PBW bases and KLR algebras

Syu Kato

syuchan@math.kyoto-u.ac.jp

Kyoto Univ.

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Quivers

$\Bbbk$: a field ($\mathbb{F}_q$ or $\mathbb{C}$)
Let $\Gamma = (I, \Omega)$ be a quiver, meaning
I : the set of vertex, Ω : the set of oriented arrows
For each $h \in \Omega$, we have $h', h'' \in I$ so that:

$$h' \xrightarrow{h} h''$$

h' : source of h, h'' : targets of h

$\Rightarrow \Gamma_0$ underlying unoriented graph of Γ

$Q^+ := \sum_{i \in I} \mathbb{Z}_{\geq 0} i \subset Q := \bigoplus_{i \in I} \mathbb{Z} i$: root lattice of type Γ_0
For $\beta \in Q$ and $i \in I$, we have $\beta_i \in \mathbb{Z}$ so that $\beta = \sum_{i \in I} \beta_i i \in Q$

$$G_\beta := \prod_{i \in I} GL(\beta_i, \Bbbk) \circlearrowright E_\beta^\Omega := \bigoplus_{h \in \Omega} \mathrm{Hom}(\Bbbk^{\beta_{h'}}, \Bbbk^{\beta_{h''}})$$

is well-defined for $\beta \in Q^+$.

Representation theory of quivers

Theorem (Gabriel)

The path algebra $\Bbbk\Gamma$ *of* Γ *satisfies:*

1. *The set*
$$\mathcal{M}(\Gamma) := \sqcup_{\beta \in Q^+} G_\beta \backslash E_\beta^\Omega$$
is the set of isomorphism classes of the representations of $\Bbbk\Gamma$*;*
2. *All* $G_\beta \backslash E_\beta^\Omega$ *are finite iff* Γ_0 *is a Dynkin diagram of type* ADE*;*
3. *If 2) holds, then* $\mathcal{M}(\Gamma)$ *is in bijection with the basis of* $U^+(\mathfrak{g}_{\Gamma_0})$*, where* $\mathfrak{g}_{\Gamma_0}$ *is the simple Lie algebra of type* Γ_0*.*

For $\beta, \beta' \in Q^+$, we have a natural correspondence:

$$E_\beta^\Gamma \times E_{\beta'}^\Gamma \stackrel{q}{\longleftarrow} Z_{\beta,\beta'}^\Gamma \stackrel{p}{\longrightarrow} E_{\beta+\beta'}^\Gamma.$$

For each $\beta \in Q^+$, we define

$$H_\beta^\Gamma := \{G_\beta - \text{invariant functions on } E_\beta^\Gamma\}.$$

A realization of $U_v(\mathfrak{g}_\Gamma)$

If $\Bbbk = \mathbb{F}_q$ with $q < \infty$,

$$E^\Gamma_\beta \times E^\Gamma_{\beta'} \xleftarrow{q} Z^\Gamma_{\beta,\beta'} \xrightarrow{p} E^\Gamma_{\beta+\beta'}$$

defines a map

$$\star : H^\Gamma_\beta \times H^\Gamma_{\beta'} \ni (f,g) \mapsto f \star g \in H^\Gamma_{\beta+\beta'},$$

where $(f \star g)(x) := \frac{1}{|G_\beta||G_{\beta'}|} \sum_{y \in p^{-1}(x)} q^*(f,g)(y)$.

Theorem (Ringel-Green)

1. *$\star$ defines $H^\Gamma := \oplus_{\beta \in Q^+} H^\Gamma_\beta$ an associative algebra structure;*
2. *If Γ_0 is of type ADE, then $U^+_v(\mathfrak{g}_{\Gamma_0}) \cong H^\Gamma$, where $U^+_v(\mathfrak{g}_{\Gamma_0})$ is the quantized enveloping algebra and $v = \sqrt{q}$;*
3. *In general, we have a natural inclusion $U^+_v(\mathfrak{g}_{\Gamma_0}) \hookrightarrow H^\Gamma$.*

A geometric realization of $U_v(\mathfrak{g}_\Gamma)$

By function-sheaf dictionary, we replace

$$H_\beta^\Gamma \mapsto \mathcal{P}_{G_\beta}(E_\beta^\Gamma) \mapsto K(\mathcal{P}_{G_\beta}(E_\beta^\Gamma)),$$

where $\mathcal{P}_{G_\beta}(E_\beta^\Gamma)$ is the G_β-equivariant category of pure sheaves of weight 0. We have natural actions:

$$\mathbb{Z} \circlearrowright \mathcal{P}_{G_\beta}(E_\beta^\Gamma) \circlearrowleft \mathbb{D}.$$

so that $1 \mapsto [1](1/2)$ ($\mathbb{D}$ is the Verdier duality).

Theorem (Lusztig)

Assume that Γ_0 is of type ADE.

1. *For each $\beta \in Q^+$, we have $K(\mathcal{P}_{G_\beta}(E_\beta^\Gamma)) \cong H_\beta^\Gamma$*
2. *$\star$ has a geometric counter-part and $\bigoplus_\beta K(\mathcal{P}_{G_\beta}(E_\beta^\Gamma)) \cong U_v^+(\mathfrak{g}_{\Gamma_0})$ with $v = [1](1/2)$ for $\mathbb{k} = \mathbb{C}$.*

Positivity of the PBW basis (for adapted words)

In the below, we assume that Γ_0 is of type ADE.
$B_\beta := G_\beta \backslash E_\beta^\Gamma$: parametrizing set of G_β-orbits
$b \in B_\beta \Rightarrow \mathbb{O}_b \subset E_\beta^\Gamma$: the corresponding orbit
For each $b \in B_\beta$, we have
IC_b : IC complex from $\mathbb{O}_b$, and $(j_b)_!$: extension-by-zero sheaf.

$$\{[\mathrm{IC}_b]\}_b \text{ and } \{[(j_b)_!]\}_b \text{ are two } \mathbb{Z}[v^{\pm 1}]\text{-bases of } K(\mathcal{P}_{G_\beta}(E_\beta^\Gamma))$$

Theorem (Lusztig)

1. $\{[\mathrm{IC}_b]\}_b$ *is the canonical basis, and* $\{[(j_b)_!]\}_b$ *is a PBW basis of* $U_v^+(\mathfrak{g}_{\Gamma_0})$*;*
2. *If we write* $[\mathrm{IC}_b] = \sum_{b'} c_{b,b'}(v)[(j_{b'})_!]$ *in* $K(\mathcal{P}_{G_\beta}(E_\beta^\Gamma))$*, then* $c_{b,b'}(v)$ *is the graded dimension of the stalk of* IC_b *along* $\mathbb{O}_{b'}$*;*
3. *In particular,* $c_{b,b'}(v)$ *is positive, and* $c_{b,b'} = 0$ *for* $b' \not\leq b$*, where* $\leq$ *is the closure ordering.*

Lusztig's conjecture

$W := W(\Gamma_0)$: the Weyl group of $\mathfrak{g}_{\Gamma_0} \circlearrowright Q$
PBW bases are bases determined by a reduced expression of the longest element $w_0 \in W$

$$w_0 = s_{i_1} s_{i_2} \cdots s_{i_\ell} \Rightarrow \mathbf{i} = (i_1, i_2, \ldots, i_\ell)$$

In fact, starting from simple root vectors $\{E_i\}_{i \in I}$, we define a PBW basis element

$$E^{\mathbf{i}}_{\mathbf{c}} := E_{i_1}^{(c_1)} (T_{i_1} E_{i_2}^{(c_2)}) (T_{i_1} T_{i_2} E_{i_3}^{(c_3)}) \cdots,$$

where $\mathbf{c} \in \mathbb{Z}^{\ell}_{\geq 0}$, $X^{(c)}$ is the divided power, and T_i is the Lusztig action
$\{E^{\mathbf{i}}_{\mathbf{c}}\}_{\mathbf{c}}$ is a basis of $U_v^+(\mathfrak{g}_{\Gamma_0})$

Conjecture (Lusztig)

For each reduced expression $\mathbf{i}$ *of* w_0*, the expansion coefficients of* $[\mathrm{IC}_b]$ *in terms of* $E^{\mathbf{i}}_{\mathbf{c}}$ *belongs to* $\mathbb{N}[v]$*.*

KLR algebras

The variety E^{Γ}_{β} has a "natural resolution" F^{Γ}_{β} (which is not connected!)
$\mathbb{D}$-selfdual constant sheaf $\dot{\mathcal{L}}$ on F^{Γ}_{β} ($\xrightarrow{\pi} E^{\Gamma}_{\beta}$) yields

$$\mathcal{L} := \pi_* \dot{\mathcal{L}} \in \mathcal{P}_{G_\beta}(E^{\Gamma}_{\beta}).$$

Definition (Khovanov-Lauda, Rouquier, Varagnolo-Vasserot)

We set the KLR-algebra of type Γ_0 as:

$$R^{\Gamma}_{\beta} := \bigoplus_{k \in \mathbb{Z}} \mathrm{Ext}^{k}_{D^{b}_{G_\beta}(E^{\Gamma}_{\beta})}(\mathcal{L}, \mathcal{L}).$$

This is a graded algebra which depends only on Γ_0. (In the below, we drop Γ from R^{Γ}_{β}.)

Induction functors between KLR-algebras

R_β-gmod : category of f.g. graded modules

For $\beta, \beta' \in Q^+$, algebraic presentation gives an inclusion

$$R_\beta \boxtimes R_{\beta'} \subset R_{\beta+\beta'} \qquad \text{RHS is free over LHS,}$$

which yields an induction functor

$$\odot : R_\beta\text{-gmod} \times R_{\beta'}\text{-gmod} \longrightarrow R_{\beta+\beta'}\text{-gmod}.$$

This is an exact functor which satisfies the Frobenius reciprocity

$$\mathrm{ext}^i_{R_{\beta+\beta'}}(M \odot N, L) \cong \mathrm{ext}^i_{R_\beta \boxtimes R_{\beta'}}(M \boxtimes N, L) \qquad \text{for every } i, M, N, L.$$

Here ext^i denotes the i-th graded extension, which is a $\mathbb{Z}$-graded vector space.

KLR algebras categorify $U_v^+(\mathfrak{g}_{\Gamma_0})$

For each $i \in I$ and $m \geq 0$, $\exists! P_{mi}$ graded projective R_{mi}-module and $\exists! L_{mi}$ simple R_{mi}-module up to grading shifts.
R_β^Γ-gproj $\subset R_\beta^\Gamma$-gmod : category of f.g. graded projective modules.

Theorem (Khovanov-Lauda, Rouquier, Varagnolo-Vasserot)

1. *The map*

$$\odot : K(R_\beta\text{-gproj}) \times K(R_{\beta'}\text{-gproj}) \longrightarrow K(R_{\beta+\beta'}\text{-gproj})$$

equips $\bigoplus_{\beta \in Q^+} K(R_\beta\text{-gproj})$ *an associative algebra structure;*

2. *We have an isomorphism* $U_v^+(\mathfrak{g}_{\Gamma_0}) \cong K(R_\beta\text{-gproj})$, *where v denotes the grading shift by one;*
3. *The canonical basis of* $U_v^+(\mathfrak{g}_{\Gamma_0})$ *corresponds to the class of an indecomposable projective module (up to a grading shift);*
4. *The upper global basis of* $U_v^+(\mathfrak{g}_{\Gamma_0})^*$ *corresponds to the class of a simple module (up to a grading shift);*

KLR-version of Lusztig's conjecture

For $\mathcal{F} \in \mathcal{P}_{G_\beta}(E_\beta^\Gamma)$, we have

$$\bigoplus_{k \in \mathbb{Z}} \mathrm{Ext}^k_{D^b_{G_\beta}(E^\Gamma_\beta)}(\mathcal{F}, \mathcal{L}) \in R_\beta\text{-gmod}.$$

Then, we have

$\mathrm{IC}_b \mapsto P_b$: indecomposable projective module
$(j_b)_! \mapsto \widetilde{E}_b$: "PBW" type module

Lusztig's positivity result turns into

$$[P_b : \widetilde{E}_{b'}] \in \mathbb{N}[v].$$

But, what is $[P_b : \widetilde{E}_{b'}]$, and why they are positive?

Standard modules and orthogonality property

- Each $\mathbb{O}_b$ has a connected stabilizer G_b;
- The Borho-MacPherson argument guarantees "triangularity" of character expansions of $\{\widetilde{E}_b\}_b$ into simples;
- $(j_b)_!$ admits a successive dist. triangle by $\mathrm{IC}_{b'}$;
- As vector spaces, Ext in dg-category of $(R^{\Gamma}_{\beta}, d = 0)$ and module category is the same (Bernstein-Lunts).

Theorem (arXiv:1207.4640)

There exists a collection of R_β-module $\{E_b\}_b$:

- *Each $\widetilde{E}_b$ is a successive extension of E_b;*
- *We have an orthogonality relation*

$$\mathrm{ext}^i_{R_\beta}(\widetilde{E}_b, E^*_{b'}) = \begin{cases} \mathbb{C} & (i = 0, b = b') \\ \{0\} & (\textit{otherwise}) \end{cases}$$

Abstract Brauer-Humphreys reciprocity

For $b \in B_\beta \Rightarrow L_b$: self-dual simple, and P_b : its projective cover
It is rephrased as

$$\mathrm{ext}^i_{R_\beta}(P_b, L^*_{b'}) = \mathrm{ext}^i_{R_\beta}(P_b, L_{b'}) = \begin{cases} \mathbb{C} & (i = 0, b = b') \\ \{0\} & (\text{otherwise}) \end{cases}.$$

Compared with

$$\mathrm{ext}^i_{R_\beta}(\widetilde{E}_b, E^*_{b'}) = \begin{cases} \mathbb{C} & (i = 0, b = b') \\ \{0\} & (\text{otherwise}) \end{cases},$$

Theorem (arXiv:1207.4640)

We have the equality of graded character expansion coefficients:

$$[P_b : \widetilde{E}_{b'}] = [E_{b'} : L_b].$$

Kashiwara's problem

On the other hand,

- succ. dist. tri. of $(j_b)_!$ by $\mathrm{IC}_{b'}$ is finite
  ($\Leftarrow$ category of sheaves have finite global dimension!)
- above argument says that E_b admits a finite projective resolution
- $\dim E_b < \infty$ (this follows from the actual definition...sorry).

Theorem (Kashiwara's problem, arXiv:1203.5254)

Every KLR-algebra corresponding to Dynkin diagram of type ADE *has finite global dimension.*

Remark

This was a problem posed by Prof. Kashiwara several times, including informal lecture at Kyoto in 2009 and intensive lecture at Tokyo Dec/2011.

Idea: Reminder

- Lusztig : positivity of [canonical:PBW] follows by the (geometric) Hall-alegbra interpretation;
- KLR-interpretation : In that case, it is also a consequence of aBH-reciprocity

$$[P_b : \widetilde{E}_{b'}] = [E_{b'} : L_b] \quad \text{RHS is positive.}$$

⇒ If we can categorify the construction of PBW bases for arbitrary **i** so that aBH holds, then we deduce Lusztig's conjecture.

⇒ Base cases are common with the above. Remains to categorify T_i (Lusztig action) nicely

BGP reflection functor

$i \in I$ is called sink/source of Ω iff i is not a target/source of any edge in Ω

In that case, $s_i\Omega$ is the set of vertices with arrow concerning i reversed

For a sink $i \in I$ of Ω and $\beta \in Q^+ \cap s_i Q^+$, we have open subsets

$$ {}_iE^{\Omega}_{\beta} \subset E^{\Omega}_{\beta}, \text{ and } {}^iE^{s_i\Omega}_{s_i\beta} \subset E^{s_i\Omega}_{s_i\beta} $$

In addition, we have an isomorphism as stacks:

$$ G_\beta \backslash {}_iE^{\Omega}_{\beta} \cong G_{s_i\beta} \backslash {}^iE^{s_i\Omega}_{s_i\beta}. $$

In particular, this induces maps $R_\beta \to {}_iR^{\Omega}_{\beta}$ and $R_{s_i\beta} \to {}^iR^{s_i\Omega}_{s_i\beta}$ so that:

Lemma

$$ {}_iR^{\Omega}_{\beta}\text{-gmod} \cong {}^iR^{s_i\Omega}_{s_i\beta}\text{-gmod}. $$

Crystal invariants ϵ_i, ϵ_i^*

Proposition (arXiv:1203.5254,1207.4640)

1. *For any Ω, the maps $R_\beta \to {}_iR_\beta^\Omega$ and $R_{s_i\beta} \to {}^iR_\beta^{s_i\Omega}$ are surjective;*
2. *The kernel is spanned by simple modules L_b with $\epsilon_i(b) = 0$ and $\epsilon_i^*(b) = 0$, respectively;*
3. *The invariants ϵ_i, ϵ_i^* of $B(\infty) = \sqcup_\beta B_\beta$ are crystal invariants, and hence does not depend on Ω (but is easy only when i is a sink/source);*
4. *In particular, the quotients $R_\beta \to {}_iR_\beta^\Omega$ and $R_{s_i\beta} \to {}^iR_{s_i\beta}^{s_i\Omega}$ does not depend on the choice of Ω.*

$\Rightarrow$ we drop Ω and $s_i\Omega$ from ${}_iR_\beta^\Omega$ and ${}^iR_{s_i\beta}^{s_i\Omega}$, and simply write ${}_iR_\beta$ and ${}^iR_{s_i\beta}$.

Saito reflection functors

For each $i \in I$, we define the functor (which is acutually defined as a composition of three functors)

$$\mathbb{T}_i : R_\beta\text{-gmod} \to {}^i R_\beta\text{-gmod} \cong {}_i R_{s_i\beta}\text{-gmod} \to R_{s_i\beta}\text{-gmod},$$

where the first functor is ${}^i R_\beta \otimes_{R_\beta} \bullet$.
Similarly, we define

$$\mathbb{T}_i^* : R_\beta\text{-gmod} \to {}_i R_\beta\text{-gmod} \cong {}^i R_{s_i\beta}\text{-gmod} \to R_{s_i\beta}\text{-gmod},$$

where the first functor is $\hom_{R_\beta}({}_i R_\beta, \bullet)$.

By construction, these functor sends simples to simples/zero.
$\mathbb{T}_i$ is right exact, while $\mathbb{T}_i^*$ is left exact.

Definition of the PBW bases

Let $\mathbf{i}$ be a reduced expression of w_0. For each $\mathbf{c} = (c_1, c_2, \cdots) \in \mathbb{Z}^{\ell}_{\geq 0}$, we define

$$\widetilde{E}^{\mathbf{i}}_{\mathbf{c}} := P_{c_1 i_1} \odot \mathbb{T}_{i_1} P_{c_2 i_2} \odot \mathbb{T}_{i_1} \mathbb{T}_{i_2} P_{c_3 i_3} \odot \cdots$$
$$E^{\mathbf{i}}_{\mathbf{c}} := L_{c_1 i_1} \odot \mathbb{T}_{i_1} L_{c_2 i_2} \odot \mathbb{T}_{i_1} \mathbb{T}_{i_2} L_{c_3 i_3} \odot \cdots .$$

These are called lower/upper PBW modules of $\oplus_{\beta \in Q^+} R_\beta$. For this, we have:

Theorem (arXiv:1203.5254)

1. *The module $\widetilde{E}^{\mathbf{i}}_{\mathbf{c}}$ is a succesive extension of $E^{\mathbf{i}}_{\mathbf{c}}$;*
2. *We have an orthogonality relation*

$$\mathrm{ext}^i_{R_\beta}(\widetilde{E}^{\mathbf{i}}_{\mathbf{c}}, (E^{\mathbf{i}}_{\mathbf{c}'})^*) = \begin{cases} \mathbb{C} & (i = 0, \mathbf{c} = \mathbf{c}') \\ \{0\} & (\textit{otherwise}) \end{cases}$$

Main Theorem

Theorem (Lusztig's conjecture, arXiv:1203.5254)

For every reduced expression $\mathbf{i}$ *of* $w_0 \in W$*, we have*

$$[P_b : \widetilde{E}^{\mathbf{i}}_{\mathbf{c}}] = [E^{\mathbf{i}}_{\mathbf{c}} : L_b].$$

In particular, the expansion coefficient of the lower global basis in terms of (lower) PBW basis belongs to $\mathbb{N}[v]$*.*

Remark

- *For "adapted"* $\mathbf{i}$*, this is due to Lusztig;*
- *Lusztig raised this as a question in his comment on his paper at his homepage (but it was widely recognized as a problem);*
- *If* $\mathbf{i}$ *is adapted,* P_b *admits a filtration whose associated graded is* $\{\widetilde{E}^{\mathbf{i}}_{\mathbf{c}}\}_{\mathbf{c}}$*.*

How the proof is going on?

1. From our characterization of ${}_iR$ and ${}^iR$, we have

$$\mathrm{ext}^k(\mathbb{T}_i M, N) \cong \mathrm{ext}^k(M, \mathbb{T}_i^* N).$$

2. From "induction theorem" imported from affine Hecke algebras,

$$\mathbb{T}_i M \odot \mathbb{T}_i N \cong \mathbb{T}_i (M \odot N).$$

3. Crystal-theoretic analysis says $M \boxtimes N \subset M \odot N$ contains "maximal" ϵ_i-part if all irreducible constituent of N has $\epsilon_i = 0$;
4. If we have $c_1 = c_1'$, we have

$$\mathrm{ext}(\widetilde{E}^{\mathbf{i}}_{\mathbf{c}}, E^{\mathbf{i}}_{\mathbf{c}'}) \cong \mathrm{ext}(P_{c_1 i_1} \odot \mathbb{T}_{i_1} E^{\mathbf{i}^\sharp}_{\mathbf{c}^\sharp}, E^{\mathbf{i}}_{\mathbf{c}'}) \cong \mathrm{ext}(P_{c_1 i_1} \boxtimes \mathbb{T}_{i_1} E^{\mathbf{i}^\sharp}_{\mathbf{c}^\sharp}, L_{c_1 i_1} \boxtimes \mathbb{T}_{i_1} E^{\mathbf{i}^\sharp}_{(\mathbf{c}')^\sharp})$$
$$\cong \mathrm{ext}(E^{\mathbf{i}^\sharp}_{\mathbf{c}^\sharp}, \mathbb{T}_{i_1}^* \mathbb{T}_{i_1} E^{\mathbf{i}^\sharp}_{(\mathbf{c}')^\sharp}) \cong \mathrm{ext}(E^{\mathbf{i}^\sharp}_{\mathbf{c}^\sharp}, E^{\mathbf{i}^\sharp}_{(\mathbf{c}')^\sharp}).$$

Previous results

There are several relevant constructions which are present in the literature:

- Kleshchev-Ram classified finite-dimensional modules of R_β in terms of cuspidal modules, which are our $\mathbb{T}_w L^{(c)}$ for w (from special $\mathbf{i}$);
- Benkart-Kang-Oh-Park constructed similar modules for non-simply laced cases, but also for a special choice of $\mathbf{i}$;
- Webster also discusses such kind of modules for arbitrary simply-laced Kac-Moody algebra with a special string.

As far as we know, they are concerned with modules with simple head, but not ext-orthogonality properties.

An instance

- Each $\mathbb{T}_{i_1} \cdots \mathbb{T}_{i_{k-1}} L_{ci_k}$ is simple;
- Each $\mathbb{T}_{i_1} \cdots \mathbb{T}_{i_{k-1}} P_{ci_k}$ is a maximal self-extension of $\mathbb{T}_{i_1} \cdots \mathbb{T}_{i_{k-1}} L_{ci_k}$;

If Γ_0 is of type A_2, one series of PBW basis is of the form

$$\widetilde{E}_1^{(c_1)} \cdot \widetilde{E}_{21}^{(c_2)} \cdot \widetilde{E}_2^{(c_3)},$$

where $\widetilde{E}_{21} = \widetilde{E}_{21}^{(1)}$ is given as:

$$0 \to \widetilde{E}_1 \cdot \widetilde{E}_2 \langle 2 \rangle \to \widetilde{E}_2 \cdot \widetilde{E}_1 \to \mathbb{T}_2 \widetilde{E}_1 \to 0.$$

This is a categorical version of

$$[\widetilde{E}_{21}] = [\widetilde{E}_2] \star [\widetilde{E}_1] - q[\widetilde{E}_1] \star [\widetilde{E}_2],$$

which satisfies the above requirement.