1&3 \end{array} + \begin{array}{llll} 2&3&& \\ 1&1&1&3 \end{array}$

and $t s_1\theta_2 = \begin{array}{llll} 2&3&& \\ 1&1&3&3 \end{array} + \begin{array}{llll} 2&2&& \\ 1&1&3&3 \end{array} + \begin{array}{llll} 2&2&& \\ 1&1&2&3 \end{array}$.



Eulerian property satisfied by the edge multiplicities:

$$2 \times 2 = 1 \times 3 + 1 \times 1$$

But, stronger statement, keeping the evaluation of tableaux instead of only their number.

One projects $\mathfrak{Free} \otimes \mathfrak{Free}$ onto $\mathfrak{Pol}^{\clubsuit} \otimes \mathfrak{Pol}$
by sending $i \otimes j$ onto $x_i^{-1} \otimes x_j$.

Since the image of π_i under the inversion of variables $\clubsuit$ is $-\theta_i$, one has

$$\big(f\clubsuit \otimes g\big)\mathbb{U}_i = -f\theta_i\clubsuit \otimes g^{s_i} + f\clubsuit \otimes g\theta_i \,.$$

Thus the operators to use on $\mathfrak{Pol} \otimes \mathfrak{Pol}$ are

$$\widetilde{\mathbb{U}}_i = -\theta_i \otimes s_i + 1 \otimes s_i$$

followed by the evaluation

$$f \otimes g \to f(1/x_1, \ldots, 1/x_n) \otimes g(x_1, \ldots, x_n)$$

Our starting point was $x^\lambda \otimes x^\lambda$. It image under $\widetilde{\mathbb{U}}_i$ is

$$x^\lambda \otimes x^\lambda\, \widetilde{\mathbb{U}}_i = -x^\lambda\theta_i \otimes x^{\lambda s_i} + x^\lambda \otimes (x^\lambda\theta_i)$$

which evaluates to 0 (one has in fact computed the image of $x^{-\lambda}x^\lambda = 1$ under the operator $\widehat{\pi}_i = \partial_i x_{i+1}$).

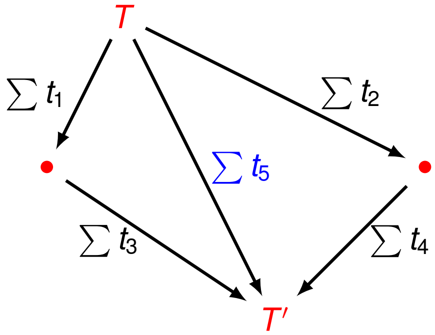
At the level of tableaux, writing $t_{41} = \begin{array}{|c|cccc}\hline 2 & \multicolumn{4}{c}{} \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\ \hline \end{array}$, $t_{14} = \begin{array}{|c|cccc}\hline 2 & \multicolumn{4}{c}{} \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} \\ \hline \end{array}$,
one has

$$t_{41}\theta_1 = \begin{array}{|c|ccc}\hline 2 & \multicolumn{3}{c}{} \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} \\ \hline \end{array} + \begin{array}{|c|ccc}\hline 2 & \multicolumn{3}{c}{} \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} \\ \hline \end{array} = t + t'\ ,$$

and the relation is the symmetry property:

$$-\frac{t_{14}}{t} - \frac{t_{14}}{t'} + \frac{t}{t_{41}} + \frac{t'}{t_{41}} \qquad \text{evaluates to } 0\ .$$

Similarly, the action of $\widetilde{\mathbb{U}}_i \widetilde{\mathbb{U}}_j$ on $x^\lambda \otimes x^\lambda$, corresponding to the figure



gives the relation

$$\sum \frac{t_3}{t_1} + \sum \frac{t_4}{t_2} - \sum \frac{T'}{t_5} - \sum \frac{t_5}{T} \sim 0 \, .$$

I shall not draw a picture for length 3 ! In general, let $\lambda \in \mathbb{N}^n$ be a strict partition, $\mathfrak{Tab}$ be the set of tableaux in $\{1, \ldots, n\}$, $\{\sigma\}$ be the set of permutation of λ.

Let $E = E_0 + E_1 + E_2 + \ldots$ be the matrix, filtered by distance, with entries

$$E[\sigma, \zeta] = \sum_t ev(t)$$

sum over all tableaux in $\mathfrak{Tab}$ with left key σ, right key ζ.
In particular, E_0 is the diagonal matrix with $ev(\sigma)$ on its diagonal.

Theorem. The inverse of E is

$$E_0^{\clubsuit} - E_1^{\clubsuit} + E_2^{\clubsuit} - E_3^{\clubsuit} + \ldots$$

For example, the matrix and its inverse corresponding to the keys $[4,2,0], [4,0,2], [2,4,0], [2,0,4]$ are

$$\begin{bmatrix} {x_2}^2{x_1}^4 & x_2x_3{x_1}^4 & {x_2}^3{x_1}^3 & x_2{x_3}^2{x_1}^3 + {x_2}^2{x_1}^3x_3 \\ \cdot & {x_3}^2{x_1}^4 & \cdot & {x_3}^3{x_1}^3 \\ \cdot & \cdot & {x_2}^4{x_1}^2 & x_2{x_3}^3{x_1}^2 + {x_2}^2{x_1}^2{x_3}^2 + {x_2}^3{x_1}^2x_3 \\ \cdot & \cdot & \cdot & {x_3}^4{x_1}^2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{{x_2}^2{x_1}^4} & -\frac{1}{x_2x_3{x_1}^4} & -\frac{1}{{x_2}^3{x_1}^3} & \frac{1}{x_2{x_3}^2{x_1}^3} + \frac{1}{{x_2}^2{x_1}^3x_3} \\ \cdot & \frac{1}{{x_3}^2{x_1}^4} & \cdot & -\frac{1}{{x_3}^3{x_1}^3} \\ \cdot & \cdot & \frac{1}{{x_2}^4{x_1}^2} & -\frac{1}{x_2{x_3}^3{x_1}^2} - \frac{1}{{x_2}^2{x_1}^2{x_3}^2} - \frac{1}{{x_2}^3{x_1}^2x_3} \\ \cdot & \cdot & \cdot & \frac{1}{{x_3}^4{x_1}^2} \end{bmatrix}$$

In summary, the preceding construction allows to understand sequences $t_1, t_2, t_3, \ldots$ of tableaux such that

$$\text{right key}(t_i) \leq \text{left key}(t_{i+1})$$

(that we call chains of tableaux).

Back to Schubert calculus:

Postulation of Schubert subvarieties of a flag manifold
= dimension of the space of sections of the powers of a line bundle L_λ over the Schubert variety $\mathfrak{S}_\sigma$

$$\Leftrightarrow\ (1 - zx^\lambda)^{-1} \mathcal{O}_{\mathfrak{S}_\sigma} \pi_\omega \Big|_{x_i=1} = (1 - zx^\lambda)^{-1} \pi_{\sigma^{-1}} \Big|_{x_i=1}$$

Combinatorially: enumerate chains of tableaux of shape λ, and count them.
In the case of Grassmannians, this problem was solved by Hodge (with the help of Littlewood for the determinantal formula giving the postulation number). One has to enumerate chains of partitions (with respect to inclusion of diagrams), or, equivalently plane partitions.
For the usual Plücker embedding, $\lambda = \rho = [n-1, \ldots, 1]$. In terms of Key polynomials, one takes a permutation v of ρ and one studies the function

$$1 + zK_v + z^2 K_{2v} + z^3 K_{3v} + \ldots \Big|_{x_i=1}$$

General fact: for the Schubert variety corresponding to σ, one obtains a rational function with denominator $(1 - z)^{\ell(\sigma)+1}$, and for numerator a positive polynomial E_σ, whose description remains to be written in terms of the tableauhedron.

For example, for $n = 4$, taking, up to inversion, all permutations which do not belong to a Young subgroup, one has the following E_σ:

[3, 1, 4, 2] : $1 + 8z + 3z^2$
[3, 4, 1, 2] : $1 + 25z + 44z^2 + 8z^3$
[4, 1, 2, 3] : $1 + 10z + 5z^2$
[4, 1, 3, 2] : $1 + 18z + 24z^2 + 3z^3$
[4, 2, 1, 3] : $1 + 19z + 25z^2 + 3z^3$
[4, 2, 3, 1] : $1 + 43z + 150z^2 + 81z^3 + 5z^4$
[4, 3, 1, 2] : $1 + 38z + 120z^2 + 58z^3 + 3z^4$
[4, 3, 2, 1] : $1 + 57z + 302z^2 + 302z^3 + 57z^4 + z^5$

There is a symmetry such that $(1 - zx^{\rho})^{-1}\pi_{\sigma}$ and $(1 - zx^{\rho})^{-1}\widehat{\pi}_{\sigma}$ have numerators reversed of each other.
Let us compute, for $n = 4$, $(1 - zx^{3210})^{-1}\,\widehat{\pi}_3\widehat{\pi}_2\widehat{\pi}_1$. There are 8 keys and 6 tableaux which are not keys:

$$t_1 = \begin{array}{|c|c|c|}\hline 3 & & \\ \hline 2 & 3 & \\ \hline 1 & 1 & 2 \\ \hline\end{array} \;,\quad t_2 = \begin{array}{|c|c|c|}\hline 4 & & \\ \hline 2 & 3 & \\ \hline 1 & 1 & 1 \\ \hline\end{array} \;,$$

$$t_3 = \begin{array}{|c|c|c|}\hline 4 & & \\ \hline 2 & 3 & \\ \hline 1 & 1 & 2 \\ \hline\end{array} \;,\quad t_4 = \begin{array}{|c|c|c|}\hline 4 & & \\ \hline 2 & 3 & \\ \hline 1 & 2 & 2 \\ \hline\end{array} \;,\quad t_5 = \begin{array}{|c|c|c|}\hline 4 & & \\ \hline 3 & 3 & \\ \hline 1 & 1 & 2 \\ \hline\end{array} \;,\quad t_6 = \begin{array}{|c|c|c|}\hline 4 & & \\ \hline 3 & 3 & \\ \hline 1 & 2 & 2 \\ \hline\end{array}$$

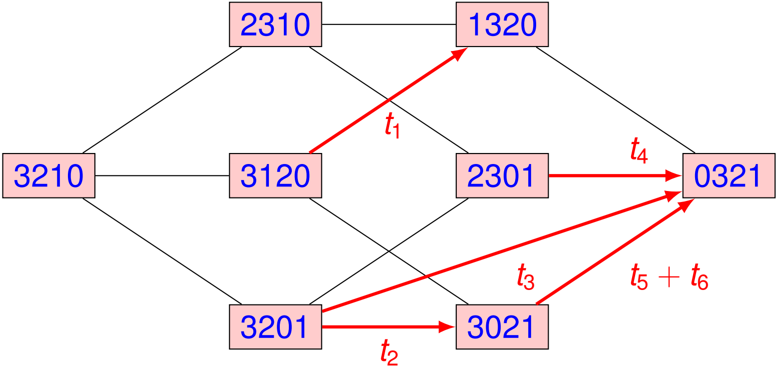
The numerator is

$$z(1 + zt_0 + z^2 t_0 t_0 + z^3 t_0 t_0 t_0 + \dots)\,\widehat{\pi}_3\widehat{\pi}_2\widehat{\pi}_1 (1 - z)^4 \Big|_{t_i = 1}$$

Explicitly, specializing the keys to 1, this numerator is

$$z(1 + t_3 + t_4 + t_5 + t_6) + z^2(4 + t_1 + t_2 - t_3 + t_4 + t_5 + t_6 + t_2 t_5 + t_2 t_6) + z^3$$

which specializes to $5z + 10z^2 + z^3$, but Euler is needed to eliminate the minus sign!



Here is the structure with which to make the preceding computation, with the Euler relation $2t_3 \sim t_2 t_5 + t_2 t_6$.