MULTIPLYING SCHUBERT POLYNOMIALS BY SCHUR POLYNOMIALS

SAMI ASSAF, NANTEL BERGERON, AND FRANK SOTTILE

The problem

The Schubert polynomial $\mathfrak{s}_w \in \mathbb{Z}[x_1, x_2, \ldots]$ is a polynomial representation for the Schubert class of w in the cohomology of the flag manifold.

Loosely, $\mathfrak{s}_w$ is the unique homogeneous polynomial of degree $\ell(w)$, the length of w, representing the Schubert cycle of w.

Schubert polynomials form an additive basis for $\mathbb{Z}[x_1, x_2, \ldots]$:

$$\mathfrak{s}_u \cdot \mathfrak{s}_v = \sum_{w \in \mathfrak{S}_\infty} c^w_{u,v} \mathfrak{s}_w$$

The coefficients $c^w_{u,v}$ enumerate flags in a suitable triple intersection of Schubert varieties, and so they are known to be nonnegative.

Fundamental Problem: Find a *positive* combinatorial construction for $c^w_{u,v}$.

Possibly, $c^w_{u,v}$ counts (saturated) chains in Bruhat order from u to w satisfying some conditions imposed by v. (Recall, Bruhat order corresponds to inclusion of Schubert varieties.)

Special case: Littlewood–Richardson Rule

The Schur functions $\{s_\lambda\}$ form an orthonormal basis for the ring of symmetric functions. For given $\lambda \mathrm{Par}(n)$, the Schur function $s_{\lambda(x_1,\ldots,x_k)}$ is the character of the irreducible representation of GL_k indexed by λ.

The Littlewood–Richardson rule states

$$s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu$$

where $c^\nu_{\lambda,\mu}$ counts the number of saturated chains in Young's lattice from λ to ν with weight μ.

The *Grassmannian permutation* associated to a partition λ and positive integer k, where $\lambda \subseteq (n-k)^k$, is the unique permutation of shape λ with a descent at k, i.e.

$$v(\lambda, k) \ : \ i \mapsto i + \lambda_{k+1-i}$$

Schubert polynomials indexed by Grassmannian permutations are Schur functions:

$$s_\lambda(x_1,\ldots,x_k) = \mathfrak{s}_{v(\lambda,k)}$$

Therefore the Littlewood–Richardson rule is indeed a special case.

The case we consider in this talk is that of a Schubert polynomimal times a Schur polynomial:

$$\mathfrak{s}_u \cdot s_\lambda(x_1,\ldots,x_k) = \sum_{w\in\mathfrak{S}_\infty} c^w_{u,v(\lambda,k)}$$

Our approach is threefold:

- Bergeron and Sottile (1998) used the *Grassmannian-Bruhat order* to translate this problem into that finding the *Schur coefficients of a certain quasisymmetric function.*
- Assaf used *dual equivalence* to find the *Schur coefficients of certain quasisymmetric functions.*
- Billey, Billera, Stanley organized a *Banff workshop on quasisymmetric functions* and invited Assaf, Bergeron and Sottile.

Grassmannian-Bruhat order

Bruhat order on $\mathfrak{S}_\infty$ as cover relations

$$u \lessdot w \text{ if } \ell(w) = \ell(u) + 1 \text{ and } w = u\cdot(a,b)$$

Young's lattice on Par has cover relations

$$\mu \lessdot \nu \text{ if } |\nu| = |\mu| + 1 \text{ and } \nu \supset \mu$$

Monk's Rule states that for $\lambda = (1)$, where $v(\lambda,k) = (k,k+1)$, we have

$$\mathfrak{s}_u \cdot (x_1 + \cdots + x_k) = \sum_{\substack{a\le k<b \\ \ell(u\cdot(a,b)=\ell(w)+1}}$$

There is a nice proof of this using insertion, similar to proving RSK.

Motivated by this rule, define the *Grassmannian-Bruhat order* on $\mathfrak{S}_\infty$ by cover relations

$$u \lessdot_k w \text{ if } \ell(w) = \ell(u) + 1 \text{ and } w = u\cdot(a,b) \text{ and } a \le k < b$$

The *Grassmannian-Bruhat interval* $[u,w]_k$ is the set of saturated chains in this order from u to w, e.g.

$$\mathcal{C} : \{u = x_1 \xrightarrow{(a_1,b_1)} x_2 \xrightarrow{(a_2,b_2)} \cdots \xrightarrow{(a_n,b_n)} x_{n+1} = w$$

Define the *descent set* of such a chain by

$$\mathrm{Des}(\mathcal{C}) = \{i \mid b_i > b_{i+1}\} \subseteq [n-1]$$

Theorem (Bergeron–Sottile 1998)

$$G_{[u,w]_k} \overset{\text{def}}{=} \sum_{\mathcal{C}\in[u,w]_k} Q_{\mathrm{Des}(\mathcal{C})} \overset{\text{thm}}{=} \sum_{\lambda} c^w_{u,v(\lambda,k)} s_\lambda$$

The idea of the proof is to use the formula

$$(x_1 + \cdots + x_k)^n = \sum_{\lambda\in\mathrm{Par}(n)} f^\lambda s_\lambda(x_1, \ldots, x_k)$$

to deduce that

$$|[u,w]_k| = \sum_{\lambda} f^\lambda c^w_{u,v(\lambda,k)}$$

Quasisymmetric functions

Quasisymmetric functions are a superset of symmetric functions with monomial basis

$$M_{1,3} = x_1x_2^2x_3 + x_1x_2^2x_4 + x_1x_3^2x_4 + x_2x_3^2x_4$$

Gessel's *fundamental basis for quasisymmetric functions* is

$$Q_D = \sum_{\substack{i_1\leq\cdots\leq i_n \\ j\in D\Rightarrow i_j<i_{j+1}}} x_{i_1}\cdots x_{i_n}$$

These two basis are triangularly related

$$Q_{1,3} = M_{1,3} + M_{1,2,3}$$

A *standard Young tableau* is a saturated chain in Young's lattice. The *descent set* of a tableau is

$$\mathrm{Des}(T) = \{i \mid \text{box for } i+1 \text{ lies above box for } i\}$$

Theorem (Gessel 1984) For λ a partition, we have

$$\sum_{T\in\mathrm{SYT}(\lambda)} Q_{\mathrm{Des}(T)} = s_\lambda$$

Dual equivalence

Haiman defined involutions d_i, $1 < i < n$, that interchanges i and $i \pm 1$, whichever is further away in the reading order.

Theorem (Haiman 1992) For $T \in \mathrm{SYT}(\lambda)$, we have

$$[T] = \mathrm{SYT}(\lambda)$$

Therefore we can shift paradigms

$$\sum_{T\in[T_\lambda]} Q_{\mathrm{Des}(T)} = s_\lambda$$

where T_λ is the superstandard tableau of shape λ.

Definition (Assaf) Given any set of combinatorial objects $\mathcal{A}$ and any descent statistic Des : $\mathcal{A} \to 2^{[n-1]}$, a family of involutions $\{\varphi_i\}$ $(1 < i < n)$ is a *strong dual equivalence* if

(i) φ_i has specific fixed points
(ii) φ_i changes descents in a specific way
(iii) $\varphi_i = \varphi_{i+1}$ under specific conditions
(iv) $\varphi_i \circ \varphi_j = \varphi_j \circ \varphi_i$ whenever $|i-j| \geq 3$
(v) a non-local minimality condition (painful to verify)

Theorem (Assaf) If $\{\varphi_i\}$ is a strong dual equivalence for $(\mathcal{A}, \mathrm{Des})$, then

$$\sum_{T \in \mathcal{A}} Q_{\mathrm{Des}(T)} = \sum_{\lambda} a_\lambda s_\lambda$$

where a_λ has a simple, explicit description corresponding to dual equivalence classes.

Definition (Assaf) A *weak dual equivalence* excludes condition (v), weakens condition (iii), and must arise "from words".

Theorem (Assaf) If $\{\varphi_i\}$ is a weak dual equivalence for $(\mathcal{A}, \mathrm{Des})$, then

$$\sum_{T \in \mathcal{A}} Q_{\mathrm{Des}(T)} = \sum_{\lambda} a_\lambda s_\lambda$$

where a_λ has an explicit description, but isn't so easy to describe.

The solution

Bergeron and Sottile developed a monoid(e) structure from the Grassmannian-Bruhat order that involves a six case involution on chains in the Grassmannian-Bruhat order.

Theorem (Assaf–Bergeron–Sottile) These involutions give a weak dual equivalence for $([u, w]_k, \mathrm{Des})$. Therefore we have a combinatorial proof that $c^w_{u,v(\lambda,k)} \in \mathbb{N}$.

Theorem (Assaf–Bergeron–Sottile) If wu^{-1} avoids six specific patterns of length up to 7, then these involutions give a strong dual equivalence. Therefore we have a nice combinatorial formula for $c^w_{u,v(\lambda,k)}$ in this case.