Equivariant quantum Schubert polynomials

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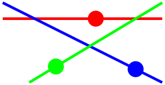
Flavors of Schubert calculus

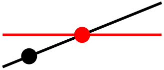
$Fl(n)$: variety of flags in $\mathbb{C}^n$, $V_\bullet = (V_1 \subset \cdots \subset V_n = \mathbb{C}^n)$.

Schubert calculus: fix 3 flags $E_\bullet$, $F_\bullet$, $G_\bullet$.

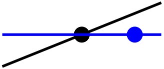


How many $V_\bullet$...

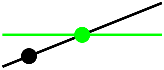
meet $E_\bullet$ like u;



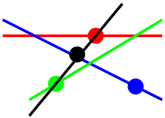
meet $F_\bullet$ like v;



and meet $G_\bullet$ like w?



Flavors of Schubert calculus



Flavors of Schubert calculus

Write

$$c_{u,v,w} = \# \text{ of } V_\bullet \text{ satisfying conditions } u, v, w$$
$$= \# \text{ of points in } \Omega_u \cap \Omega_v \cap \Omega_w.$$

Reindexing a bit, make a ring out of these numbers: $H^* Fl(n)$ has basis elements σ_w (w in S_n), multiplication

$$\sigma_u \cdot \sigma_v = \sum_w c_{u,v}^w \sigma_w.$$

Flavors of Schubert calculus

Quantum Schubert calculus is similar, but use

$$c^d_{u,v,w} = \# \text{ of degree } d \text{ curves through } \Omega_u, \Omega_v, \Omega_w.$$

Similar reindexing, get (associative!) ring $QH^*Fl(n)$ with basis elements σ_w and multiplication

$$\sigma_u \star \sigma_v = \sum_{w,d} q^d \, c^{w,d}_{u,v} \sigma_w.$$

(Degree is $d = (d_1, \ldots, d_{n-1})$, and the parameter is $q = (q_1, \ldots, q_{n-1})$. So $QH^*Fl(n)$ is an algebra over $\mathbb{Z}[q]$.)

Flavors of Schubert calculus

The equivariant versions, H_T^* and QH_T^*, encode the action of the torus $T = (\mathbb{C}^*)^n$.

Have product

$$\sigma_u \circ \sigma_v = \sum_{w,d} q^d \, c_{u,v}^{w,d}(t) \sigma_w.$$

in $QH_T^* Fl(n)$, similar to before.

The *equivariant quantum structure constants* $c_{u,v}^{w,d}(t)$ are polynomials in $t = (t_1, \ldots, t_n)$.
(Defined as equivariant integrals on moduli space of maps $\mathbb{P}^1 \to Fl(n)$.)

Flavors of Schubert calculus

Key features of equivariant cohomology:

1. $H_T^*(\mathrm{pt}) = \mathbb{Z}[t] = \mathbb{Z}[t_1, \ldots, t_n]$ is nontrivial. $H_T^* X$ is an algebra over $\mathbb{Z}[t]$.

2. For some spaces (e.g. $Fl(n)$), can compute via restriction to fixed points X^T. ($H_T^* X \to H_T^* X^T$ is *injective*.)

Defined via *Borel construction*. (Fiber bundle, with fiber X.)

Equivariant quantum Schubert calculus

Why care about *equivariant* Schubert calculus?

- General principle: extra structure, extra information.
- The ring $QH^*_T Fl(n)$ specializes to all the others.
- $QH^*_T Fl(n)$ related to $H^T_* \mathcal{G}r$, double k-Schur functions. [Peterson, Lam-Shimozono]

Equivariant quantum Schubert calculus

Goal: algebraic model for $QH_T^* Fl(n)$.

Need

1. presentation of the ring, and
2. polynomial representatives for the basis elements σ_w.

Then computing the *equivariant quantum structure constants* $c_{u,v}^{w,d}(t)$ is "reduced" to multiplying polynomials.

Equivariant quantum Schubert calculus

Presentations:

- For $H^* Fl(n)$, $H_T^* Fl(n)$ [Borel]
- For $QH^* Fl(n)$ [Givental-Kim]
- For $QH_T^* Fl(n)$ [Kim]

$$QH_T^*(Fl(n)) = \mathbb{Z}[q, x, t]/(e_1^q(x) - e_1(t), \ldots, e_n^q(x) - e_n(t)),$$

where e_i is an elementary symmetric polynomial, and e_i^q is a **quantum elementary polynomial**.

To get the others, set t, q, or both to zero.

Equivariant quantum Schubert polynomials

Polynomial representatives for σ_w in...

- $H^* Fl(n)$: $\mathfrak{S}_w(x)$ [Lascoux-Schützenberger, BGG, ...]
- $H_T^* Fl(n)$: $\mathfrak{S}_w(x, t)$ [Lascoux-Schützenberger, Fehér-Rimányi, Knutson-Miller, ...]
- $QH^* Fl(n)$: $\mathfrak{S}_w^q(x)$ [Fomin-Gelfand-Postnikov, Ciocan-Fontanine, Chen]

Equivariant quantum Schubert polynomials

Theorem (A.-Chen)

In $QH_T^* Fl(n)$, we have

$$\sigma_w = \mathfrak{S}_w^q(x, t),$$

a specialization of Fulton's universal double Schubert polynomial. (Also equal to Kirillov-Maeno's polynomial.)

(To be explained...)

(Lam-Shimozono also gave a different proof of this. Theirs is more combinatorial; ours is more geometric.)

Equivariant quantum Schubert polynomials

Many ways to define $\mathfrak{S}_w^q(x,t)$.

Notation: $e_i(k)$ is the ith elementary symmetric polynomial in variables $x_1, \ldots, x_k$.

The *Schubert polynomial* of Lascoux-Schützenberger can be written uniquely

$$\mathfrak{S}_w(x) = \sum_{i_1,\ldots,i_n} a_{i_1,\ldots,i_n} e_{i_1}(1) \cdots e_{i_n}(n),$$

for some integers $a_{i_1,\ldots,i_n}$.

Equivariant quantum Schubert polynomials

The *quantum Schubert polynomial* of Gelfand-Fomin-Postnikov is defined as

$$\mathfrak{S}_w^q(x) = \sum_{i_1,\ldots,i_n} a_{i_1,\ldots,i_n} e_{i_1}^q(1)\cdots e_{i_n}^q(n),$$

where $e_i^q(k)$ is a quantum elementary polynomial.

The *equivariant quantum Schubert polynomial* is equal to

$$\mathfrak{S}_w^q(x,t) = \sum_{v^{-1}u=w} \mathfrak{S}_u^q(x)\,\mathfrak{S}_v(-t).$$

(There are other definitions, starting from the *universal Schubert polynomials* and specializing.)

Proofs

Proofs, aka, "why wasn't this easy?"

Proofs for $QH^* Fl(n)$ use moving lemmas not available equivariantly.

Key input:

Theorem (A.-Chen)

The equivariant quantum coefficients $c_{u,v}^{w,d}(t)$ from

$$\sigma_u \circ \sigma_v = \sum_{w,d} q^d \, c_{u,v}^{w,d}(t) \, \sigma_w$$

are equal to certain (equivariant) integrals on a quot scheme.

Proof uses *equivariant moving lemma*, and "almost transitive" action of a group Γ on $\mathbb{E}T \times^T Fl(n)$.

Consequence: can use equivariant techniques on the quot scheme $\mathcal{Q}_d$ to study $QH_T^* Fl(n)$.

Equivariant geometry of $\mathcal{Q}_d$ is (somewhat) understood. [Braden-Chen-Sottile]

More applications of $H_T^* \mathcal{Q}_d$??

Aside: All the above works for partial flags, too. Even for Grassmannians, the relation between $c_{u,v}^{w,d}(t)$ and $H_T^* \mathcal{Q}_d$ is new.

As with $H_T^* Fl(n)$ and $K_T Fl(n)$, the equivariant moving lemma implies positivity in $QH_T^* Fl(n)$:

Corollary (Mihalcea, A.-Chen)

Written as a sum of monomials in variables $(t_1 - t_2), (t_2 - t_3), \ldots, (t_{n-1} - t_n), t_n$, the polynomial $c_{u,v}^{w,d}(t)$ has ≥ 0 coefficients.

Examples

Equivariant quantum Schubert polynomials for $Fl(3)$.

w	$\mathfrak{S}_w^q(x,t)$
123	1
213	$x_1 - t_1$
132	$x_1 + x_2 - t_1 - t_2$
231	$x_1\,x_2 + q_1 - (x_1 + x_2)\,t_1 + t_1^2$
312	$x_1^2 - q_1 - x_1\,(t_1 + t_2) + t_1\,t_2$
321	$(x_1 - t_2)\left(x_1\,x_2 + q_1 - (x_1 + x_2)\,t_1 + t_1^2\right)$

These multiply like Schubert classes—on the nose! So get (somewhat efficient) algorithm for computing $c_{u,v}^{w,d}$.