The Puzzle conjecture for 2-step flag varieties

Anders Buch*, Andrew Kresch, Kevin Purbhoo, Harry Tamvakis.

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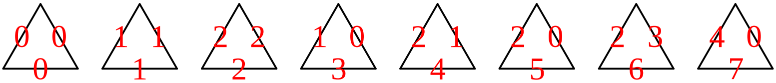

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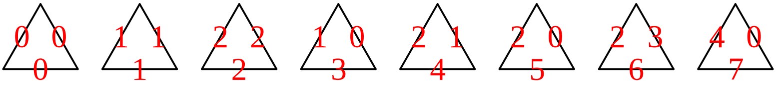

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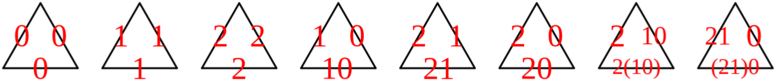
Interpretation of labels as **decreasing trees of integers**:

Simple labels:
0 1 2

Composed labels:
3 = 10 4 = 21, 5 = 20, 6 = 2(10), and 7 = (21)0



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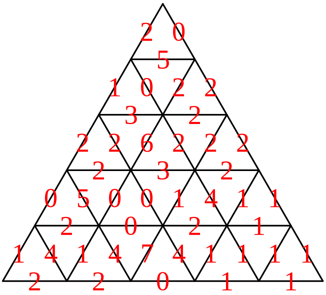

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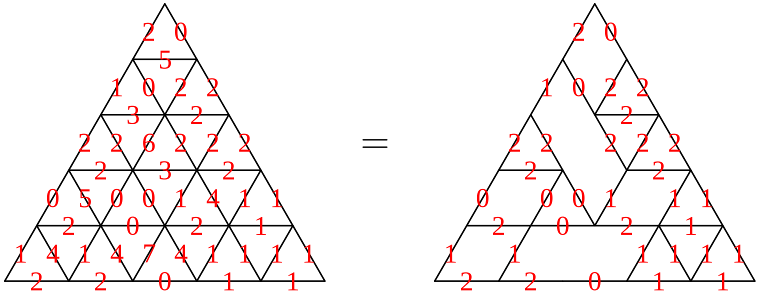


Note: The composed labels are uniquely determined by simple labels.

Two-step flag variety: Let $0 < a < b < n$.

$X = \mathrm{Fl}(a,b;n) = \{(A,B) \mid A \subset B \subset \mathbb{C}^n \;;\; \dim(A) = a \;;\; \dim(B) = b\}$

Def: A **012-string** for X is a permutation of $\mathbf{0} := 0^a\, 1^{b-a}\, 2^{n-b}$.

E.g. $u = 10212$ is a 012-string for $\mathrm{Fl}(1,3;5)$.

$\mathbb{C}^n$ has basis $\{e_1, e_2, \dots, e_n\}$. $u = (u_1, u_2, \dots, u_n)$ 012-string.

Set $A_u = \mathrm{Span}\{e_i : u_i = 0\}$ and $B_u = \mathrm{Span}\{e_i : u_i \leq 1\}$.

$\mathbf{B} \subset \mathrm{GL}(\mathbb{C}^n)$ lower triangular matrices.

Schubert variety: $X_u = \overline{\mathbf{B}.(A_u, B_u)}$

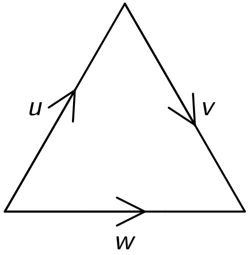
$\mathrm{codim}(X_u; X) = \ell(u) = \#\{i < j \mid u_i > u_j\}$

Schubert structure constants:

$H^*(X) = \bigoplus_u \mathbb{Z}[X_u] \quad ; \quad [X_u] \cdot [X_v] = \sum_w c_{u,v}^w [X_w]$

Conjecture (Knutson) / **Theorem** (BKPT) :

$c^{w}_{u,v} = \#$ puzzles with border labels u, v, w :



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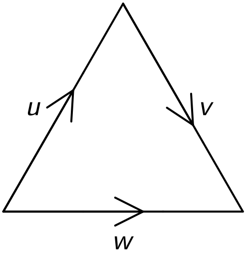
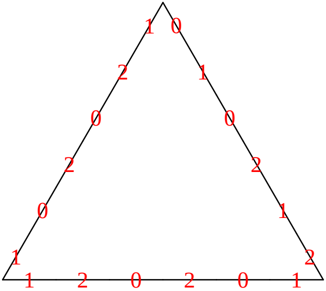


Example: $u = 102021$, $v = 010212$, $w = 120201$: $c_{u,v}^{w} =$?



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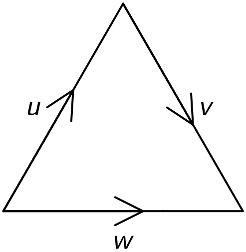
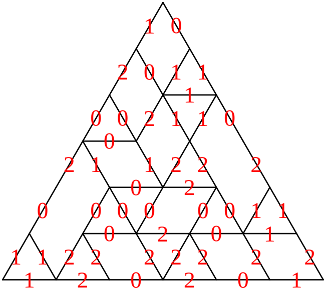
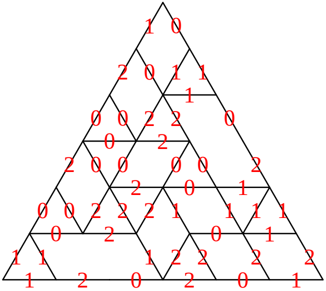


Example: $u = 102021$, $v = 010212$, $w = 120201$: $c_{u,v}^{w} = 2$





1999: Knutson circulated puzzle conjecture for all partial flag varieties $SL(n)/P = Fl(a_1, a_2, \ldots, a_m; n)$.

Shortly after: Knutson found counter example for $Fl(1, 2, 3, 4; 5)$.

2001: Knutson, Tao, Woodward proved puzzle rule for $Gr(m, n)$.

2001: Knutson and Tao proved generalization for $H_T^*(Gr(m, n))$.

2002: Buch, Kresch, Tamvakis: All (3-point, genus zero) Gromov-Witten invariants of degree d on $Gr(m, n)$ are equal to Schubert structure constants $c_{u,v}^w$ of $Fl(m-d, m+d; n)$.

Suggested that conjecture is true for two-step flag varieties.

Verified conjecture for all $Fl(a, b; n)$ with $n \leq 16$.

2007: Coskun proved different LR rule for $Fl(a, b; n)$ using Mondrian tableaux.

2010: Knutson and Purbhoo proved that special case of Knutson's original conjecture for $SL(n)/P$ computes Belkale-Kumar coefficients.

Exercise:

Let R be an associative ring with unit 1.
Let $S \subset R$ be a subset that generates R as a $\mathbb{Z}$-algebra.
Let M be a left R-module.
Let $\mu : R \times M \to M$ be any $\mathbb{Z}$-bilinear map.
Assume that for all $r \in R$, $s \in S$, and $m \in M$ we have

(1) $\mu(1, m) = m$ and

(2) $\mu(rs, m) = \mu(r, sm)$.

Then $\mu(r, m) = rm$ for all $r \in R$ and $m \in M$.

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Application:

Def: $C_{u,v}^{w} = \#$ puzzles with border labels u, v, w.

Def: $\mu : H^*(X) \times H^*(X) \to H^*(X)$ by

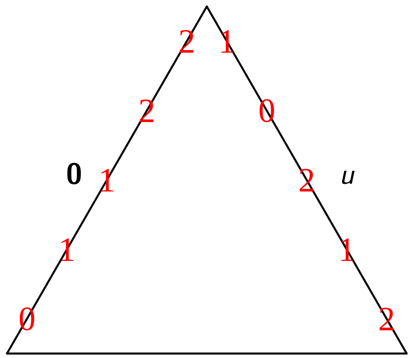
$$\mu([X_u], [X_v]) = \sum_w C_{u,v}^{w} \, [X_w]$$

Enough to show: $\mu([X_u], [X_v]) = [X_u] \cdot [X_v]$

Multiplication by 1

$\mathbf{0} = 0^a\, 1^{b-a}\, 2^{n-b} = \underbrace{0000}_{a}\underbrace{11111}_{b-a}\underbrace{2222}_{n-b}$; $[X_{\mathbf{0}}] = 1 \in H^*(X)$

Claim: $\mu(1, [X_u]) = [X_u]$



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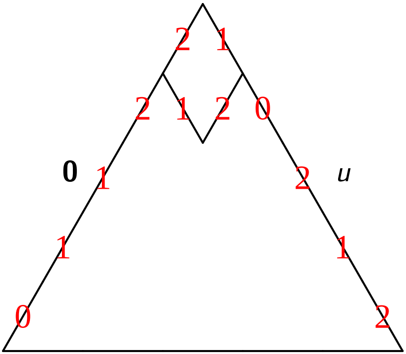

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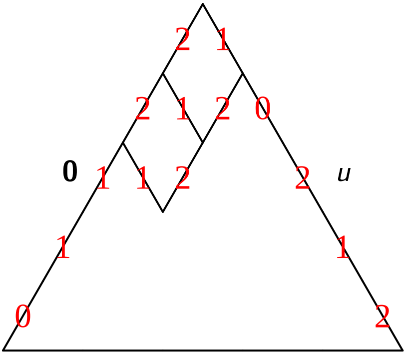

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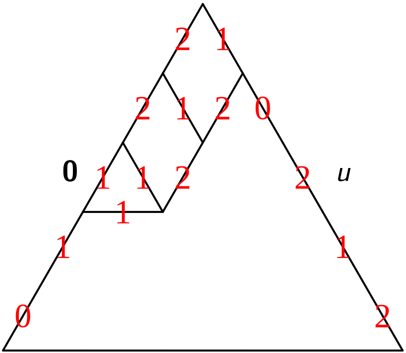

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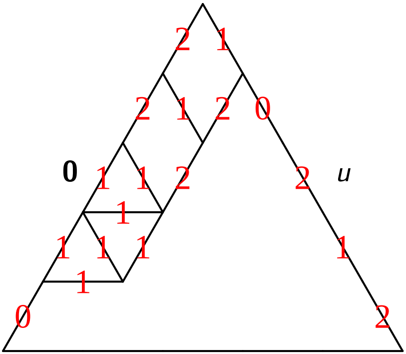

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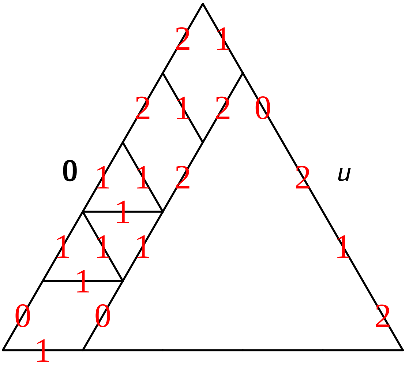

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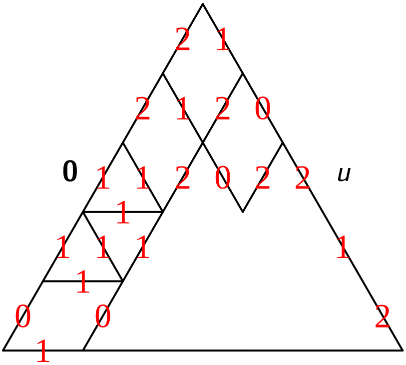

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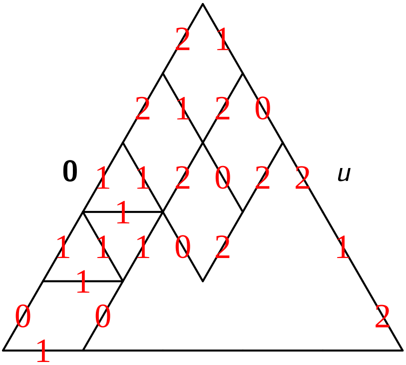

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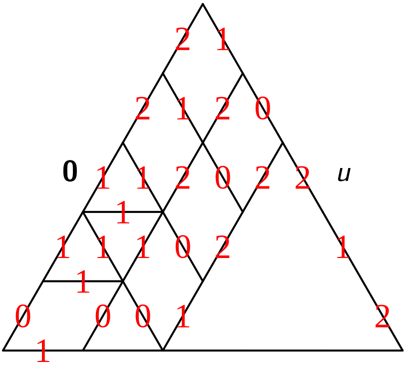

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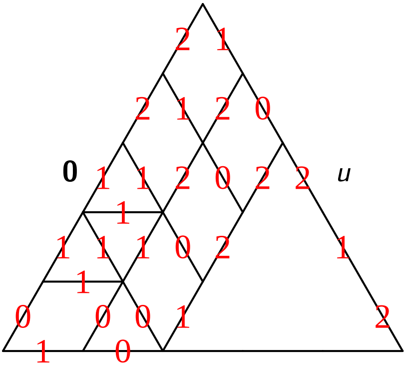

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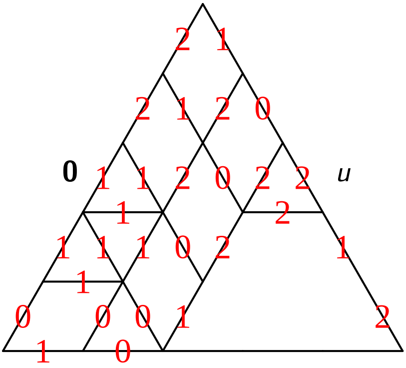

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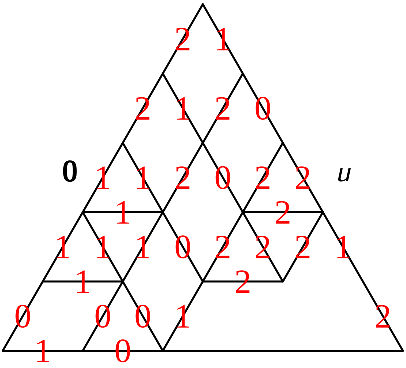

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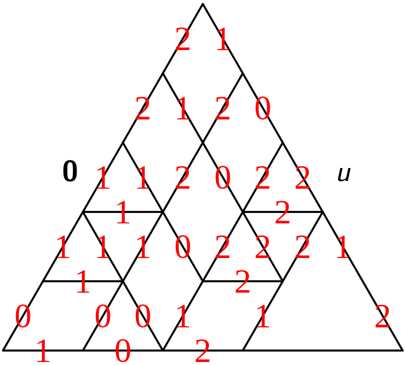

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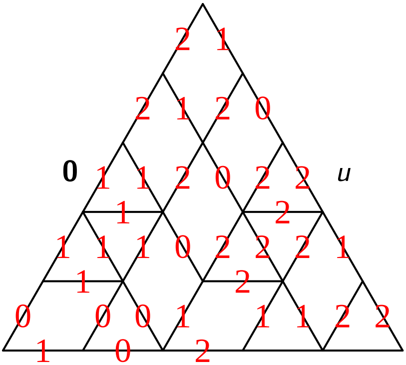

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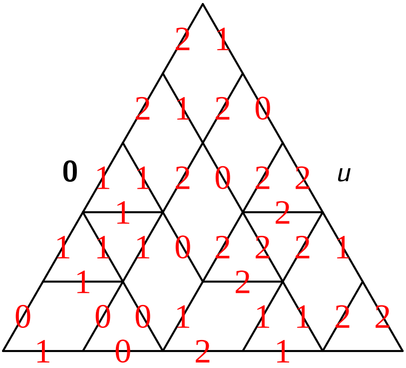

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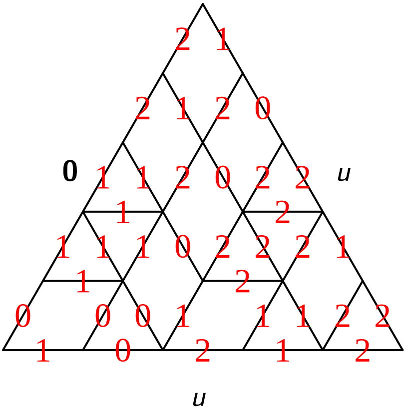

Pieri rule

Let u and u' be 012-strings.

Def: Write $u \xrightarrow{1} u'$ if u' is obtained from u by a substitution

$$02 \mapsto 20 \qquad \text{or} \qquad 100\ldots02 \mapsto 200\ldots01$$

Def: $u \xrightarrow{1} u'$ has **index** (i, j) if $i < j$ and $u_i \neq u'_i$ and $u_j \neq u'_j$.

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Let $r \in \mathbb{N}$.

Def: Write $u \xrightarrow{r} u'$ if $\exists\ u = u^0 \xrightarrow{1} u^1 \xrightarrow{1} \cdots \xrightarrow{1} u^r = u'$ such that if $u^{t-1} \xrightarrow{1} u^t$ has index (i_t, j_t) then $j_{t-1} \leq i_t$ for each t.

Example: $12021022 \xrightarrow{3} 22011202$ because:

$$12021022 \xrightarrow{1}$$
$$21021022 \xrightarrow{1}$$
$$22011022 \xrightarrow{1}$$
$$22011202$$

Pieri rule: $X = \mathrm{Fl}(a, b; n)$

Given $r \in [0, n-b]$, identify $r = \underbrace{00000}_{a}\underbrace{1111}_{b-a-1}\underbrace{222}_{r}\mathbf{1}\underbrace{222}_{n-b-r}$

Given $p \in [0, a]$, define $\widetilde{p} = \underbrace{000}_{a-p}\mathbf{1}\underbrace{000}_{p}\underbrace{1111}_{b-a-1}\underbrace{22222}_{n-b}$

Special Schubert classes:

$[X_r] = c_r(\mathcal{B}/\mathbb{C}^n_X)$ and $[X_{\widetilde{p}}] = c_p(\mathcal{A}^\vee)$ where $\mathcal{A} \subset \mathcal{B} \subset \mathbb{C}^n_X = \mathbb{C}^n \times X$ tautological flag on X.

$H^*(X)$ is generated by $S = \{[X_1], [X_2], \ldots, [X_{n-b}], [X_{\widetilde{1}}], [X_{\widetilde{2}}], \ldots, [X_{\widetilde{a}}]\}$

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Theorem: (Lascoux–Schützenberger 1982, Sottile 1996):

$$[X_r] \cdot [X_u] = \sum_{u \xrightarrow{r} u'} [X_{u'}]$$

Similar formula for $[X_{\widetilde{p}}] \cdot [X_u]$

Must show: For each $[X_r] \in S$ and 012-strings u and v, we have

$$\mu([X_u] \cdot [X_r]\ ,\ [X_v]) \ = \ \mu([X_u]\ ,\ [X_r] \cdot [X_v])$$

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$$\Leftrightarrow \quad \sum_{u \xrightarrow{r} u'} \mu([X_{u'}], [X_v]) \ = \ \sum_{v \xrightarrow{r} v'} \mu([X_u], [X_{v'}])$$

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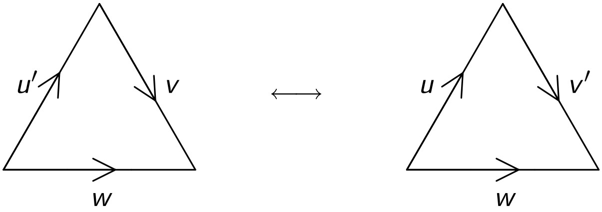
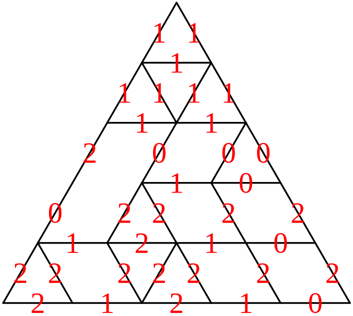
TODO: Given 012-strings u, v, w, enough to construct bijection between puzzles with border u', v, w such that $u \xrightarrow{r} u'$, and puzzles with border u, v', w such that $v \xrightarrow{r} v'$.



Easiest case: Assume $r = 1$.

$u =$ 20121 , $v =$ 11022.



$\longleftrightarrow$?

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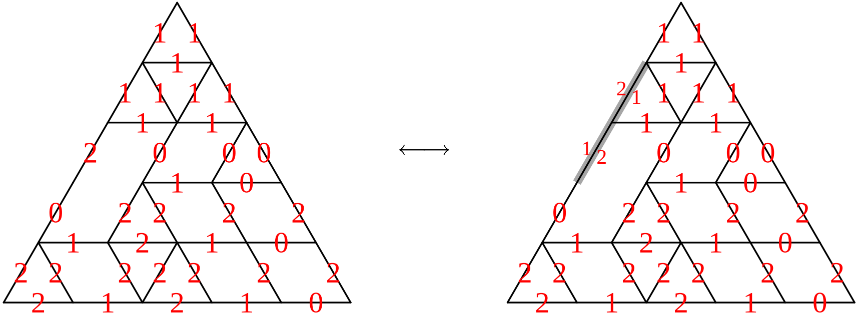

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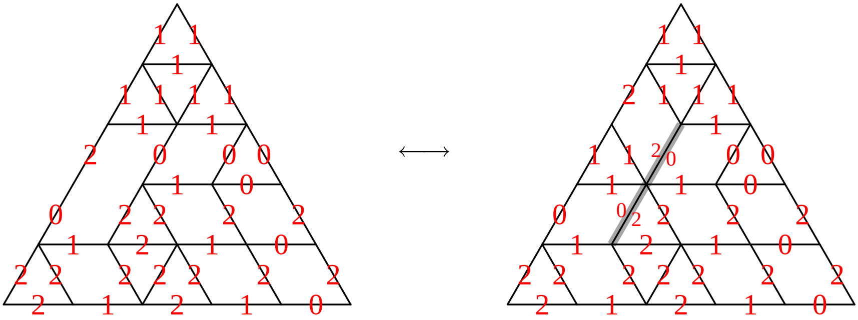

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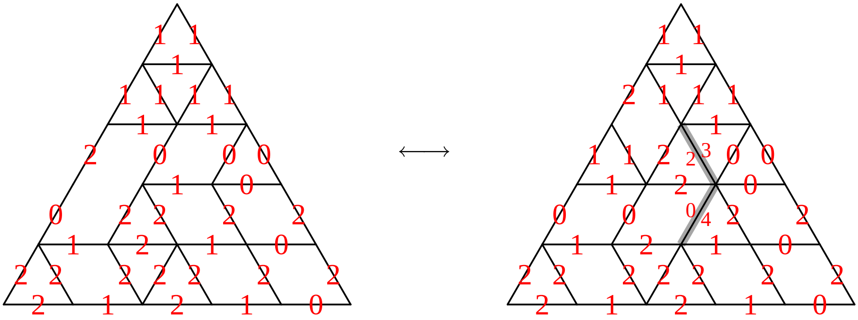

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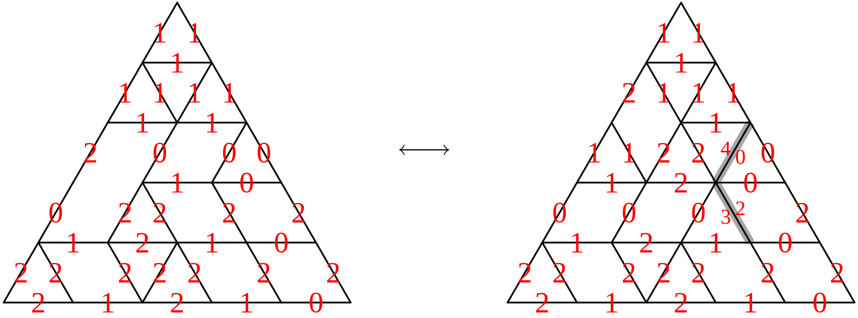
$u = 20121$, $v = 11022$.



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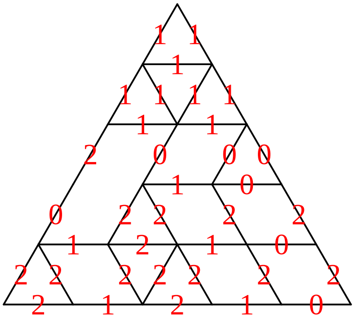
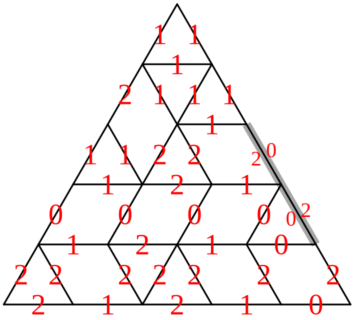


$\longleftrightarrow$



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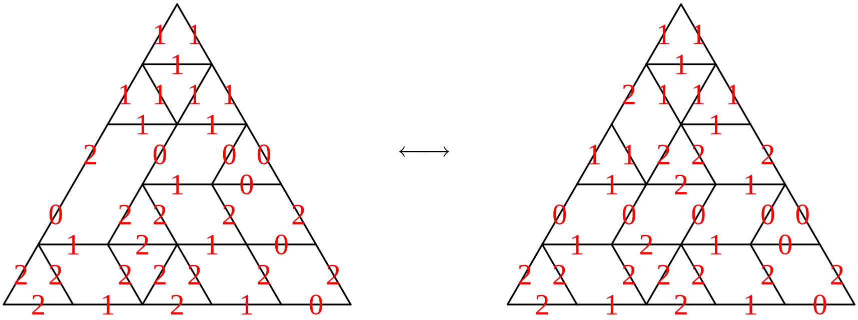
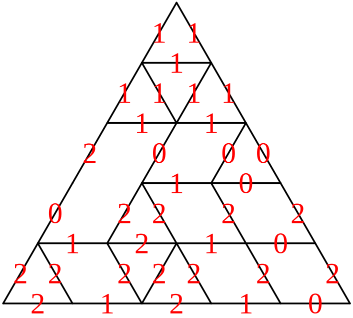


OK !

Next case: Assume $r = 2$.

$u = 10221$, $v = 11022$.



$\longleftrightarrow$?

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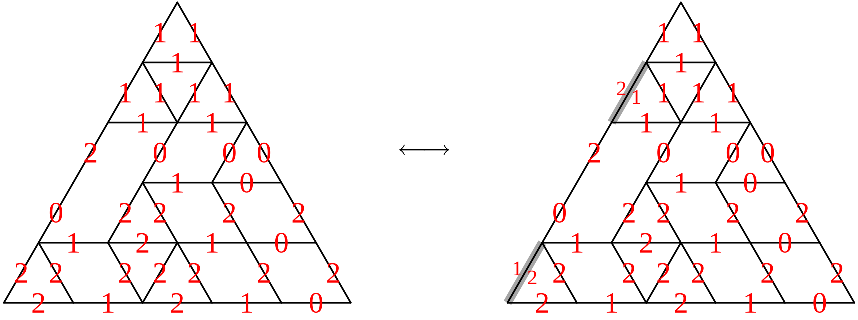

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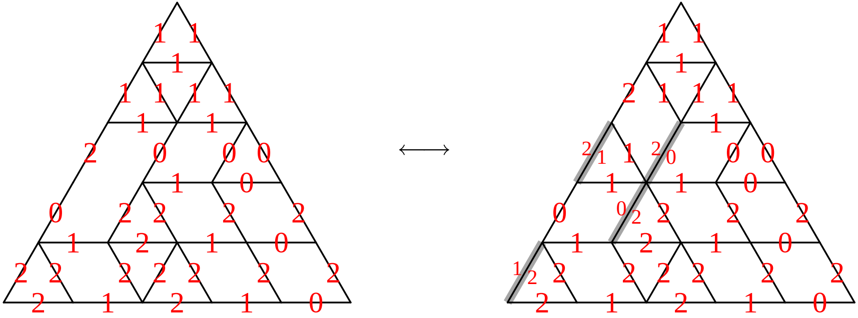

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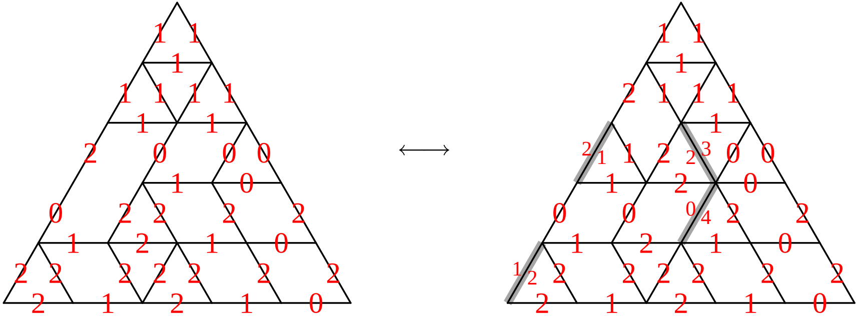

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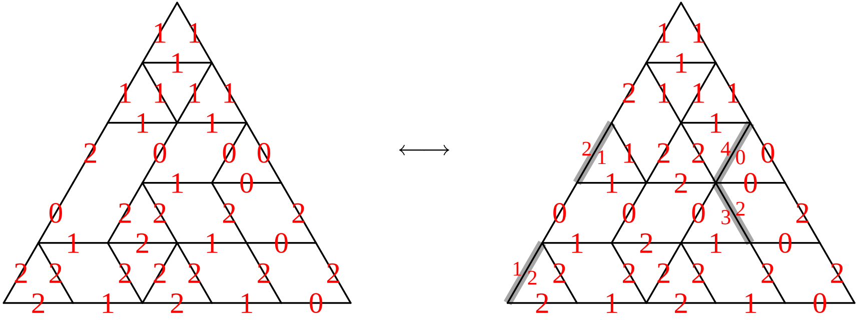

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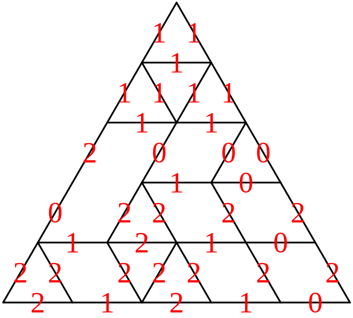
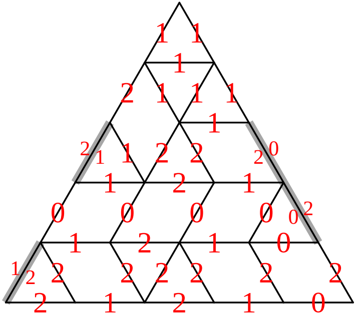


$\longleftrightarrow$



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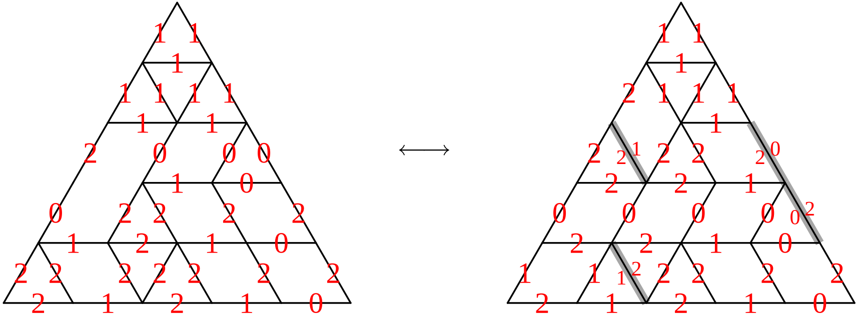

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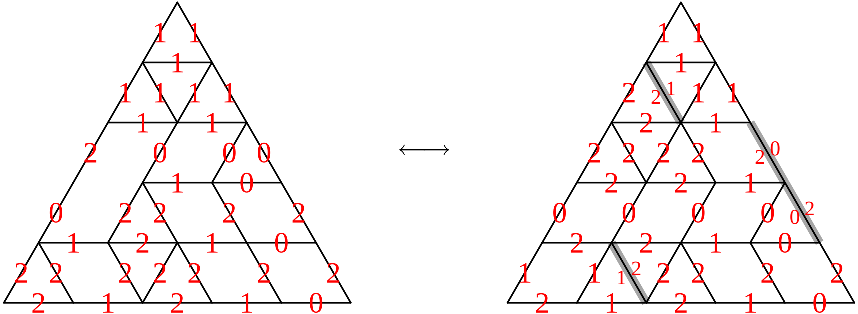

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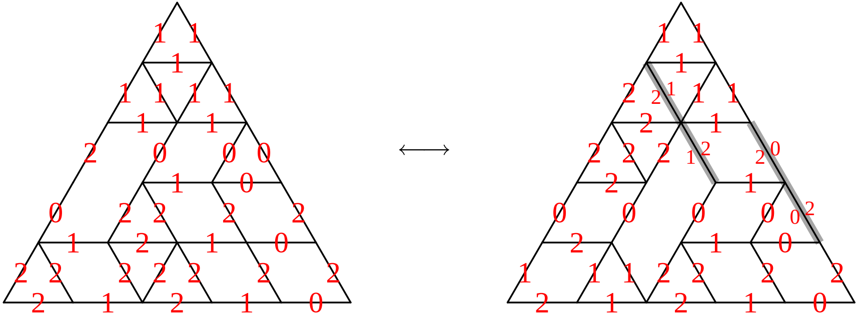

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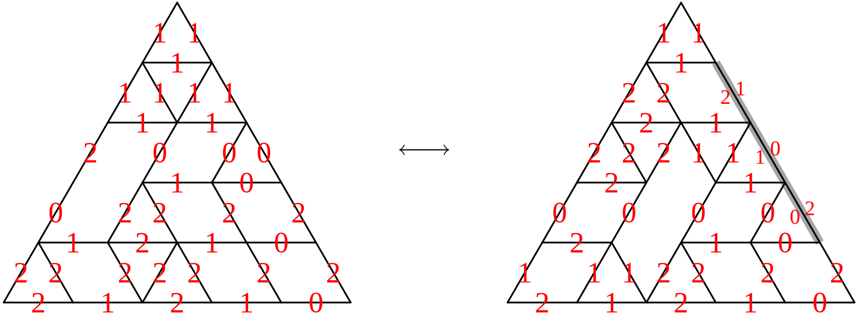

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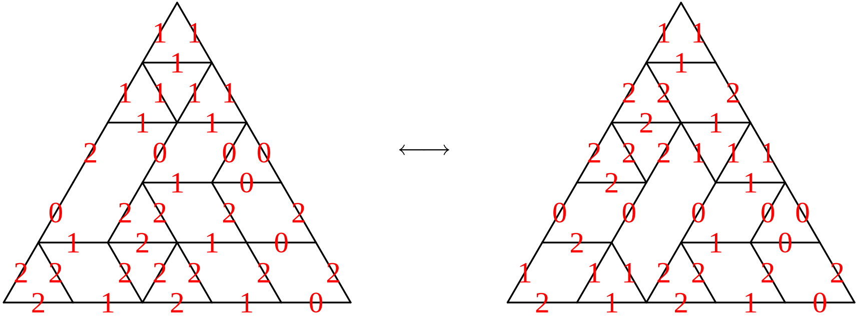
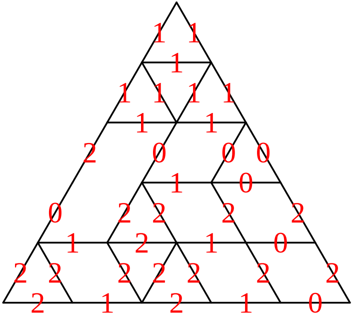


Problem: We have $v' = 12102$, but $v \not\xrightarrow{2} v'$.

Towards a solution: Assume $r = 2$.

$u = 10221$, $v = 11022$.



$\longleftrightarrow$?

Towards a solution: Assume $r = 2$.

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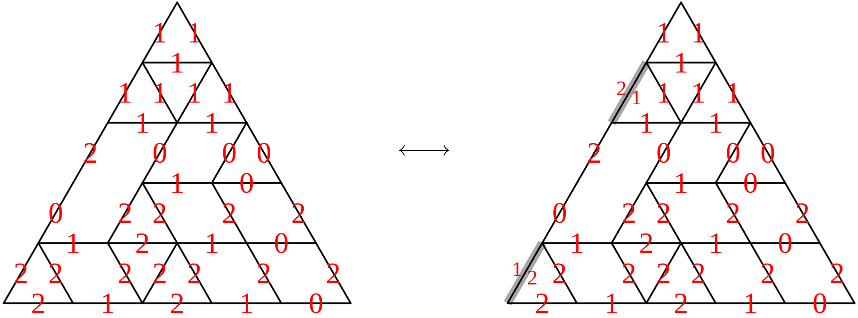

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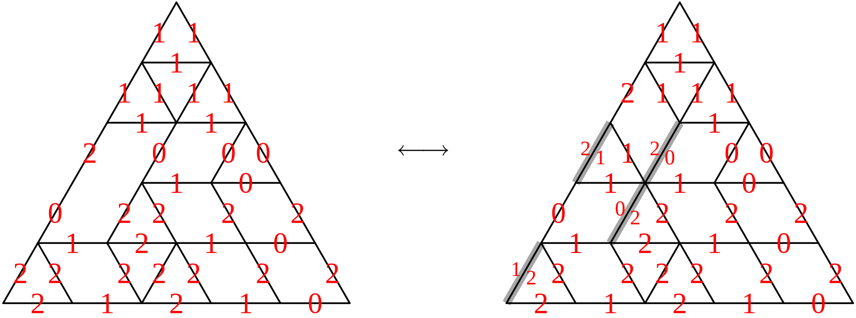

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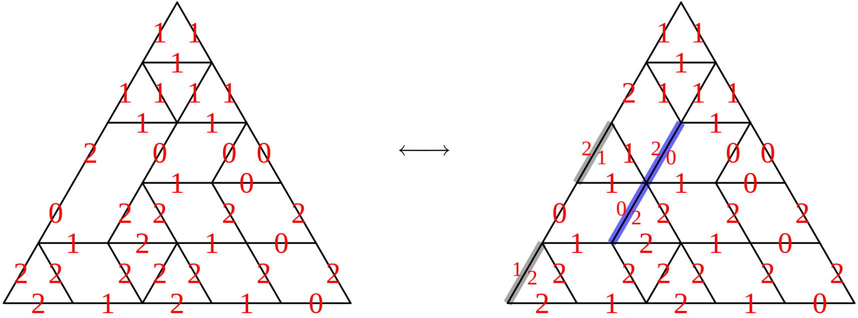

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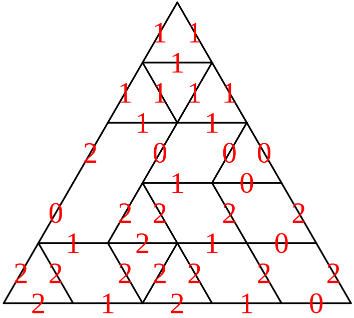

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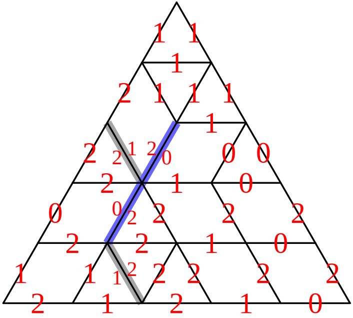

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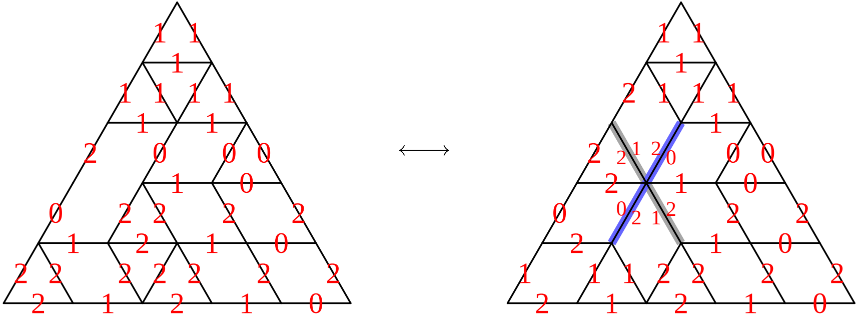

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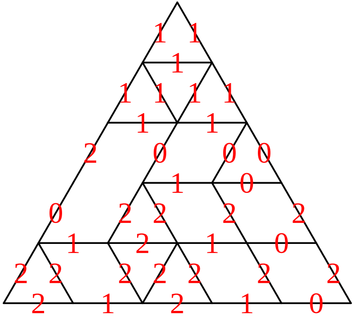

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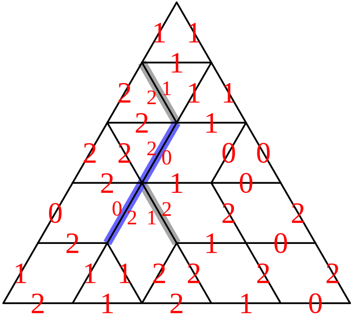

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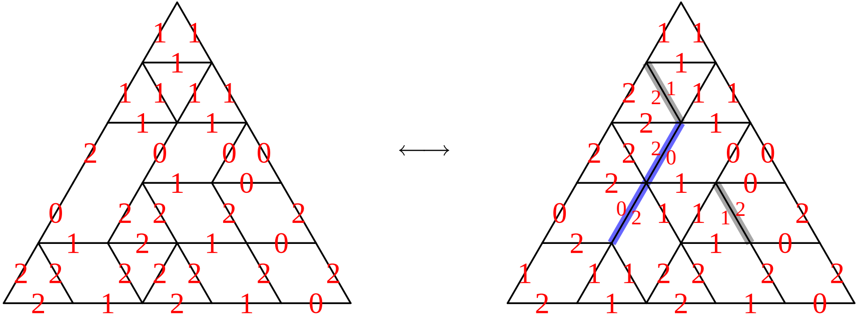

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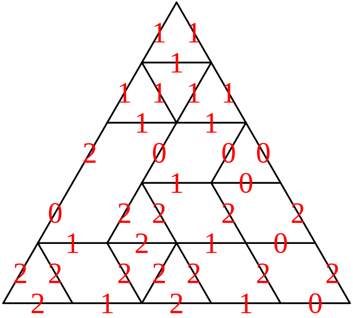

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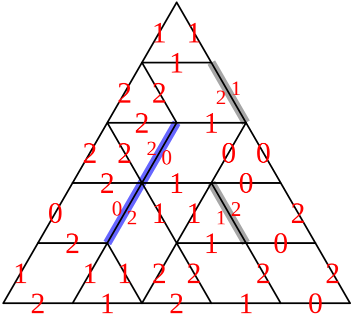

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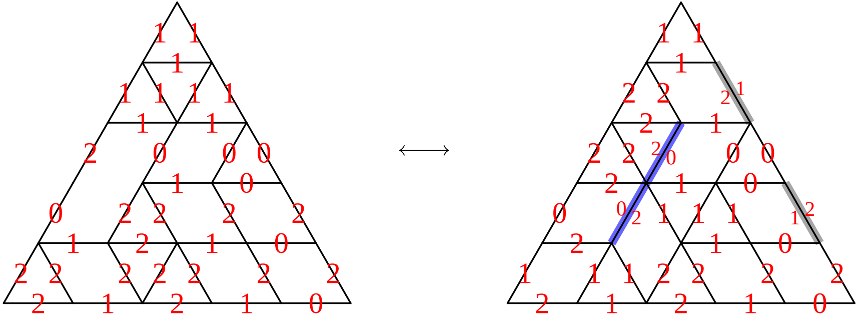

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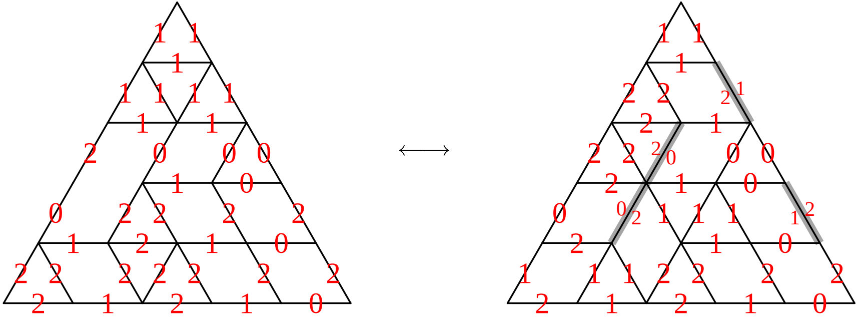

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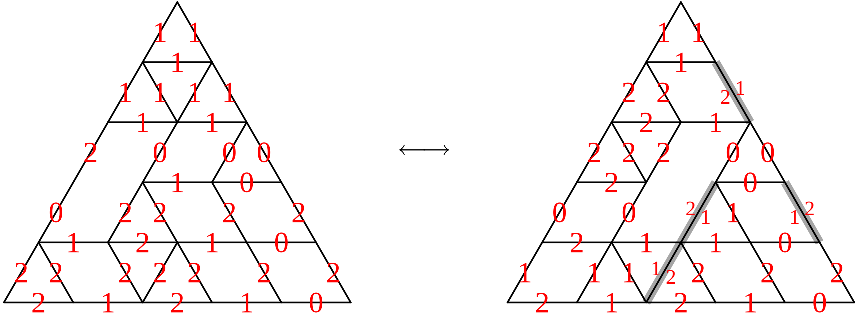

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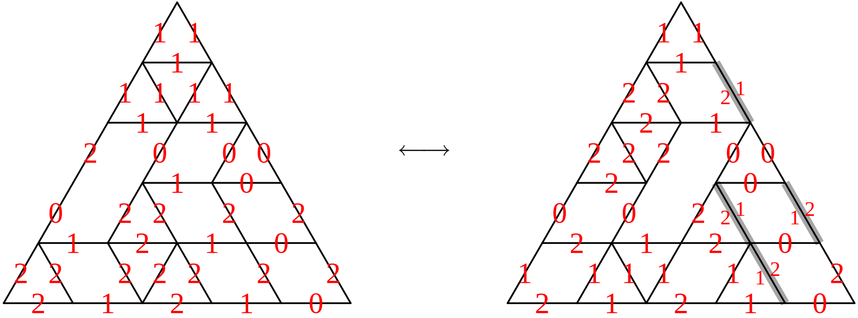

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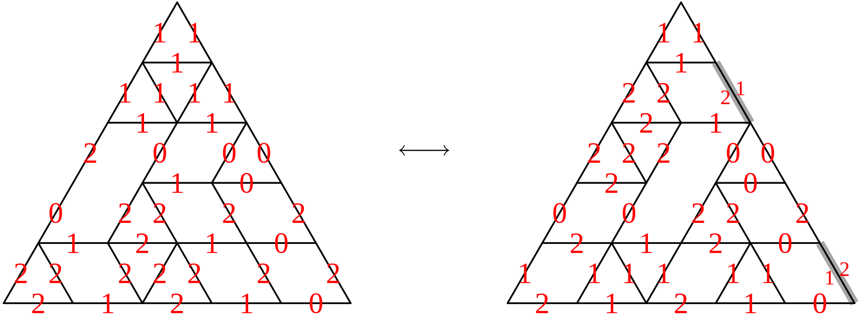

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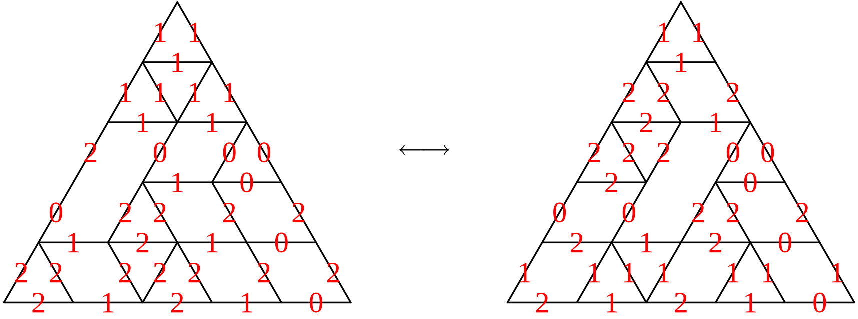
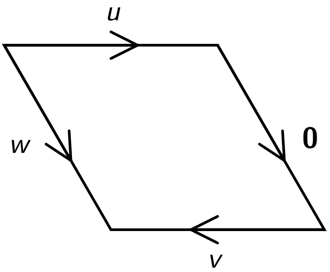


This time we have $v' =$ 12021 and $v \xrightarrow{2} v'$. **OK !!**

Lemma: $C_{u,v}^{w} = \#$ rhombus shaped puzzles with border u, $\mathbf{0}$, v, w:



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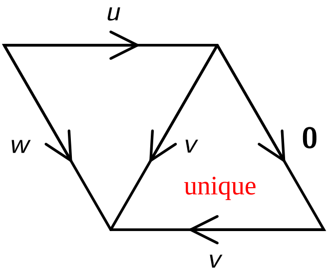

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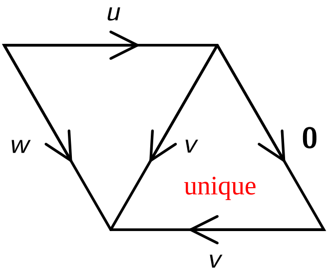
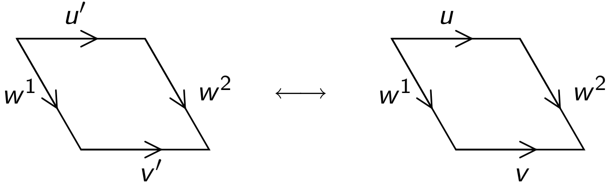


TODO: Given 012-strings u, v', w^1, w^2, and $r \in \mathbb{N}$, construct bijection between puzzles with border (w^1, u', v', w^2) such that $u \xrightarrow{r} u'$, and puzzles with border (w^1, u, v, w^2) such that $v \xrightarrow{r} v'$.



Generalized Pieri relation

Def: A **label string** is any finite sequence of integers from $[0,7] = \{0,1,2,3,4,5,6,7\}$.

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Rule: $\mathcal{R} = \frac{a_1}{b_1} \; S* \; \frac{a_2}{b_2}$ where $a_1, b_1, a_2, b_2 \in [0,7]$ and $S \subset [0,7]$.

Def: Write $u \xrightarrow{\mathcal{R}} u'$ if u' is obtained from u by a substitution

$(a_1, s_1, \ldots, s_k, a_2) \mapsto (b_1, s_1, \ldots, s_k, b_2)$, where $s_j \in S$.

We say $u \xrightarrow{\mathcal{R}} u'$ has index (i,j) if $i < j$ and $u_i \neq u'_i$ and $u_j \neq u'_j$.

Example: $\mathcal{R} = \frac{1}{2} \; 03* \; \frac{5}{7}$

Then $704\mathbf{1303}\mathbf{5}62 \xrightarrow{\mathcal{R}} 704\mathbf{2303}\mathbf{7}62$

Index: $(4,8)$

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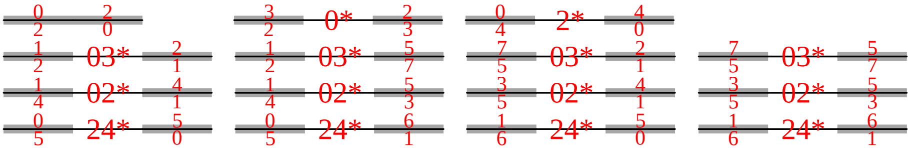
Example: $\mathcal{R} = \frac{1}{2} \; 03* \; \frac{5}{7}$ Then $704\mathbf{13035}62 \xrightarrow{\mathcal{R}} 704\mathbf{23037}62$
Index: $(4,8)$

Def: Write $u \xrightarrow{1} u'$ iff $u \xrightarrow{\mathcal{R}} u'$ for some rule $\mathcal{R}$ from the following list:

$\frac{0}{2} \; \frac{2}{0}$	$\frac{3}{2} \; 0* \; \frac{2}{3}$	$\frac{0}{4} \; 2* \; \frac{4}{0}$	
$\frac{1}{2} \; 03* \; \frac{2}{1}$	$\frac{1}{2} \; 03* \; \frac{5}{7}$	$\frac{7}{5} \; 03* \; \frac{2}{1}$	$\frac{7}{5} \; 03* \; \frac{5}{7}$
$\frac{1}{4} \; 02* \; \frac{4}{1}$	$\frac{1}{4} \; 02* \; \frac{5}{3}$	$\frac{3}{5} \; 02* \; \frac{4}{1}$	$\frac{3}{5} \; 02* \; \frac{5}{3}$
$\frac{0}{5} \; 24* \; \frac{5}{0}$	$\frac{0}{5} \; 24* \; \frac{6}{1}$	$\frac{1}{6} \; 24* \; \frac{5}{0}$	$\frac{1}{6} \; 24* \; \frac{6}{1}$

Basic rules:



Def: Write $u \xrightarrow{r} u'$ iff $\exists\ u = u^0 \xrightarrow{1} u^1 \xrightarrow{1} \cdots \xrightarrow{1} u^r = u'$, such that if $u^{t-1} \xrightarrow{1} u^t$ has index (i_t, j_t), then $j_1 < j_2 < \cdots < j_r$.

Example: $04730202245 \xrightarrow{5} 40720522015$ because:

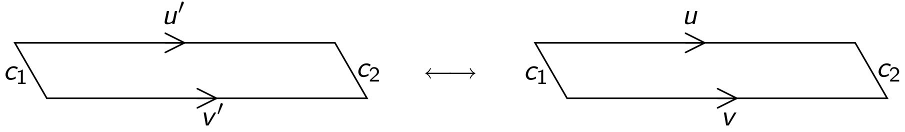
$04730202245 \xrightarrow{1}$
$40730202245 \xrightarrow{1}$
$40720302245 \xrightarrow{1}$
$40720320245 \xrightarrow{1}$
$40720322045 \xrightarrow{1}$
40720522015

Exercise:

This relation restricts to the classical Pieri relation on 012-strings.

Main Technical Result:

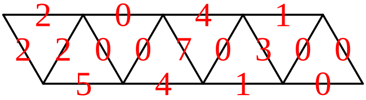
Let u and v' be label strings, let $c_1, c_2 \in \{0, 1, 2\}$, and let $r \in \mathbb{N}$.
There is an explicit bijection between
single-row puzzles with border (c_1, u', v', c_2) such that $u \xrightarrow{r} u'$, and
single-row puzzles with border (c_1, u, v, c_2) such that $v \xrightarrow{r} v'$.



Method: Propagate one gash at the time. 80 rules are required.

Example of single propagation:

$u = 0241$, $v = 5410$, $r = 2$.



$\longleftrightarrow$?

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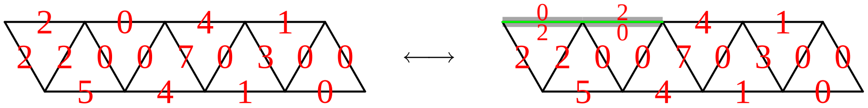

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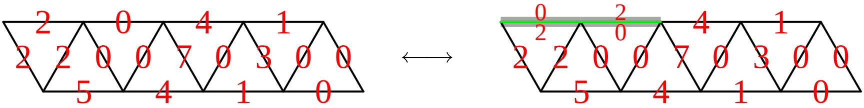
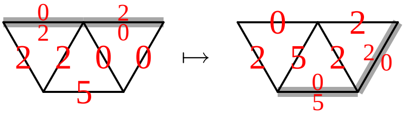


Propagation rules:



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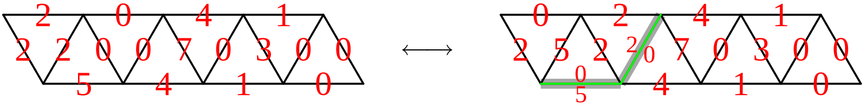

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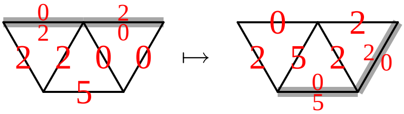

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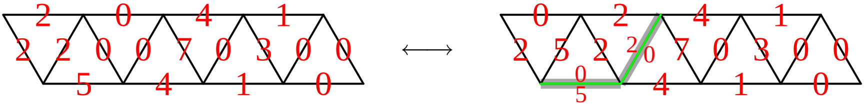

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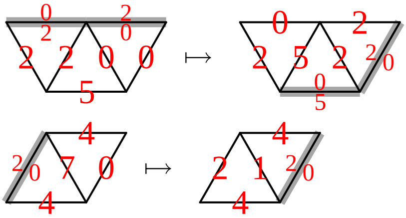

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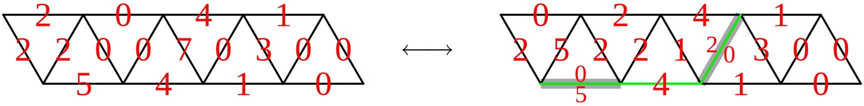

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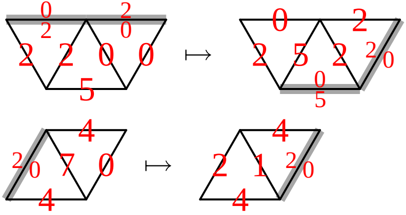

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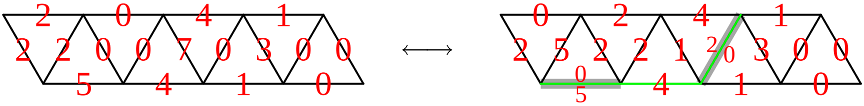
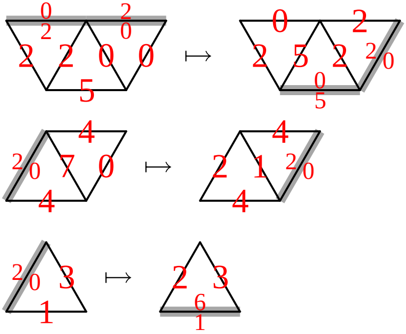
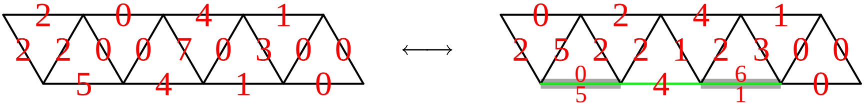


Propagation rules:



Example of single propagation:

$u = 0241$, $v = 5410$, $r = 2$.



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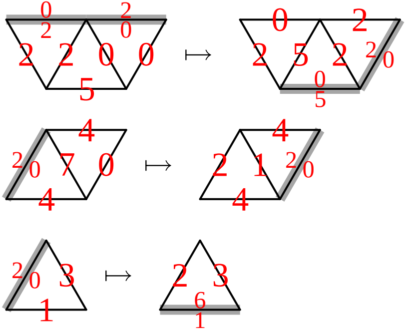

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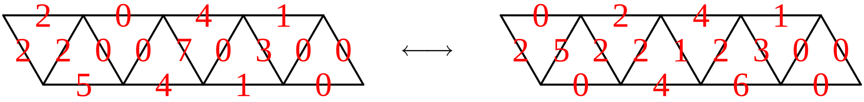
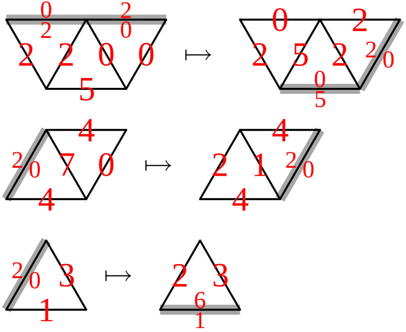


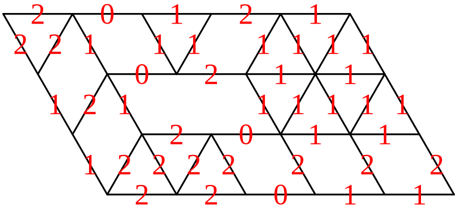
Propagation rules:

77 additional rules.



Example of resulting bijection:

$u = 10212$, $v = 22011$, $r = 2$.



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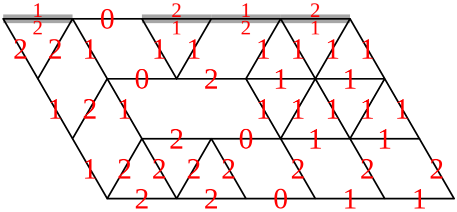

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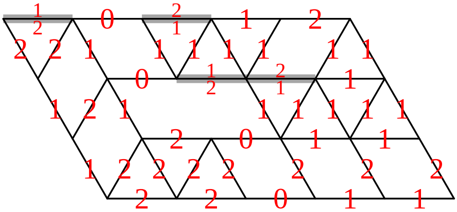

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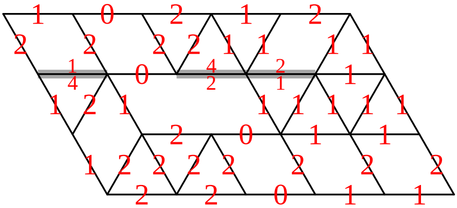

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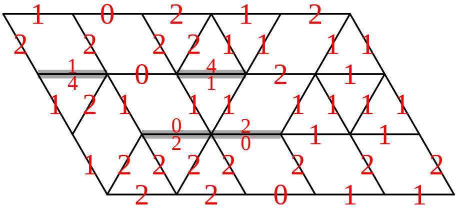

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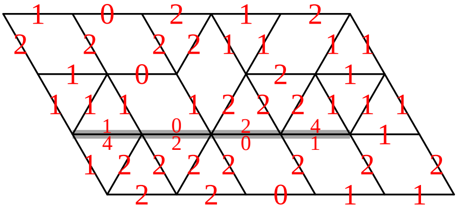

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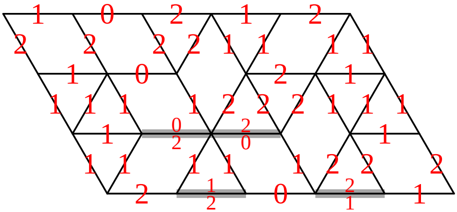

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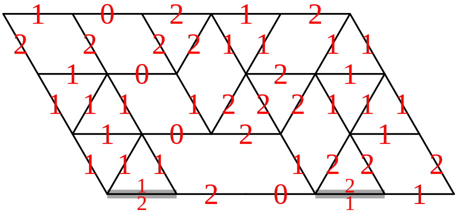

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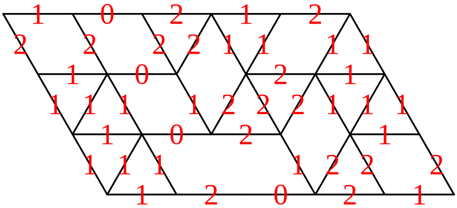
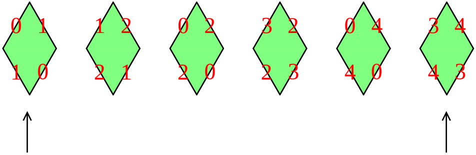


Question: What does the braid group element mean?

Equivariant cohomology of two-step flag variety $X = \mathrm{Fl}(a, b; n)$

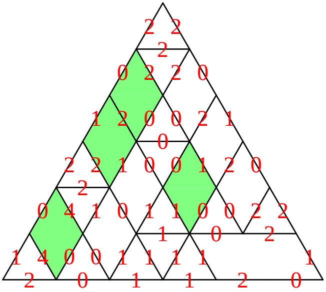
Equivariant pieces: (May NOT be rotated.)



Knutson & Tao

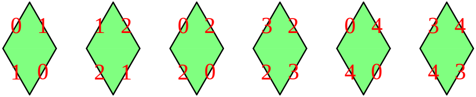
Surprising (3=01 , 4=12)

Equivariant puzzle:



Equivariant cohomology of two-step flag variety $X = \mathrm{Fl}(a, b; n)$

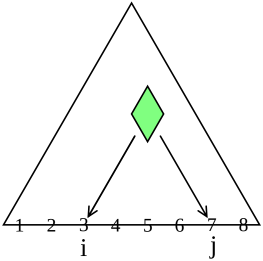
Equivariant pieces: (May NOT be rotated.)



Conjecture for $H_T^*(X)$ (Buch, printed in Coskun–Vakil's 2006 survey)

$$c_{u,v}^{w} = \sum_{P} \prod_{\lozenge \in P} \mathrm{weight}(\lozenge)$$

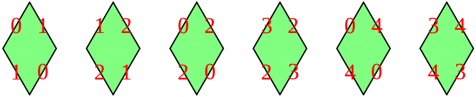
sum over equivariant puzzles P
with border labels u, v, w.



$\mathrm{weight}(\lozenge) = y_j - y_i$

Equivariant cohomology of two-step flag variety $X = \mathrm{Fl}(a, b; n)$

Equivariant pieces: (May NOT be rotated.)



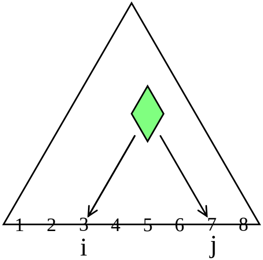
Conjecture for $H_T^*(X)$ (Buch, printed in Coskun–Vakil's 2006 survey)

$$c_{u,v}^{w} \;=\; \sum_{P} \prod_{\lozenge \in P} \mathrm{weight}(\lozenge)$$

(In the printed formula, each $\lozenge$ is a small green diamond:  )

sum over equivariant puzzles P
with border labels u, v, w.

FALSE ! ! !



weight() $= y_j - y_i$

Equivariant cohomology of two-step flag variety $X = \mathrm{Fl}(a, b; n)$

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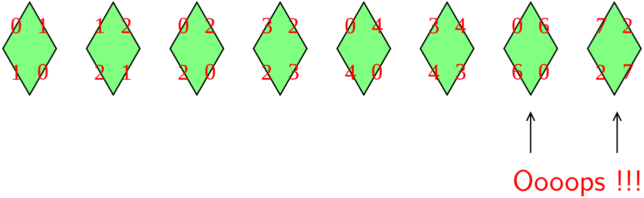

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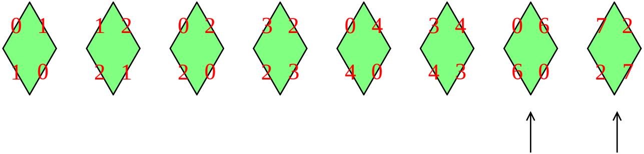


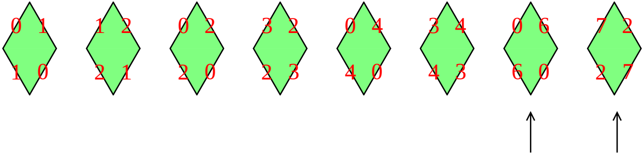
Oooops !!!

Theorem (Buch)

$$c_{u,v}^{w} = \sum_{P} \prod_{\lozenge \in P} \mathrm{weight}(\lozenge)$$

Equivariant cohomology of two-step flag variety $X = \mathrm{Fl}(a, b; n)$

Equivariant pieces: (May NOT be rotated.)



Oooops !!!

Theorem (Buch)

$$c_{u,v}^{w} = \sum_{P} \prod_{\diamond \in P} \mathrm{weight}(\diamond)$$

Consequence: Equivariant quantum Littlewood-Richardson rule for $QH_T(\mathrm{Gr}(m, n))$.

This uses [Buch–Mihalcea 2011].