Convex geometric Demazure operators

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Convex polytopes in algebraic geometry and in representation theory

0. Toric geometry

Newton (or moment) polytopes

1. Representation theory

Gelfand–Zetlin polytopes and *string polytopes*
(Berenstein–Zelevinsky, Littelmann, 1998)

2. Algebraic geometry

Newton–Okounkov convex bodies
(Kaveh–Khovanskii, Lazarsfeld–Mustata, 2009)

1 & 2. Schubert calculus

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Flag varieties

Definition

The *flag variety* X is the variety of complete flags in $\mathbb{C}^n$:

$$X = \{\{0\} = V^0 \subset V^1 \subset \ldots \subset V^{n-1} \subset V^n = \mathbb{C}^n | \dim V^i = i\}$$

Remark

Alternatively, $X = GL_n(\mathbb{C})/B$, where B denotes the group of upper-triangular matrices (*Borel subgroup*). In this form, the definition can be extended to arbitrary connected reductive groups.

Dimension

$$\dim X = \frac{n(n-1)}{2}$$

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Flag varieties and representation theory

Definition

A collection of integers $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ is a *strictly dominant weight* of the group $GL_n(\mathbb{C})$ if $\lambda_i < \lambda_{i+1}$ for all $i = 1, \dots, n-1$.

Fact

very ample line bundles on $X \longleftrightarrow$ irreducible representations of $GL_n(\mathbb{C})$ with strictly dominant weights.

Construction

- V_λ — the irreducible GL_n-module with the highest weight λ
  $\Longrightarrow X \hookrightarrow \mathbb{P}(V_\lambda)$, $g \mapsto gv_\lambda$ — embedding;
- $\mathcal{L}$ — very ample line bundle
  $\Longrightarrow H^0(X, \mathcal{L})^* = V_\lambda$

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Representation theory and Gelfand–Zetlin polytopes

Gelfand–Zetlin polytope

For each strictly dominant weight λ, define a convex polytope $P_\lambda \subset \mathbb{R}^d$ (where $d = n(n-1)/2$) with integer vertices.

Origins

Gelfand and Zetlin constructed a natural basis in V_λ. The basis elements are parameterized by the integer points inside and at the boundary of P_λ.

Dimension

$$\dim P_\lambda = d = \dim X$$

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The Gelfand–Zetlin polytope P_λ is defined by inequalities:

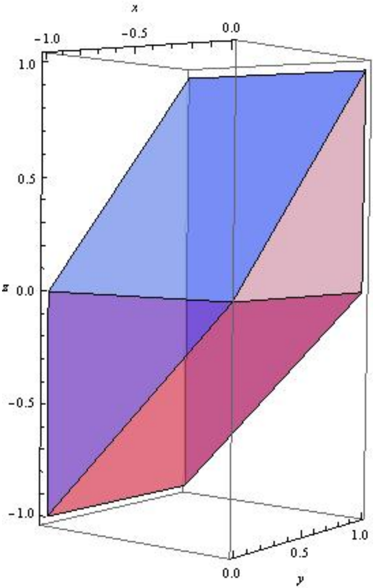
$$\begin{array}{ccccccccc}
\lambda_1 & & \lambda_2 & & \lambda_3 & & \ldots & & \lambda_n \\
& x_1^1 & & x_2^1 & & \ldots & & x_{n-1}^1 & \\
& & x_1^2 & & \ldots & & x_{n-2}^2 & & \\
& & & \ddots & \ldots & & & & \\
& & & x_1^{n-2} & & x_2^{n-2} & & & \\
& & & & x_1^{n-1} & & & &
\end{array}$$

where $(x_1^1, \ldots, x_{n-1}^1; \ldots; x_1^{n-1})$ are coordinates in $\mathbb{R}^d$, and the notation

$$\begin{array}{ccc}
a & & b \\
& c &
\end{array}$$

means $a \le c \le b$.

Gelfand–Zetlin polytopes



A Gelfand–Zetlin polytope for GL_3:

$$\begin{array}{ccccc} -1 & & 0 & & 1 \\ & x & & y & \\ & & z & & \end{array}$$

Flag varieties and Gelfand–Zetlin polytopes

Goal

Use combinatorics of P_λ to study geometry of X.

Results

- Relation between Schubert varieties and preimages of rc-faces of P_λ under the Guillemin–Sternberg moment map $X \to P_\lambda$ (Kogan, 2000)
- Degenerations of Schubert varieties to (reducible) toric varieties given by (unions of) faces of P_λ (Kogan–Miller, Knutson–Miller, 2003)
- Description of $H^*(X, \mathbb{Z})$ using *volume polynomial* of P_λ (Kaveh, 2003)
- Schubert calculus: intersection product of Schubert cycles in $H^*(X, \mathbb{Z})$ = intersection of faces in P_λ (K.–Smirnov–Timorin, 2011)

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Schubert calculus and Gelfand–Zetlin polytopes

Definition

For each permutation $w \in S_n$, the *Schubert variety* $X_w \subset X$ is

$$X_w = \overline{BwB},$$

where w acts on the standard basis vectors e_i by the formula $e_i \mapsto e_{w(i)}$.

Dimension

$\dim X_w = \ell(w)$

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The *Schubert cycle* $[X_w]$ is the class of X_w in $H^*(X, \mathbb{Z})$.

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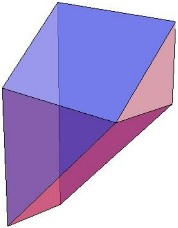
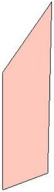
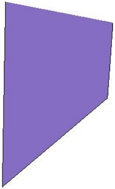
$[X_{s_1 s_2 s_1}]=$ 

$[X_{s_1 s_2}]=$ 

$[X_{s_2 s_1}]=$ 

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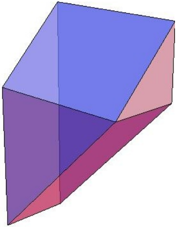
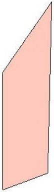
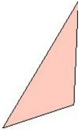
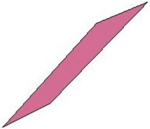
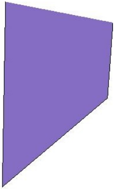
$[X_{s_1 s_2 s_1}] =$ 

$[X_{s_1 s_2}] =$  $=$  $+$ 

$[X_{s_2 s_1}] =$ 

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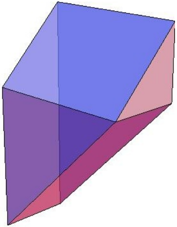
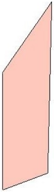
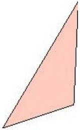
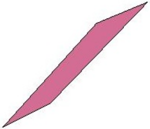
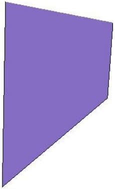
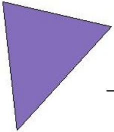
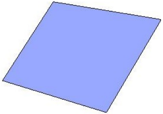
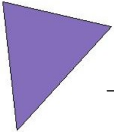
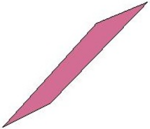
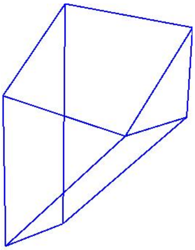
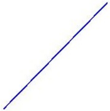
$[X_{s_1 s_2 s_1}] =$ 

$[X_{s_1 s_2}] =$  $=$  $+$ 

$[X_{s_2 s_1}] =$  $=$  $+$  $=$  $+$ 

Schubert calculus and Gelfand–Zetlin polytopes



$[X_{s_1}] =$ $=$  $[X_{s_2}] =$  $=$

 $= [X_{s_1}] + [X_{s_2}]$

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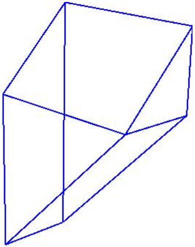
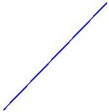
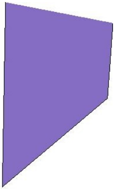
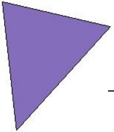
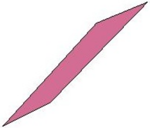
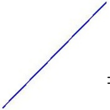


$[X_{s_1}] =$ $=$ 

$[X_{s_2}] =$  $=$

$[X_{s_2 s_1}]^2 =$  $\cdot \Big($  $+$  $\Big) =$  $= [X_{s_1}]$

Schubert calculus and Gelfand–Zetlin polytopes

GL_n

Any two Schubert cycles $[X_w]$ and $[X_{w'}]$ can be represented as sums of faces so that every face appearing in the decomposition of $[X_w]$ is transverse to every face appearing in the decomposition of $[X_{w'}]$[1].

Corollary

Intersection of any two Schubert cycles can be represented by linear combinations of faces with nonnegative coefficients.

Question

Why intersecting faces is better than multiplying Schubert polynomials?

[1] see arXiv:1101.0278v1 [math.AG] for precise formulas

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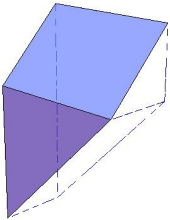
Answer

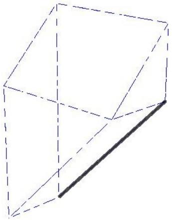
Faces are more "positive" than monomials: all computations with faces are cancelation free.

Example for GL_3

Compute $[X_{s_1}] \cdot [X_{s_2 s_1}]$ in two ways: via Schubert polynomials and via faces.

$$x_1x_2(x_1+x_2) = x_1^2x_2 - x_1x_2^2 = 1 - 1 = 0 \text{ (cancelation)}$$





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Schubert calculus and Gelfand–Zetlin polytopes

Answer

Faces are more "positive" than monomials: all computations with faces are cancelation free.

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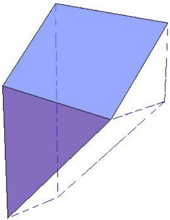

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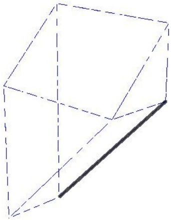

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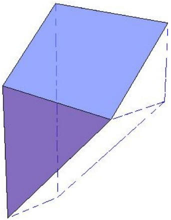

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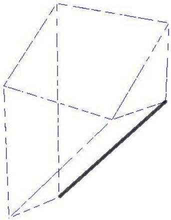

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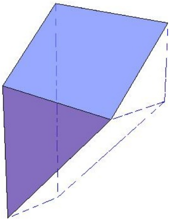

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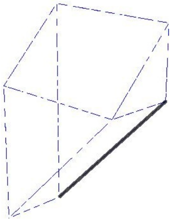

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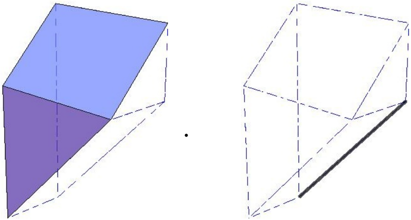

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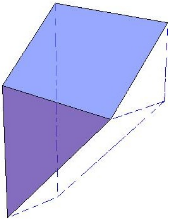

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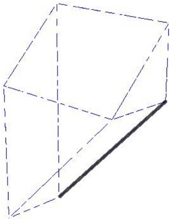

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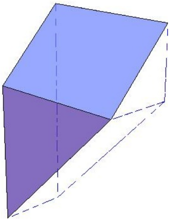

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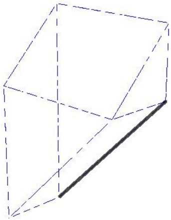

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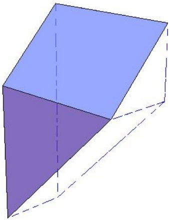

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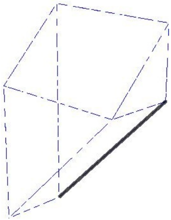

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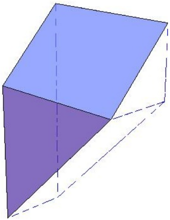

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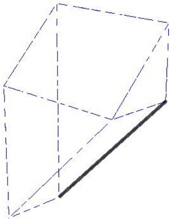

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Arbitrary reductive groups

How to relate Schubert cycles to (unions of) faces of polytopes?

Main tool

A convex geometric incarnation of divided difference operators.

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Tool: divided difference operators

Definition (for GL_n)

Divided difference operator δ_i (for $i = 1, \ldots, n-1$) acts on $\mathbb{Z}[x_1, \ldots, x_n]$ as follows:

$$\delta_i : f \mapsto \frac{f - s_i(f)}{x_i - x_{i+1}}.$$

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Divided difference operators in cohomology

Theorem (Bernstein–Gelfand–Gelfand, Demazure, 1971)

Let $w = s_{i_1} \ldots s_{i_\ell}$ be a reduced representation. In the Borel presentation,

$$[X_w] = \delta_{i_\ell} \ldots \delta_{i_1} [X_{id}],$$

where $[X_{id}]$ is the class of a point.

Remark

For GL_n,

$$[X_{id}] = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$

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Topological meaning of divided difference operators

Gysin morphism

Let P_i be the minimal parabolic subgroup, and $p_i : G/B \to G/P_i$ the natural projection. Then the action of δ_i on $H^*(G/B, \mathbb{Z})$ coincides with the action of $p_i^* \circ p_{i*}$:

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Generalizations of divided difference operators

Generalized cohomology theories

Let A^* be a generalized oriented cohomology theory. Define *generalized divided difference operator* δ_i^A as the composition

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- classical cohomology H^* or Chow ring CH^*
- K-theory K_0^*
- complex cobordism MU^* or algebraic cobordism Ω^*

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Is there an algebraic formula for δ_i^A?

Formal group law

There exists a formal power series $F_A(x, y) = x + y + \dots$ with coefficients in A^0 such that

$$F(c_1^A(L), c_1^A(M)) = c_1^A(L \otimes M)$$

in $A^*(X)$ for any pair of line bundles L and M on a variety X.

Examples

CH^* $F(x, y) = x + y$

K_0^* $F(x, y) = x + y - xy$

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Theorem

$$\delta_i^A = (1 + s_i)\frac{1}{x_i -_A x_{i+1}}$$

Applications

Formulas for "Schubert cycles" in the "Borel presentation" for $A^*(G/B)$. Algorithms for multiplying "Schubert cycles".

- H^* Bernstein–Gelfand–Gelfand, Demazure, 1973
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Demazure operators

Notation

Let G be a connected reductive group of semisimple rank r, Λ_G the weight lattice of G, and $\mathbb{Z}[\Lambda_G]$ the group ring. Simple roots of G are denoted by $\alpha_1, \ldots, \alpha_r$.

Remark

Elements of Λ_G are written in the form

$$\sum_{\mu \in \Lambda_G} m(\mu) e^{\mu}.$$

Definition

Demazure operator D_i (for $i = 1, \ldots, n$) acts on $\mathbb{Z}[\Lambda_G]$ as follows:

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$$D_1(e^{\alpha_2}) = \frac{e^{\alpha_2} - e^{\alpha_1} e^{\alpha_1+\alpha_2}}{1 - e^{\alpha_1}} = e^{\alpha_2} + e^{\alpha_1+\alpha_2}$$

Exercise

Define (λ, α_i) by the identity $s_i(\lambda) = \lambda - (\lambda, \alpha_i)\alpha_i$.

$$\begin{cases} D_i(e^\lambda) = e^\lambda(1 + e^{\alpha_i} + \ldots + e^{-(\lambda,\alpha_i)\alpha_i}), & (\lambda, \alpha_i) \leq 0 \\ D_i(e^\lambda) = 0, & (\lambda, \alpha_i) = 1 \\ D_i(e^\lambda) = -e^\lambda(1 + e^{-\alpha_i} + \ldots + e^{-((\lambda,\alpha_i)-2)\alpha_i}), & (\lambda, \alpha_i) > 1 \end{cases}$$

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$$D_1(e^{\alpha_2}) = \frac{e^{\alpha_2} - e^{\alpha_1}e^{\alpha_1+\alpha_2}}{1 - e^{\alpha_1}} = e^{\alpha_2} + e^{\alpha_1+\alpha_2}$$

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Define (λ, α_i) by the identity $s_i(\lambda) = \lambda - (\lambda, \alpha_i)\alpha_i$.

$$\begin{cases} D_i(e^{\lambda}) = e^{\lambda}(1 + e^{\alpha_i} + \ldots + e^{-(\lambda,\alpha_i)\alpha_i}), & (\lambda, \alpha_i) \leq 0 \\ D_i(e^{\lambda}) = 0, & (\lambda, \alpha_i) = 1 \\ D_i(e^{\lambda}) = -e^{\lambda}(1 + e^{-\alpha_i} + \ldots + e^{-((\lambda,\alpha_i)-2)\alpha_i}), & (\lambda, \alpha_i) > 1 \end{cases}$$

Demazure operators

Remark

For $G = GL_n$, put $x_i := 1 + e^{\chi_i}$ where the character χ_i is given by the i-th entry of the diagonal torus. Then

$$D_i = -\delta_i^K,$$

that is, the Demazure operator is equal up to a sign to the K-theory divided difference operator (= isobaric divided difference operator).

Demazure characters

Definition

Demazure B-module $V_{\lambda,w} := H^0(X_w, \mathcal{L}_\lambda|_{X_w})^*$ is the dual space to the space of global sections of the line bundle $\mathcal{L}_\lambda$ on G/B (corresponding to V_λ) restricted to X.

Definition

Demazure character $\chi_w(\lambda)$ of $V_{\lambda,w}$ is the sum over all basis weight vectors of the exponentials of the corresponding weights:

$$\chi_w(\lambda) := \sum_{\mu\in\Lambda} m_{\lambda,w}(\mu)e^\mu$$

Demazure character formula [Andersen, 1985, ...]

Let $w = s_{i_1} \dots s_{i_\ell}$ be a reduced representation. Then

$$\chi_w(\lambda) = D_{i_1} \dots D_{i_\ell} e^\lambda$$

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Examples

- $V_{\lambda,id} = \mathbb{C}_\lambda$, $\chi_{id}(\lambda) = e^\lambda$
- $V_{\lambda,w_0} = V_\lambda$, $\chi_{w_0}(\lambda)$ — Weyl character

Remark

For GL_n, the definition of Gelfand–Zetlin polytopes implies that

$$\chi_{w_0}(\lambda) = \sum_{x \in P_\lambda \cap \mathbb{Z}^d} e^{p(x)},$$

where $p(x) := (\sum_{j=1}^{n-1} x_j^1)\alpha_1 + (\sum_{j=1}^{n-2} x_j^1)\alpha_2 + \ldots + x_1^{n-1}\alpha_{n-1}$ is the weight of x.

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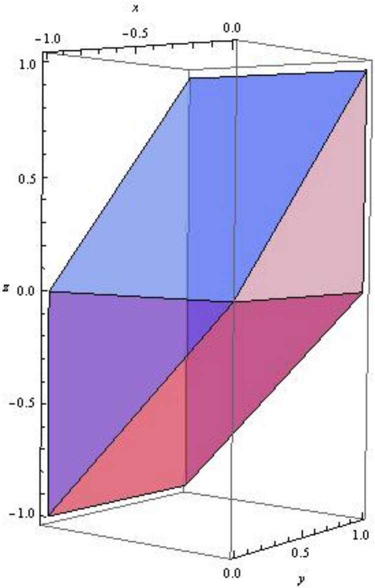
Demazure characters and Gelfand–Zetlin polytopes

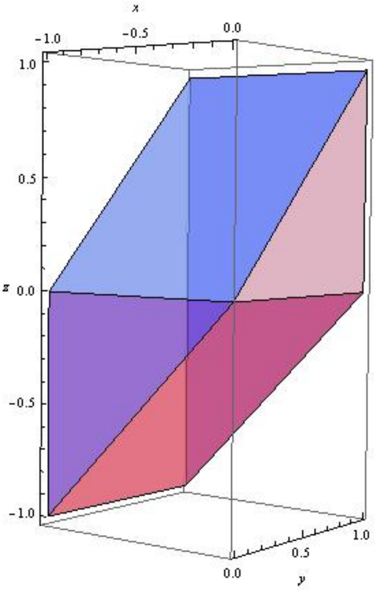


A Gelfand–Zetlin polytope for GL_3:

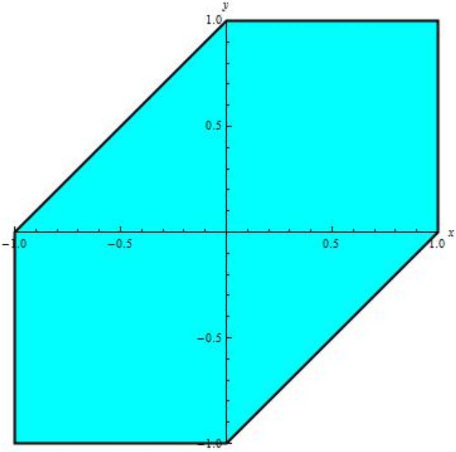
$$\begin{array}{ccccc} -1 & & 0 & & 1 \\ & x & & y & \\ & & z & & \end{array}$$

$$p(x, y, z) = (x + y, z)$$

Demazure characters and Gelfand–Zetlin polytopes



The weight polytope (=image of P_λ under the projection p):



Convex geometric Demazure operators

Goal

Define operators $D_1, \ldots, D_r$ on convex polytopes in $\mathbb{R}^d$ and a weight map $\mathbb{R}^d \to \mathbb{R}^r$ such that for any reduced decomposition $w = s_{i_1} \ldots s_{i_\ell}$ the sequence of polytopes

$$pt(\lambda) \xrightarrow{D_{i_1}} P_1(\lambda) \xrightarrow{D_{i_2}} P_2(\lambda) \xrightarrow{D_{i_3}} \ldots \xrightarrow{D_{i_\ell}} P_\ell(\lambda)$$

yields the sequence of the Demazure characters

$$e^\lambda \xrightarrow{D_{i_1}} \chi_{s_{i_1}}(\lambda) \xrightarrow{D_{i_2}} \chi_{s_{i_1} s_{i_2}}(\lambda) \xrightarrow{D_{i_3}} \ldots \xrightarrow{D_{i_\ell}} \chi_w(\lambda)$$

that is,

$$\chi_w(\lambda) = \sum_{x \in P_\ell(\lambda) \cap \mathbb{Z}^d} e^{p(x)}.$$

Convex geometric Demazure operators

Definition

A *root space* of *rank* r is a real vector space $\mathbb{R}^d$ together with a direct sum decomposition

$$\mathbb{R}^d = \mathbb{R}^{d_1} \oplus \ldots \oplus \mathbb{R}^{d_r}$$

and a collection of linear functions $l_1, \ldots, l_r \in (\mathbb{R}^d)^*$ such that l_i vanishes on $\mathbb{R}^{d_i}$.

Definition

A convex polytope $P \subset \mathbb{R}^d$ is called a *parapolytope* if for all $i = 1, \ldots, r$, and any vector $c \in \mathbb{R}^d$ the intersection of P with the parallel translate $c + \mathbb{R}^{d_i}$ of $\mathbb{R}^{d_i}$ is a *coordinate parallelepiped*.

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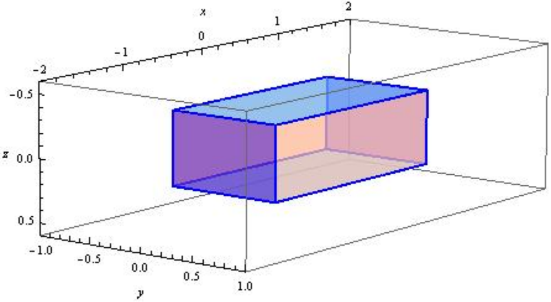


A coordinate parallelepiped in $\mathbb{R}^3$

Notation

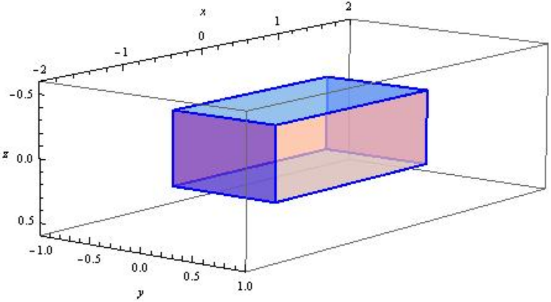
Coordinates in $\mathbb{R}^d$: $(x_1^1, \ldots, x_{d_1}^1; \ldots; x_1^n, \ldots, x_{d_n}^n)$.

Definition

A *coordinate parallelepiped* in $\mathbb{R}^{d_i}$ is

$$\Pi(\mu, \nu) = \{(x_1^i, \ldots, x_{d_i}^i) \in \mathbb{R}^{d_i} | \mu_j \leq x_j^i \leq \nu_j, j = 1, \ldots, d_i\}.$$

Convex geometric Demazure operators



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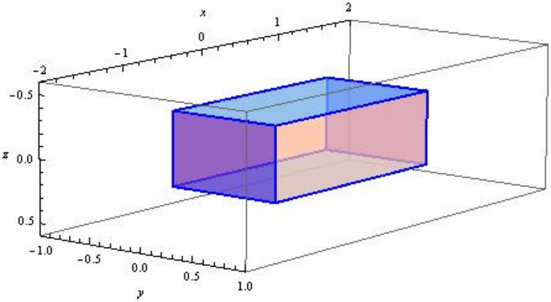

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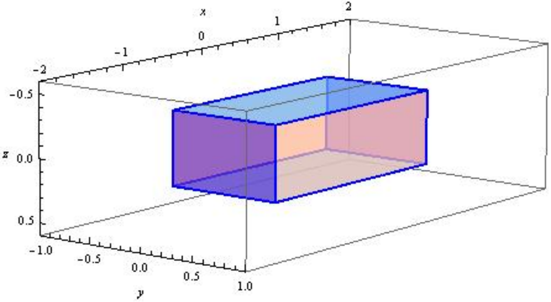

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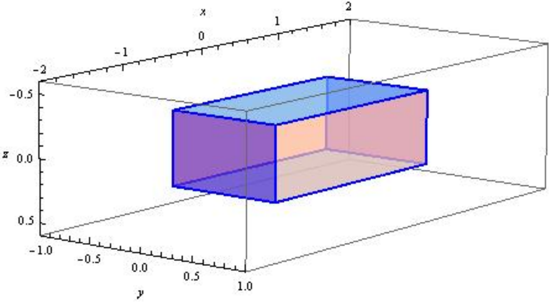

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Convex geometric Demazure operators

Example

$d = \frac{n(n-1)}{2}$, $r = (n-1)$

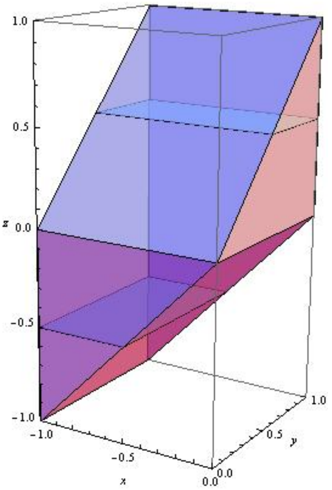
$$\mathbb{R}^d = \mathbb{R}^{n-1} \oplus \mathbb{R}^{n-2} \oplus \ldots \oplus \mathbb{R}^1$$

Exercise

The Gelfand–Zetlin polytope P_λ is a parapolytope.

$$\begin{array}{ccccccccc}
\lambda_1 & & \lambda_2 & & \lambda_3 & & \ldots & & \lambda_n \\
& x_1^1 & & x_2^1 & & \ldots & & x_{n-1}^1 & \\
& & x_1^2 & & \ldots & & x_{n-2}^2 & & \\
& & & \ddots & \ldots & & & & \\
& & & x_1^{n-2} & & x_2^{n-2} & & & \\
& & & & x_1^{n-1} & & & &
\end{array}$$

Convex geometric Demazure operators



$\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$

The slices $\{z = \frac{1}{2}\}$ and $\{z = -\frac{1}{2}\}$ are coordinate rectangles.

Convex geometric Demazure operators

Definition

1. $P = \Pi(\mu, \nu) \subset (c + \mathbb{R}^{d_i})$ — coordinate parallelepiped
   Choose the smallest $j = 1, \ldots, d_i$ such that $\mu_j = \nu_j$. Put

$$D_i(P) := \Pi(\mu, \nu'),$$

   where $\nu'_k = \nu_k$ for all $k \neq j$ and ν'_j is defined by the equality

$$\sum_{k=1}^{d_i} (\mu_k + \nu'_k) = l_i(c).$$

2. P — any parapolytope

$$D_i(P) = \bigcup_{c \in \mathbb{R}^d} \{D_i(P \cap (c + \mathbb{R}^{d_i}))\}$$

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Examples

$\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$, $l_1(x, y, z) = z$, $l_2(x, y, z) = x + y$

$P = \{(a, b, c)\}$ — a point

Convex geometric Demazure operators

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$P = \{(a, b, c)\}$ — a point

$$D_1(P) = [(a, b, c), (a', b, c)],$$

where a' is defined by the equality

$$a + b + a' + b = l_1(a, b, c),$$

that is,

$$a' = c - a - 2b$$

Convex geometric Demazure operators

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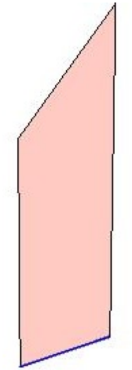
$P = [(a,b,c),(a',b,c)]$ — segment



$$D_2(P) = \bigcup_{x\in[a,a']} [(x,b,c),(x,b,c'(x))],$$

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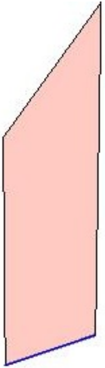

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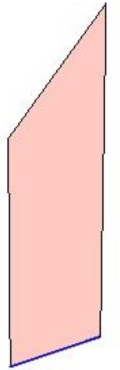

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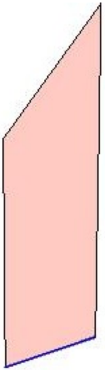

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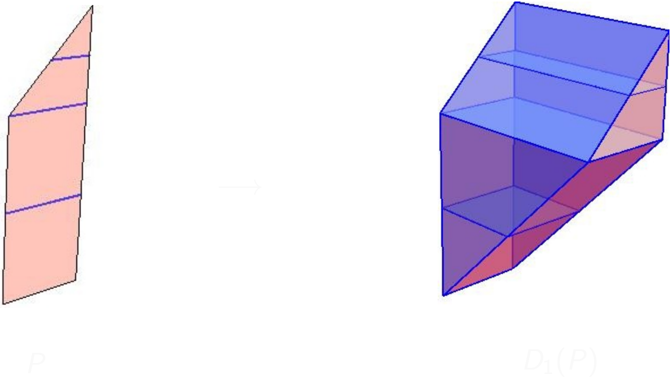
$P = D_2 D_1(\{(\lambda_1, \lambda_2, \lambda_1)\})$ — trapezoid



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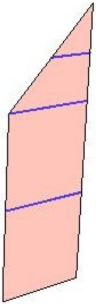
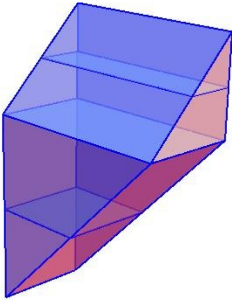


P

$\longrightarrow$



$D_1(P)$

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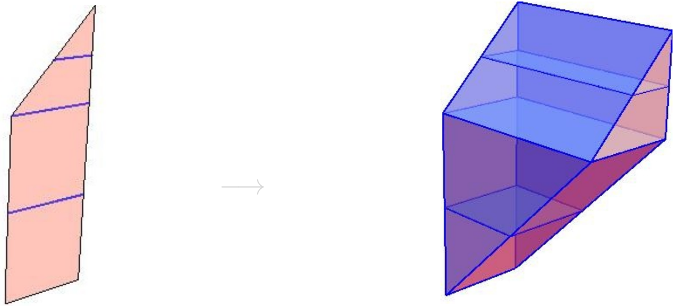

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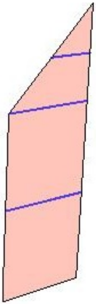

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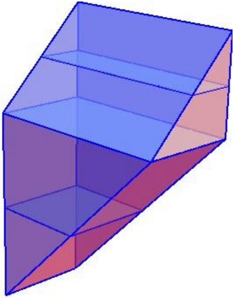

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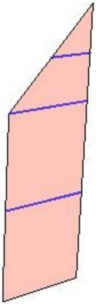
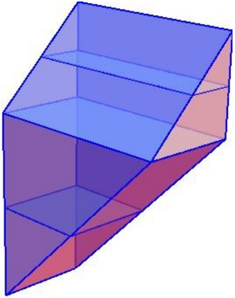
$P = D_2 D_1(\{(\lambda_1, \lambda_2, \lambda_1)\})$ — trapezoid



P

$\longrightarrow$



$D_1(P)$

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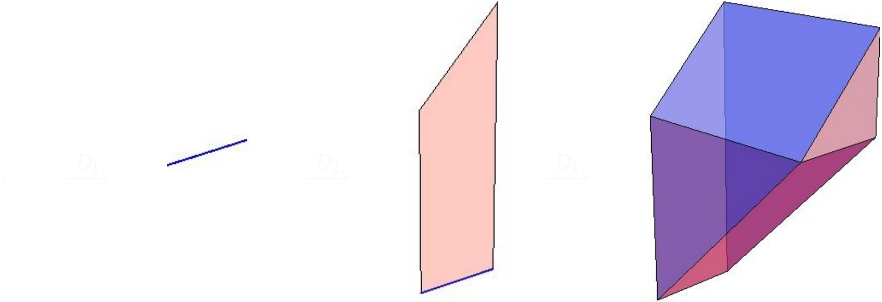
Exercise

$D_1 D_2 D_1(\{(\lambda_1, \lambda_2, \lambda_1)\})$ — Gelfand–Zetlin polytope for $\lambda = (\lambda_1, \lambda_2, -\lambda_1 - \lambda_2)$.



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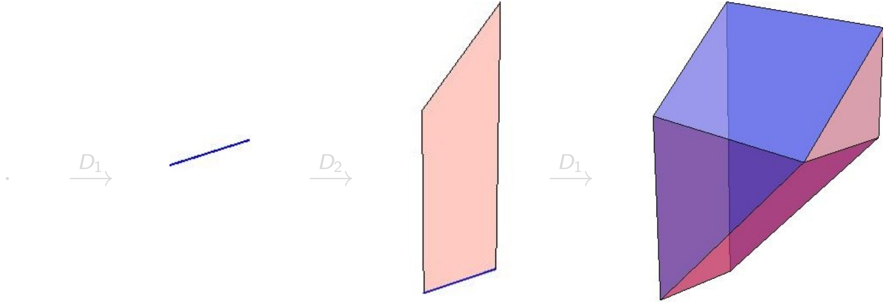

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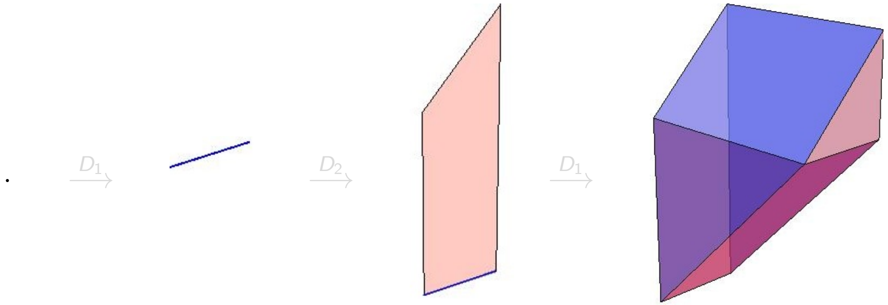

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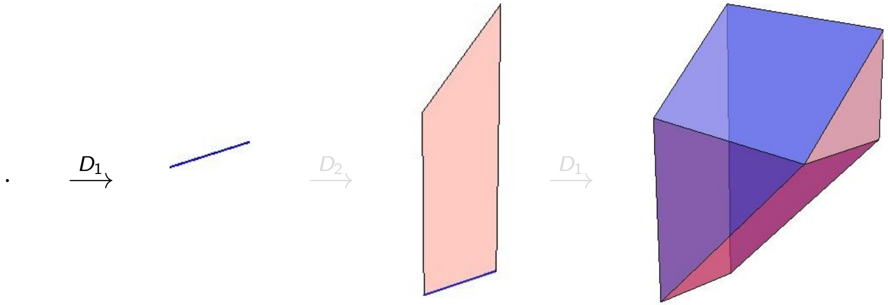

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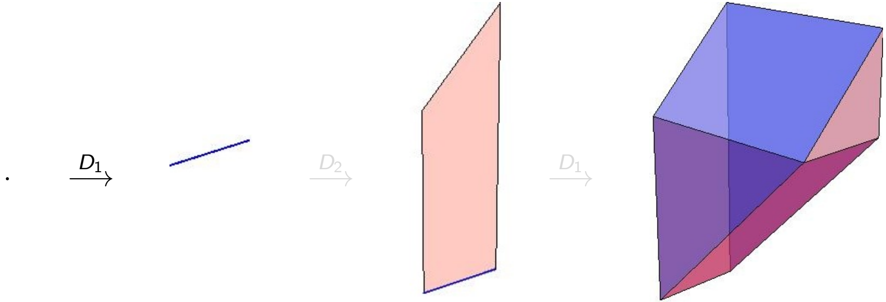

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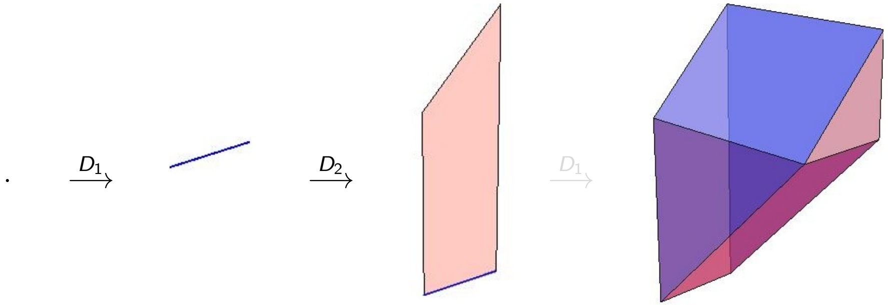

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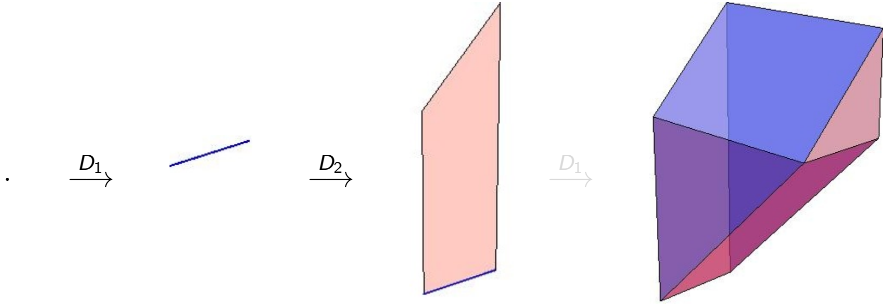

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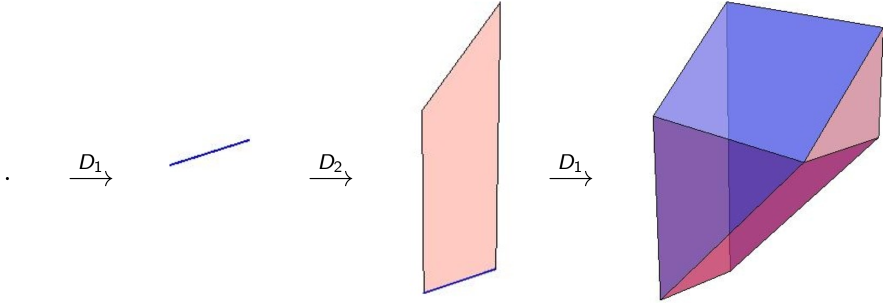

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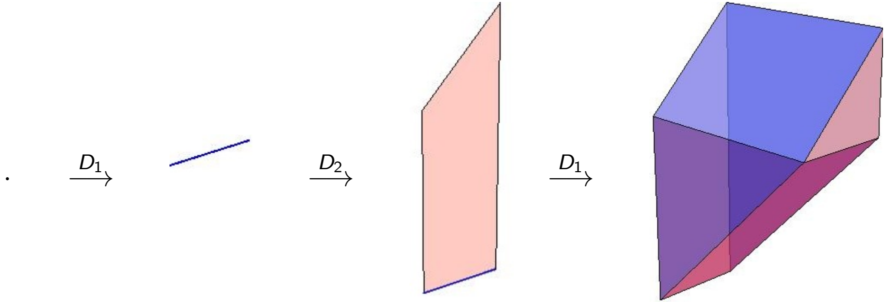

Convex geometric Demazure operators

GL_n root space

$\mathbb{R}^d = \mathbb{R}^{n-1} \oplus \mathbb{R}^{n-2} \oplus \ldots \oplus \mathbb{R}^1$

Functions l_i

$$l_i(x) = \sigma_{i-1}(x) + \sigma_{i+1}(x),$$

where $\sigma_i(x) = \sum_{j=1}^{d_i} x_j^i$ (=sum of coordinates in the i-th row) for $i = 1, \ldots, n-1$ and $\sigma_0 = \sigma_n = 0$

$$\begin{array}{ccccccc}
x_1^1 & & x_2^1 & & \ldots & & x_{n-1}^1 \\
& x_1^2 & & \ldots & & x_{n-2}^2 & \\
& & \ddots & \ldots & & & \\
& & x_1^{n-2} & & x_2^{n-2} & & \\
& & & x_1^{n-1} & & &
\end{array}$$

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\end{array}$$

Convex geometric Demazure operators

Proposition

The Gelfand–Zetlin polytope P_λ coincides with

$$[(D_1 \dots D_{n-1})(D_1 \dots D_{n-2}) \dots (D_1)]\,(p),$$

where $p \in \mathbb{R}^d$ is the point $(\lambda_1, \dots, \lambda_{n-1}; \lambda_1, \dots, \lambda_{n-2}; \dots ; \lambda_1)$.

Convex geometric Demazure operators

$\overline{w}_0$

Fix a reduced decomposition $w_0 = s_{i_1} \dots s_{i_d}$ of the longest element in the Weyl group of G.

$(G, \overline{w}_0)$ root space

$$\mathbb{R}^d = \mathbb{R}^{d_1} \oplus \dots \oplus \mathbb{R}^{d_r},$$

where d_i is the number of s_{i_j} in $\overline{w}_0$ such that $i_j = i$.

$$l_i(x) = \sum_{k \neq i} (\alpha_k, \alpha_i) \sigma_k(x).$$

Example

For $G = GL_n$ and $w_0 = (s_1 \dots s_{n-1})(s_1 \dots s_{n-2}) \dots (s_1)$, we get GL_n root space.

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Convex geometric Demazure operators

Theorem

For each dominant weight λ of G, there exists a point $p_\lambda \in \mathbb{R}^d$ such that the polytope

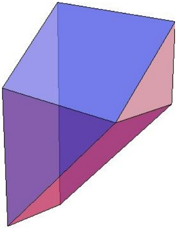
$$P := D_{i_1} \dots D_{i_d}(p_\lambda)$$

yields the Weyl character $\chi(V_\lambda)$ of the irreducible G-module V_λ, namely,

$$\chi(V_\lambda) = \sum_{x \in P \cap \mathbb{Z}^d} e^{p(x)}.$$

Geometric mitosis

$D_1 D_2 D_1(p) \quad =$



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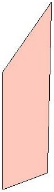

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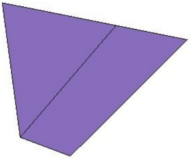

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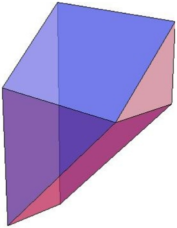

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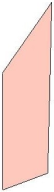

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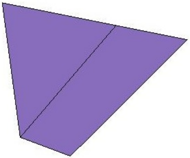

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