Summer School

Takeshi Ikeda - *Equivariant Schubert polynomials*

The aim of this lecture is to give a reasonably self-contained account of the theory of Schubert polynomials(in a wider sense) for all the classical Lie types in the setting of torus equivariant K-theory, as well as equivariant cohomology. As a powerful combinatorial device, we use the restriction maps to the set of torus fixed points throughout the lectures.

Lecture 1: Grassmannians and Factorial Schur functions. I will start with recalling echelon forms of matrices, which give Schubert cells in Grassmannians. Chang-Skjelbred's condition enables us to "define" equivariant cohomology ring in a purely combinatorial manner. Schubert classes in this situation are identified with the so-called factorial Schur functions.

Lecture 2: Schur's Q-functions and the maximal isotropic Grassmannians. I will try to provide a workable introduction to Q-functions. The functions play a crucial part in the theory of Schubert polynomials of all classical types.

Lecture 3: Equivariant multiplicities. In this lecture, I will discuss equivariant multiplicity which is closely related to the nil-Hecke algebra of Kostant-Kumar, and also degenerations of Schubert varieties. If time permits, I will show how equivariant Schubert polynomials are related to degeneracy loci formulas.

Lecture 4: Extension to equivariant K-theory. I will show how the combinatorial approach to Schubert polynomials so far is extended to equivariant K-theory.

Allen Knutson - *Schubert calculus and puzzles*

The most fundamental, and long since solved, problem in Schubert calculus is to compute the cohomological intersection of two Schubert varieties in the Grassmannian. Since those intersections usually aren't transverse, one instead intersects Schubert varieties with opposite Schubert varieties, obtaining Richardson varieties. Then the geometric problem (solved by Vakil) becomes to degenerate Richardson varieties to unions of Schubert varieties.

In my first lecture I'll present a family of varieties interpolating between Schubert and Richardson, called "interval positroid varieties".

In the second I'll explain how Vakil's degeneration technique lets one compute the homology class of any interval positroid variety, by counting "puzzles".

In the third I'll define equivariant cohomology and K-theory, and show how to extend the puzzle rule to study these classes as well.

In the fourth I'll move beyond the Grassmannian to other partial flag manifolds GL_n/P. These require a different set of puzzles, that computes not the whole cohomology product, but just its Belkale-Kumar degeneration. (This work is joint with Kevin Purbhoo.) Finally, I'll give a conjectural puzzle rule for the actual product on 2-step flag manifolds.

Sara Billey - *Consequences of the Lakshmibai-Sandhya Theorem; the ubiquity of permutation patterns in Schubert calculus and related geometry*

In 1990, Lakshmibai and Sandhya showed that Schubert varieties in the flag manifold of type A are smooth if and only if the indexing permutations avoid 3412 and 4231. This elegant characterization made a huge computational improvement in determining smoothness. It also opened a whole new area of Schubert geometry related to pattern avoidance. Over the past 20 years this theorem has been extended to all finite Lie types, and the notion of pattern avoidance has been generalized in many ways to encompass a wide variety of computational and geometrical properties related to Schubert varieties. We will review some of the many interesting developments including recent extensions to affine Schubert varieties.

Lecture 1: Characterizing smoothness, tangent space bases and singular loci of Schubert varieties.

Lecture 2: Pattern avoidance for any Coxeter group. This lecture will include analogs of the theorems from Lecture 1, some are known and some are open.

Lecture 3: Many other pattern related results. This lecture will include the Woo-Yong theorem characterizing Gorenstein Schubert varieties using patterns and Schubert Calculus, a theorem of Gasharov-Reiner on Schubert varieties defined by inclusions, and connections to the cohomology rings of Schubert varieties.

Lecture 4: Computations on Schubert varieties. We will present learning algorithms for properties characterized by pattern avoidance or more generally by the marked mesh patterns building on the work of Branden, Claesson, and Úlfarsson. This presentation will be based on Sage code. A primer is available here: `http://wstein.org/books/sagebook/sagebook.pdf`

Thomas Lam

Lecture 1: k-Schur functions and affine Stanley symmetric functions In this lecture, I'll review the tableau definition of Schur functions, and then introduce (Lapointe-Lascoux-Morse's) k-Schur functions and (my) affine Stanley symmetric functions via (Lam-Lapointe-Morse-Shimozono) strong and weak tableaux. In the process, I will discuss properties of the weak order and Bruhat order of the affine symmetric group.

Lecture 2: Kostant and Kumar's nilHecke ring In this lecture, I'll define and explain the main properties of Kostant and Kumar's nilHecke ring. I will then discuss the relations between the nilHecke ring and equivariant Schubert calculus of Kac-Moody flag varieties.

Lecture 3: (Co)homology of affine Grassmannians In this lecture, I will synthesize the first two lectures to discuss the Schubert calculus of the affine Grassmannian. I will explain Peterson's idea to study the homology of the affine Grassmannian as a subalgebra of the nilHecke ring. I will sketch the proof that k-Schur functions and affine Schur functions represent (co)homology Schubert classes of the affine Grassmannian.

Lecture 4: Affine to quantum In this lecture, I will discuss Peterson's theorem comparing Schubert calculus of the affine Grassmannian with quantum Schubert calculus of (partial) flag manifolds. I will also explain how these results are interconnected with the Toda lattice.

Frank Sottile - *Experimentation in the Schubert Calculus*

The Schubert calculus provides a rich and well-structured class of problems in enumerative algebraic geometry, which can be used as a laboratory for exploring ill-understood phenomena. This lecture series will discuss two such phenomena, reality of solutions to geometric problems and intrinsic structure as detected by Galois groups. They will describe what we know and what we suspect, and the large-scale computer experimentation that has driven these questions and results.

Lecture 1: The Shapiro Conjecture and its proof. This lecture will cover the genesis, evidence for, and resolution of the remarkable conjecture of Boris and Michael Shapiro that all solutions to a Schubert problem on a Grassmannian are real when the flags giving the problem are tangent to a rational normal curve.

Lecture 2: Beyond the Shapiro Conjecture. While the Shapiro Conjecture may be formulated for any flag manifold, it typically fails in this generality. In some cases these failures are quite interesting and the conjecture may be repaired. Also in some cases, if the tangent flags are replaced by disjoint secant flags, reality seems to hold. This lecture will survey the emerging landscape beyond the Shapiro Conjecture.

Lecture 3: Lower bounds and computing Schubert problems. Another phenomenon that was discovered en route to the proofs of the Shapiro Conjecture are lower bounds on the number of real solutions to Schubert problems. While experimentation reveals this to be wide-spread, this phenomenon is not understood well enough to formulate a conjecture. In addition to explaining these lower bounds, this lecture will discuss how the massive (several teraHertz years of computing) experiments studying these phenomena are conducted.

Lecture 4: Intrinsic structure in the Schubert calculus. The symmetries of a geometric problem are encapsulated in its Galois group. While Schubert problems are quite structured, it is unknown which problems possess intrinsic structure in that their Galois groups are not the full symmetric group. This lecture will discuss some preliminary results and the different methods that are being deployed to study Galois groups of the millions of computable Schubert problems.

Research conference

Anders Buch - *TBA*

Alexander Yong - *Schubert calculus via root datum and sliding laws*

The sliding law of [Schutzenberger '77], part of his celebrated jeu de taquin theory, controls Schubert calculus of Grassmannians. Using work of [Proctor '04], [Thomas-Yong '09] showed how root datum and sliding could be extended to minuscule Schubert calculus. I'll discuss further progress from this perspective as one varies the cohomology theory and/or flag variety G/P.

Peter Fiebig - *Parity sheaves, Moment graphs and KL-polynomials*

We show how moment graphs can be used to calculate the local cohomology of parity sheaves on flag manifolds in almost all characteristics. The multiplicities are conjecturally given by Kazhdan-Lusztig polynomials. We also briefly discuss how this conjecture implies Lusztig's conjecture on simple characters of reductive groups in positive characteristics.

Mark Shimozono - *Chevalley rule for equivariant K-theory of Kac-Moody flag manifolds*

Explicit combinatorial formulas are given for the product of a dominant or antidominant line bundle with a Schubert class, in the torus-equivariant K-theory of a Kac-Moody flag manifold. The formulas are given in terms of Lakshmibai-Seshadri paths and in terms of the alcove model of Lenart and Postnikov. The latter can conjecturally be used for a Chevalley rule in equivariant quantum K-theory of finite-dimensional flag manifolds. This is joint with Cristian Lenart.

Dave Anderson -*Equivariant quantum Schubert polynomials, transversality and positivity*

Just as the Schur functions represent cohomology classes in the Grassmannian, the Schubert polynomials represent classes in the flag variety. The last two decades have seen generalizations to analogues for equivariant cohomology and quantum cohomology. In this talk, I will describe joint work with Linda Chen which provides a common generalization of both theories: the ”equivariant quantum Schubert polynomials” are polynomials in three sets of variables which represent Schubert classes in equivariant quantum cohomology, and specialize to all the previous versions. Our approach uses the geometry of the Quot scheme, and opens the way to further applications of the rich combinatorial structure of this space. I'll also explain the technique of equivariant transversality, and its implications for positivity in various flavors of Schubert calculus.

Leonardo Mihalcea - *Curve neighborhoods of varieties in flag manifolds*

If X is a Schubert variety in a flag manifold, its curve neighborhood is defined to be the union of the rational curves of a fixed degree passing through X. It turns out that this is also a Schubert variety, and I will explain how to identify it explicitly in terms of the combinatorics of the Weyl group and of the associated (nil-)Hecke product. I will also show how the geometry and combinatorics of this and more general curve neighborhoods is reflected in computations in quantum cohomology of flag manifolds. This is part of several joint projects with A. Buch, P.E. Chaput, C. Li and N. Perrin.

K. N. Raghavan - *RSK Correspondence and KL Cells*

Starting from a basic question in classical invariant theory of the general linear group, we are led to questions in the ordinary and modular representation theory of the symmetric group. The Hecke algebra and Kazhdan-Lusztig basis are involved in the answers (although not always in the questions themselves).

Masaki Nakagawa - *K-homology of the space of loops on a symplectic group*

Let $SU = SU(\infty)$, $Sp = Sp(\infty)$, and $SO = SO(\infty)$ be the infinite special unitary, symplectic, and special orthogonal group respectively, and ΩSU, ΩSp, and $\Omega_0 SO$ denote its loop space (Ω_0 means the connected component of the identity). Let $E^*(\cdot)$ be a complex oriented generalized cohomology theory, and $E_*(\cdot)$ the corresponding homology theory. As is well known, the E-homology $E_*(\Omega SU)$ can be identified with $\Lambda \otimes_{\mathbb{Z}} E_*$, where Λ is the ring of symmetric functions and $E_* := E_*(\mathrm{pt})$. In the first half of this talk, I will show that the E-homology $E_*(\Omega Sp)$ (resp. $E_*(\Omega_0 SO)$) can be naturally realized as a Hopf sub-algebra of $E_* \otimes_{\mathbb{Z}} \Lambda$, which may be considered as a ring of "E-homology Schur P- (resp. Q)-functions". In the latter half, I will focus on the case $E = K$, namely the K-theory. Motivated by the construction of *stable dual Grothendieck polynomials* g_λ (λ a partition) due to Lam-Pylyavskyy, we, H. Naruse and myself, constructed a family of "symmetric" polynomials gp_λ, gq_λ (λ a strict partition). I will explain the basic properties of these polynomials, and the relationship between the Schubert classes of the K-homology of ΩSp and $\Omega_0 SO$. This is joint work with H. Naruse.

Bumsig Kim - *J-functions of GIT quotients*

The J-function plays an important role in Gromov-Witten theory, generating solutions to quantum differential equations. We introduce the conjectural formula of the small J-funcitons for GIT quotients $V//G$ of a vector space. Such GIT quotients include smooth toric varieties and partial flag varieites. In fact, the formula is obtained by applying the conjectural abelianization which expresses J for $V//G$ by J for $V//T$ with twisting, where T is a maximal abelain subgroup of G. Combined with the quantum Lefschetz, it yields a formula of J-function for complete intersections in $V//G$. In this talk, we explain the procedures. The talk is based on joint works with Bertram, Ciocan-Fontanine, Maulik, Sabbah, and Diaconescu.

Valentina Kiritchenko - *Convex geometric Demazure operators*

Divided difference (or Demazure) operators play an important role in Schubert calculus and representation theory. I will talk about convex geometric analogs of Demazure operators. Geometric Demazure operators act on polytopes (more generally on convex chains) and take a polytope to a polytope of dimension one greater. For instance, Gelfand-Zetlin polytopes in type A can be obtained by applying a suitable composition of geometric Demazure operators to a point.

This construction can be extended to arbitrary reductive groups and their full flag varieties. It produces polytopes similar to string polytopes of Littelmann-Bernstein-Zelevinsky. I will describe a simple algorithm for representing Schubert cycles by the unions of faces in these polytopes (in type A, this is closely related to mitosis of Knutson-Miller).

Geometric Demazure operators can be defined not only for arbitrary reductive groups but in a more general setting. In particular, they can be used to construct twisted polytopes of Grossberg-Karshon for Bott towers.

Alain Lascoux - *Eulerian structure of the Symmetric Group and Tableaux*

The symmetric group, with its Ehresmann-Bruhat order, is Eulerian, that is, its Moebius function takes values ±1. Young tableaux may be considered as chains in the Bruhat order between two permutations. This allows to interpret the symmetric group as a directed graph with edges labelled by tableaux. We describe the Eulerian properties of this new structure.

Syu Kato - *PBW bases and KLR algebras*

Khovanov-Lauda and Rouquier introduced a certain collection of algebras (which we refer as the KLR-algebras) to each of symmetrizable Kac-Moody algebra which categorify the positive half of the correspnding quantum group. In case the Kac-Moody algebra is symmetric, Varagnolo-Vasserot showed the correspondence between simple/projective modules of the KLR algebras with the lower/upper global bases of the quantum groups.

In this talk, we categorify every PBW base of a finite quantum group of type ADE as a complete collection of "standard/dual standard" modules of the KLR-algebras. This verifies a problem of Kashiwara on the finiteness of the global dimensions of the KLR algebras, and a conjecture of Lusztig on the positivity of their expansion coefficients in these cases.

Nicolas Perrin - *Spherical multiple flags*

Joint work with P. Achinger. For G a reductive group and B a Borel subgroup, the B-orbit closures in projective varieties homogeneous under G are the Schubert varieties. They play an important role in Schubert calculus as well as in representation theory. In my talk I will consider the B-orbit closures in products of homogeneous spaces under G and focus on the case where the product is spherical. I will explain when the orbit closures are normal and how the rank of these orbits is related to a distance appearing in some quantum cohomology computations.

Masaki Kashiwara - *Infinite-dimensional Schubert cells versus finite-dimensional Schubert cells*

Abstract For the flag manifolds of affine Lie algebras, we can consider two kinds of flag maqnifolds; one is an ind-variety, which is the union of finite-dimensional Schubert cells, the other is an infinite-dimansional (non quasi-compact) scheme whose Schubert cells are foinite-codimensional. I want to discuss their relations.

Satoshi Naito - *On an intrinsic description of level-zero Lakshmibai-Seshadri paths and the quantum Schubert calculus*

In this talk, I will give an intrinsic description (or, a characterization) of the set of level-zero Lakshmibai-Seshadri paths in terms of the quantum Bruhat graph introduced by Brenti-Fomin-Postnikov in the study of the quantum cohomology of (finite-dimensional) flag manifolds. For a level-zero dominant integral weight λ, the set $\mathbb{B}(\lambda)_{cl}$ of Lakshmibai-Seshadri paths, which are identified modulo multiples of null roots, of shape λ provides an explicit combinatorial realization of the crystal basis of the quantum Weyl module (or, the standard module) $W(\lambda)$ over the quantum affine algebra. Here, $\mathbb{B}(\lambda)$ denotes the set of Lakshmibai-Seshadri paths of shape λ, and so the set $\mathbb{B}(\lambda)_{cl}$ can be thought of as a quotient of it. However, we have not obtained an intrinsic description of this set $\mathbb{B}(\lambda)_{cl}$ until quite recently. In this talk, we explain how the set $\mathbb{B}(\lambda)_{cl}$ can be descripbed quite explicitly in terms of the quantum Bruhat graph associated to the Weyl group of a finite dimensional simple Lie algebra. The quantum Bruhat graph was introduced by Brenti-Fomin-Postnikov in their study of the quantum (T-equivariant) cohomology of flag varieties. More precisely, it is well-known that the quantum T-equivariant Chevalley formula, which describes the multiplication rule by the quantum equivariant class $[\lambda]$ of a line bundle with weight λ (with respect to the basis of quantum equivariant Schubert classes), can be stated in terms of the quantum Bruhat graph. In view of the facts above, it seems very likely that the (finite-dimensional) representation theory of quantum affine algebras is hidden behind the quantum cohomology of flag varieties in the same way as the representation theory of semi-simple Lie algebras are hidden behind the uasual Schubert calculus. This talk is based on a joint work with C. Lenart, D. Sagaki, A. Schilling, and M. Shimozono.

Sami Assaf - *Multiplying Schubert polynomials by Schur functions*

One of the most fundamental problems in algebraic combinatorics is to find a positive combinatorial construction for the structure constants of Schubert polynomials. A special case of this is the Littlewood-Richardson rule for the structure constants of Schur functions, which appear as Grassmannian Schubert polynomials. In joint work with N. Bergeron and F. Sottile, we give a combinatorial construction for the Schubert expansion of a Schubert polynomials times a Schur function. In this talk, I'll present the combinatorics of the k-Bruhat interval that led Bergeron and Sottile to construct a new quasisymmetric function whose Schur expansion encodes these structure constants and show how we used dual equivalence to obtain a combinatorial formula for the Schur coefficients.

Kelli Talaska - *Networks and the Deodhar decomposition of real Grassmannians*

We will discuss some of the combinatorics of the Deodhar decomposition of a real Grassmannian, which is a refinement of Postnikov's positroid stratification, which is in turn a refinement of the Schubert cell decomposition. The cells of the positroid stratification are indexed by Le-diagrams, and when we consider their intersections with the totally nonnegative part of the Grassmannian, we obtain a nice parametrization of each cell using planar networks, in the sense that each Plucker coordinate can be written as a generating function for certain families of paths in the network. In the Deodhar decomposition, cells are indexed by ”Go-diagrams”, a generalization of Le-diagrams. In this setting, we can also construct a (not necessarily planar) network which parametrizes the corresponding cell of the Deodhar decomposition. Thus we obtain a network characterization for the entire Grassmannian. This is a joint project with Lauren Williams, motivated by her work with Yuji Kodama.

Julianna Tymoczko - *Billey's formula and geometric Schubert calculus*

In 1999, Billey discovered a combinatorial formula associated a nonnegative integer polynomial to each ordered pair of Weyl group elements. Billey's formula is now recognized as key combinatorial interpretation of geometric Schubert calculus: it describes the image of an equivariant Schubert class in the equivariant cohomology of the flag variety under the map induced by inclusion of the torus-fixed points into the flag variety. We describe how this formula can be used to extend Schubert calculus to different subvarieties of the flag variety.

Maxim Kazarian - *Polylinear expansions in Schubert calculus*

We present a new technique of polylinear expansions in the computation of the Gysin homomorphism associated with the resolution of various kinds of degeneracy loci. Although the problem itself is well studied, the new method allows one to unify and to simplify essentially computations of degeneracy classes in many cases. The efficiency of the method will be demonstrated on the reviewed computations of classical degeneracy classes as well as in the derivation of closed formulas for a number of Thom polynomials.

Toshiaki Shoji - *Character sheaves on a symmetric space and Kostka polynomials*

It is known by Lusztig that Kostka polynomials has a geometric interpretation in terms of the intersection cohomology associated to the closure of nilpotent orbits in GL_n. This fact connects the representation theory of $GL_n(F_q)$ (F_q : finite field) with Kostka polynomials through the theory of character sheaves on GL_n. On the other hand, Bannai-Kawanaka-Song studied the Hecke algebras associated to the pair $(GL_{2n}(F_q), Sp_{2n}(F_q))$, and showed that its character theory has a close relationship with the theory of $GL_n(F_q)$. In this talk, I would like to give a geometric interpretation of the results of BKS in terms of the theory of character sheaves on the symmetric space GL_{2n}/Sp_{2n}. Also I will give an apporach to Kato's exotic nilpotent cone, whcih is a symmetric space analogue of the nilpotent cone for GL_n, based on the character sheaves on the symmetric space, and discuss the relationship with Kostka polynomials labelled by pairs of partitions. This is a joint work with K. Sorlin.

Piotr Pragacz - *Positivity in Schubert calculus*

Positivity results were a part of Schubert calculus already from the beginning. The Bertini-Kleiman transversality theorem implies positivity of many numbers appearing in Schubert calculus. Today positivity is a vivid area of research of algebraic geometry thanks to the monograph by Lazarsfeld. I shall report on some modern posivity results of Graham, Anderson, Pragacz-Weber, Mikosz-Pragacz-Weber. I shall put an emphasis on Thom polynomials and the “relative Schubert calculus”.