Arrangements (introduction)

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Introduction to the introduction

The goal of this introduction is to try to convince the novices that Hyperplane Arrangement theory interacts with a vast variety of areas of mathematics. Trying to do this in two hours restricts the means very much, for instance, no proofs will be given. Some of these interactions will be discussed in more depth in other lectures of the conference.

Setup and notation

By a hyperplane arrangement we understand the set $\mathcal{A}$ of several hyperplanes of an ℓ - dimensional affine space V over a field K. If all the hyperplanes are linear, i.e., passing through a common point (called 0) , then $\mathcal{A}$ is central. If 0 is the only common point then $\mathcal{A}$ is essential. Often we will order $\mathcal{A}$ and then write $\mathcal{A} = \{H_1, \ldots, H_n\}$.

Any time when it is convenient, we fix a linear basis $(x_1, \ldots, x_\ell)$ of V^* and identify V with K^ℓ using the dual basis in V. Then for each hyperplane H of K^ℓ we fix a degree 1 polynomial $\alpha_H \in S = K[x_1, \ldots, x_\ell]$ such that H is the zero locus of α_H. This polynomial is uniquely defined up to multiplication by a nonzero element from K. If $\mathcal{A}$ is central all α_H are homogeneous. We will abbreviate α_{H_i} as α_i.

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Combinatorics of arrangements

For many invariants of arrangements hyperplanes themselves are not needed; these invariants are determined by the combinatorics of arrangements. There are two essentially equivalent combinatorial objects that $\mathcal{A}$ determines: a geometric lattice and a simple matroid. We will briefly discuss the former referring the listener for details to the lectures by Richard Stanley.

For an arrangement $\mathcal{A}$ its intersection lattice $L = L(\mathcal{A})$ consists of intersections of all subsets of hyperplanes from $\mathcal{A}$ (including V itself as the intersection of the empty set of hyperplanes). The partial order on L is the reverse inclusion of subspaces. In particular the unique minimal element of L is V and the unique maximal element is $\bigcap_{i=1}^{n} H_i$ (even if it is $\emptyset$). $\mathcal{A}$ itself becomes the set of all elements of L following the minimal element, called atoms.

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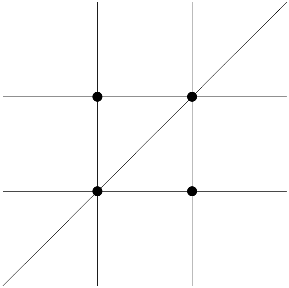
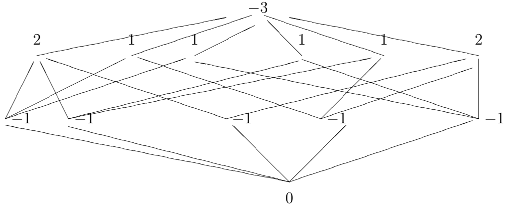
Example of L





$1 + 6t + 8t^2 + 3t^3$

Geometric lattices

The poset L is far from arbitrary. Let us collect the following three properties of L. We are assuming for simplicity that $\mathcal{A}$ is central.

(i) it is atomic, i.e., its every element is the join (the least upper bound) of some atoms;
(ii) it is ranked, i.e., every nonrefinable flag (chain) $(V < X_1 < \cdots < X_r = X)$ from V to a fixed $X \in L$ has the same number of elements, namely the codimension of X; (in lattice theory, this number is called the rank of X and denoted by $\operatorname{rk} X$);

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Geometric lattices

(iii) for every $X, Y \in L$ the following semimodular inequality holds

$$\operatorname{rk} X + \operatorname{rk} Y \geq \operatorname{rk}(X \vee Y) + \operatorname{rk}(X \wedge Y)$$

where the symbols $\vee$ and $\wedge$ denote respectively the join and meet (i.e., the greatest lower bound).

Lattices satisfying the above properties are called geometric.

The rank $\operatorname{rk} L$ of a geometric lattice L is the maximal rank of its elements. Clearly $\operatorname{rk} L \leq \ell$ and $\operatorname{rk} L = \ell$ if and only if the arrangement is essential.

Möbius function

An important invariant of L (as of every poset) is its Möbius function. It is the function $\mu : L \times L \to \mathbb{Z}$ satisfying the conditions $\mu(X, X) = 1$, $\mu(X, Z) = 0$ unless $X \leq Z$, and

$$\sum_{Y \in L, X \leq Y \leq Z} \mu(X, Y) = 0$$

for every $X, Z \in L$, $X < Z$. We put $\mu(X) = \mu(V, X)$ for every $X \in L$.

Example

If H is an atom of L then $\mu(H) = 1$. If $X \in L$ is of rank 2 with precisely k atoms below it then $\mu(X) = (k - 1)$.

For L the following generating function $\pi_L(t) = \sum_{X \in L} \mu(X)(-t)^{\operatorname{rk} X}$ is called the characteristic or Poincarè polynomial of L.

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Homology of a of a poset

For an arbitrary poset, all its flags (i.e., the linearly ordered subsets) form an (abstract) simplicial complex called the order complex on the set of all its elements. The homotopy invariants of this complex are attributed to the poset itself. If the poset is a lattice then by its homotopy invariants one usually means those of its subposet with the largest and smallest elements deleted. If again the poset is a lattice than the order complex can be substituted by another (usually much smaller) simplicial complex Ξ. It is defined on the set of all atoms $\mathcal{A}$ and a subset $\sigma \subset \mathcal{A}$ is a simplex (of dimension $|\sigma| - 1$) if $\bigvee(\sigma)$ is not the greatest element. This complex is called atomic and it is homotopy equivalent to the order complex. Below we will need the pair (Δ, Ξ) of complexes where Δ is the simplex on all the atoms.

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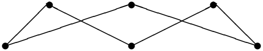
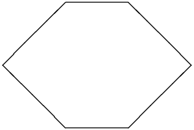
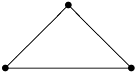
Complexes of a poset



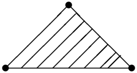


The order complex



Atomic complex

$\subset$



The pair of complexes

Exterior algebra

The lattice L determines the most important algebra associated with an arrangement. Let $\mathcal{A} = (H_1, \ldots, H_n)$ be an arrangement and k an arbitrary field (not necessarily equal to K). Let E be the exterior algebra over k with generators $e_1, \ldots, e_n$ in degree 1. Notice that the indices define a bijection from $\mathcal{A}$ to the generating set. Sometimes we will denote the generator corresponding to $H \in \mathcal{A}$ by e_H.

The algebra E is graded via $E = \oplus_{p=1}^{n} E^p$ where $E^1 = \oplus_{j=1}^{n} ke_j$ and $E^p = \bigwedge^p E^1$. The linear space E_j has the distinguished basis consisting of monomials $e_S = e_{i_1} \cdots e_{i_p}$ where $S = \{i_1, \ldots, i_p\}$ is running through all the subsets of $[n] = \{1, 2, \ldots, n\}$ of cardinality p and $i_1 < i_2 < \cdots < i_p$.

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DGA

The graded algebra E is a (commutative) DGA with respect to the differential ∂ of degree -1 uniquely defined by the conditions: linearity, $\partial e_i = 1$ for every $i = 1, \ldots, n$, and the graded Leibniz formula. Then for every $S \subset [n]$ of cardinality p

$$\partial e_S = \sum_{j=1}^{p} (-1)^{j-1} e_{S_j}$$

where S_j is the complement in S to its jth element.

For every $S \subset [n]$, put $\bigcap S = \bigcap_{i \in S} H_i$ and call S dependent if $\bigcap S \neq \emptyset$ and the set of linear polynomials $\{\alpha_i | i \in S\}$ is linearly dependent. Notice that being dependent is a combinatorial property - a set of atoms S is such if and only if $\operatorname{rk} \bigvee S < |S|$.

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OS algebra

Definition

Consider the ideal $I = I(\mathcal{A})$ of E generated by all e_S with $\bigcap S = \emptyset$ and all ∂e_S with S dependent. The algebra $A = A(\mathcal{A}) = E/I(\mathcal{A})$ is called the Orlik-Solomon (abbreviated as OS) algebra of $\mathcal{A}$. This algebra has been called also Brieskorn, BOS, and Arnold-Brieskorn.

Clearly the ideal I is homogeneous whence A is a graded algebra; we write $A = \oplus_p A^p$ where A^p is the component of degree p. In particular the linear spaces E^1 and A^1 are isomorphic and we will identify them.

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Notice that for any nonempty $S \subset [n]$ and $i \in S$ one has $e_i \partial e_S = \pm e_S$ whence I contains e_S for every dependent set S. This implies that A is generated as a linear space by the emages of e_S such that $\bigcap S \neq \emptyset$ and S is independent.

Sometimes it is useful to write the generators of I in a different form making clear that they are pure tensors in E. Indeed if $S = \{1, 2, \ldots, p\}$ then

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Homological interpretation

Since the Orlik - Solomon algebra is determined by L it is not very surprising that it has an interpretation in terms of the homology of L.

Let us first define the relative atomic complex $\nabla = \nabla(L)$. It is the chain complex over k with a linear basis consisting of all $\sigma \subset \mathcal{A}$; $\deg \sigma = |\sigma|$. The differential is defined by

$$d(\sigma) = \sum_{i | \bigvee(\sigma \setminus \{H_i\}) = \bigvee(\sigma)} (-1)^i \sigma \setminus \{H_i\}.$$

The chain complex ∇ is graded by L, i.e., $\nabla = \oplus_{X \in L} \nabla_X$ where ∇_X is the subcomplex generated by all σ with $\bigvee \sigma = X$.

It is easy to notice that ∇_X is the relative chain complex for the pair $(\Delta, \Xi)_X$ for the lattice $L_X = \{Y \in L | Y \leq X\}$.

Also notice that if σ is an independent subset of $\mathcal{A}$ then it is a cycle in ∇. Denote by $[\sigma]$ its homology class.

Multiplication

Define

$$\sigma \cdot \tau = \begin{cases} 0, & \text{if } \bigvee(\sigma \cup \tau) = \emptyset, \\ 0, & \text{if } \mathrm{rk}(\bigvee(\sigma \cup \tau)) \neq \mathrm{rk}(\bigvee(\sigma)) + \mathrm{rk}(\bigvee(\tau)), \\ \epsilon(\sigma, \tau)\sigma \cup \tau, & \text{otherwise.} \end{cases} \quad (1)$$

where $\epsilon(\sigma, \tau)$ is the sign of the permutation of $\sigma \cup \tau$ putting all elements of τ after elements of σ and preserving fixed orders inside these sets (the *shuffle* of σ and τ).

Theorem

The multiplication defined above converts ∇ to a (commutative) DGA graded by L. The correspondence $e_i \mapsto [\{i\}]$ generates an isomorphism $A \to H_*(\nabla; k)$ of graded algebras.

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Corollary

The following statements follow from the previous theorem.

Corollary

(i) The algebra A is graded by L, i.e., $A = \oplus_{X \in L} A_X$ where A_X is a graded linear subspace of A (in fact homogeneous) generated by e_S with $\bigvee S = X$ and $A_X A_Y \subset A_{X \vee Y}$.

(ii)The Hilbert series $H(A, t) = \pi_L(t)$.

This corollary uses not only the theorem but also certain property of homology of geometric lattices (namely the Folkman theorem).

Cohomology of M

From know on we assume $K = k = \mathbb{C}$.

Theorem

Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^\ell$ and M its complement,
(i) The de Rham homomorphism for M restricts to an isomorphism of the graded algebras $\mathcal{F}$ and $H^*(M, \mathbb{C})$ where $\mathcal{F}$ is the subalgebra of the algebra of closed holomorphic forms on M generated by all the forms $\frac{d\alpha_H}{\alpha_H}$ $(H \in \mathcal{A})$;
(ii) Let $[\omega]$ be the cohomology class of a form ω. Then the correspondence $[\frac{d\alpha_H}{\alpha_H}] \mapsto e_H$ defines a graded algebra isomorphism $H^*(M, \mathbb{C}) \simeq A$.

Remark. The theorem still holds if one defines all three algebras over $\mathbb{Z}$. In particular the cohomology of M is torsion free.

Poincaré polynomial

That theorem has important corollaries.

Corollary

(i) The Poincarè polynomial of M coincides with the characteristic polynomial $\pi_L(t)$ *of L.*
(ii) Space M is formal, i.e., the DGA of differential holomorphic forms on it is quasi-isomorphic to its cohomology algebra.

Braid arrangements

Example. Fix a positive integer n and consider the arrangement $\mathcal{A}_{n-1}$ in $\mathbb{C}^n$ given by linear functionals $x_i - x_j,\ 1 \leq i < j \leq n$. In fact $\mathcal{A}_{n-1}$ consists of all reflecting hyperplanes of the Coxeter group of type A_{n-1}. The complement M_n of this arrangement can be identified with the configuration space of n distinct ordered points in $\mathbb{C}$. Considering loops in this space makes it pretty clear that $\pi_1(M_n)$ is the pure braid group on n strings.

The natural way to study M_n is to project it to M_{n-1} ignoring the last coordinate of points in $\mathbb{C}^n$. This defines a fiber bundle projection that is the restriction to M_n of the projection $\mathbb{C}^n \to \mathbb{C}^n/X$ where X is a coordinate line (M_n is linearly fibered). The fiber of the projection is $\mathbb{C}$ without $n-1$ points. Repeating this process one obtains a sequence of such projections with decreasing n that ends at projecting to a point.

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Fiber-type arrangements

Generalizing that construction one obtains a recursive definition of a fiber-type arrangement.
For $X \in L$ we put $\mathcal{A}_X = \{H \in \mathcal{A} | H \supset X\}$ (cf. L_X).

Definition

An arrangement $\mathcal{A}$ in V is fiber-type if there is a line $X \in L(\mathcal{A})$ for which A_X is fiber-type and $M(\mathcal{A})$ is linearly fibered over $M(\mathcal{A}_X)$. Also arrangement of one hyperplane is fiber-type.

Using the sequence of consecutive fiber bundles it is possible to prove for every fiber-type arrangement that
(1) M is a $K[\pi, 1]$-space;
(2) $\pi_1(M)$ is a semidirect product of free groups.

Supersolvable lattices

It turns out that being fiber-type is a combinatorial property of an arrangement.

Let L here be an arbitrary geometric lattice. Then $X \in L$ is modular if

$$\operatorname{rk} X + \operatorname{rk} Y = \operatorname{rk}(X \vee Y) + \operatorname{rk}(X \wedge Y)$$

for every $Y \in L$. The lattice L is supersolvable if it contains a maximal flag of modular elements. If for an arrangement $\mathcal{A}$ its lattice is supersolvable we say $\mathcal{A}$ is.

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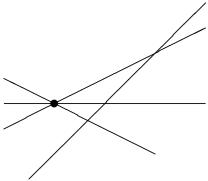
Examples

(1) All central arrangements of lines are supersolvable.

(2) A non-central arrangement $\mathcal{A}$ of lines is supersolvable if and only if there is a point $P \in L(\mathcal{A})$ such that for any other point $Q \in L(\mathcal{A})$ the line $PQ \in \mathcal{A}$.

S/S and not S/S



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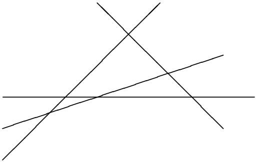

$not\ S/S$

(3) A Coxeter arrangement, i.e., the arrangement of all reflecting hyperplanes of a Coxeter groups, may be supersolvable or not. For instance, types A_n and B_n are supersolvable for all n, but type D_n is not for $n \geq 4$.

Remark. In spite of the last comment about D_n, for every reflection arrangement $\mathcal{A}$, i.e., the arrangement of all reflecting hyperplanes of a finite reflection group, $M(\mathcal{A})$ is $K[\pi, 1]$.

Modules of derivations

We consider central arrangements in $V \simeq \mathbb{C}^\ell$. Recall that $S = \mathbb{C}[x_1, \ldots, x_\ell]$ (or, invariantly, the symmetric algebra of the dual space V^*) .

Definition

A derivation of S is a $\mathbb{C}$-linear map $\theta : S \to S$ satisfying the Leibniz condition

$$\theta(fg) = \theta(f)g + g\theta(f)$$

for every $f, g \in S$.

The set $\mathrm{Der}(S)$ of all derivations is naturally an S-module. This is the graded free module of rank ℓ with a basis consisting of partial derivatives $D_i = \frac{\partial}{\partial x_i}$, $(i = 1, \ldots, \ell)$.

Module $D(\mathcal{A})$

The following S-module relates closer to the arrangement. Let $\mathcal{A}$ be a central arrangement and $Q = \prod_{H \in \mathcal{A}} \alpha_H$ its defining polynomial.

Definition

The module of $\mathcal{A}$-derivations is

$$D(\mathcal{A}) = \{\theta \in \mathrm{Der}(S) | \theta(Q) \in QS\}.$$

$D(\mathcal{A})$ is a graded submodule of $\mathrm{Der}(S)$ which is not necessarily free though. For every $\theta \in D(\mathcal{A})$ we still have $\theta = \sum_i \theta_i D_i$ with uniquely defined $\theta_i \in S$ but in general $D_i \notin D(\mathcal{A})$. θ is homogeneous if $\deg \theta_i$ does not depend on i and then this degree is called the degree of θ.

Free arrangements

Definition

A central arrangement $\mathcal{A}$ is free if the S-module $D(\mathcal{A})$ is free.

$D(\mathcal{A})$ is free if and only if it is generated by ℓ homogeneous generators. Another (Saito's) criterion says $D(\mathcal{A})$ is free if and only if it contains a system of ℓ homogeneous linearly independent over S derivations with the sum of there degrees equal $n = |\mathcal{A}|$.

Examples.

(1) Every central arrangement of lines is free.

(2) Consider the arrangement in $\mathbb{C}^3$ given by $Q = xyz(x + y + z)$. Then it can be seen that $D(\mathcal{A})$ does not contain two linearly independent derivations of degree less or equal one (one of such is the Euler derivation $\theta_E = \sum_i x_i D_i$). Since $n = 4$ Saito's criterion cannot be satisfied and $\mathcal{A}$ is not free.

Free arrangements

Definition

A central arrangement $\mathcal{A}$ is free if the S-module $D(\mathcal{A})$ is free.

$D(\mathcal{A})$ is free if and only if it is generated by ℓ homogeneous generators. Another (Saito's) criterion says $D(\mathcal{A})$ is free if and only if it contains a system of ℓ homogeneous linearly independent over S derivations with the sum of there degrees equal $n = |\mathcal{A}|$.

Examples.

(1) Every central arrangement of lines is free.

(2) Consider the arrangement in $\mathbb{C}^3$ given by $Q = xyz(x + y + z)$. Then it can be seen that $D(\mathcal{A})$ does not contain two linearly independent derivations of degree less or equal one (one of such is the Euler derivation $\theta_E = \sum_i x_i D_i$). Since $n = 4$ Saito's criterion cannot be satisfied and $\mathcal{A}$ is not free.

Free arrangements and combinatorics

The following results (by Terao) about free arrangements shows that the freeness is related to combinatorics.

Theorem

(i) Every supersolvable arrangement is free.
(ii) If an arrangement $\mathcal{A}$ is free then the characteristic polynomial

$$\pi_{L(\mathcal{A})}(t) = \prod_{i=1}^{\ell}(1 + b_i t)$$

where b_i are the degrees of the homogeneous generators of $D(\mathcal{A})$.

Terao Conjecture. The property of arrangement being free is combinatorial, i.e., it is determined by $L(\mathcal{A})$.
There are many partial results supported the conjecture.

Multiarrangements

A multiarrangement is a pair $(\mathcal{A}, m)$ where $\mathcal{A}$ is a central arrangement and $m : \mathcal{A} \to \mathbb{Z}_{\geq 0}$ is a multiplicity function.
There are several instances in arrangement theory where a multiplicity function appears naturally. Here are two examples.
(i) Suppose $\mathcal{A}$ is a central arrangement and $H_0 \in \mathcal{A}$. The restriction of $\mathcal{A}$ to H_0 is the arrangement $\mathcal{A}_{H_0} = \{H \cap H_0 \mid H \in \mathcal{A} \setminus \{H_0\}\}$. For every $\bar{H}$ from the restriction there is the natural mutiplicities $m(\bar{H}) = |\{H \in \mathcal{A} \mid H \cap H_0 = \bar{H}\}|$.
(ii) Let $\mathcal{A}$ be a reflection arrangement of a finite reflection group G. In the theory of invariants of G the following multiplicity is often used: $m(H) = o_H - 1$ where $H \in \mathcal{A}$ and o_H is the order in G of the reflection at H.
We refer to Yoshinaga's lecture for his criterion of freeness using multiarrangements.

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Resonance varieties

Let $\mathcal{A}$ be an essential central arrangement in $\mathbb{C}^\ell$ and $A = A(\mathcal{A})$. If $x \in A^1$ then $x^2 = 0$ whence the multiplication by x defines the differential $A \to A$ of degree +1, i.e., converts A into the cochain complex that we denote by (A, x).

Definition. A *p-th resonance variety* $R^p = R^p(M)$ is the (determinantal) subvariety of A^1 defined as $R^p = \{x \in A^1 \mid H^p(A, x) \neq 0\}$.
There are several properties of varieties R^p for arrangement complements that hold for all p, $0 \leq p \leq \ell$.

• (linearity of components) R^p is almost always reducible. Its irreducible components are linear subspaces of A^1.

• (general position) If $\sum_i x_i \neq 0$ then $R^p = 0$ for all p.

•(propagation of cohomology) If $H^p(A, x) \neq 0$ for some p then $H^q(A, x) \neq 0$ (i.e., $R^p \subset R^q$) for every q, $p \leq q \leq \ell$.

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Aomoto complex

Although the propagation of cohomology is a simple statement no straightforward proof of it is known (except for $p = 1$).

The existing proof is based on the fact which seems to be interesting by itself. Put $T = \mathbb{C}[x_1, \ldots, x_n]$ and consider the complex of free T-modules

$$A \otimes T : \; 0 \to A^0 \otimes T \to A^1 \otimes T \to \cdots \to A^\ell \otimes T \to 0$$

with differentials $a \otimes 1 \mapsto \sum_{i=1}^n e_i a \otimes x_i$ for every $a \in A^p$ and every p. The complex is called the Aomoto complex.

The interesting fact implying the propagation is that the Aomoto complex is exact except in $A^\ell \otimes S$ whence is a free resolution of an S-module $\mathcal{F}(\mathcal{A})$. This follows from a very general duality (called the BGG-correspondence) using a non-trivial result that A has a linear injective resolution as the exterior algebra module.

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The first resonance variety

The propagation property allows one to focus on the first non-vanishing R^p. For instance, for R^1 much more results are known.

First we projectivize the linear space and study an arrangement of projective hyperplanes in the complex projective space. The cohomology algebra of the porjectivized complement (that we still denote by A and call the OS algebra) is the graded subalgebra of the OS algebra of the central arrangement generated by $e_i - e_j$ for $i \neq j$.

In order to get most results about R^1 it suffices to work with a arrangements of lines in $\mathbb{P}^2$. Indeed intersect arbitrary $\mathcal{A}$ with a generic plane and apply the Lefschetz hyperplane section theorem.

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Local components

Here we describe the simplest components of R^1 (trivial in some sense).
Suppose $P \in \mathbb{P}^2$ is a point where k lines of $\mathcal{A}$ intersect, $k \geq 3$. Denote by $e_1, \ldots, e_k$ the respective generators of A. Then the linear subspace

$$V_P = \{x = \sum_{i=1}^{k} x_i e_i | \sum x_i = 0\}$$

of A^1 is a component (of dimension $k - 1$) of R^1. This component is called local component of R^1.

Notice that there is no uniformed upper bound for the dimensions of local components of arrangements. This makes these components different from all others.

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Nets in $\mathbb{P}^2$

Now we will discuss the non-local components. Our goal is to state the theorem that gives at least two different characterizations of them. For that we need to define the terms to be used.

First we need some special configurations of lines and points in $\mathbb{P}^2$.

Definition An arrangement $\mathcal{A}$ of lines partitioned in k blocks $\mathcal{A} = \cup_{j=1}^{k} \mathcal{A}_j$ is a *k-net* if for every point P which is the intersection of lines from different blocks there is a precisely one line from each block passing through P.
It is easy to see that all blocks have the same number of lines. We denote this number by d and say that the net is a (k, d)-net.

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Examples of nets

Nets can be defined purely combinatorially using an incidence relation. Then after identifying two blocks of a (k, d) -net with each other, every other block gives a Latin square of size d and these squares are pairwise orthogonal. If $k = 3$ identifying all blocks gives a multiplication table of a quasi-group.

A $(k, 1)$-net consists of k lines passing through a point with each block consisting of one line . This case is considered to be trivial. Clearly $(k, 1)$-nets correspond to local components of R^1.

Examples of nets

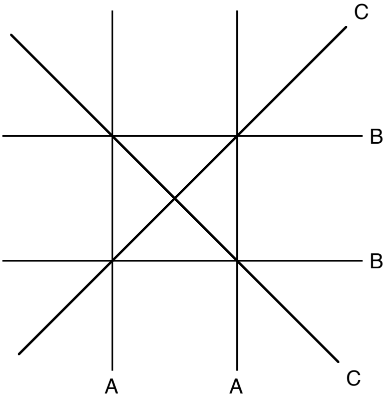
The combinatorial nets that can be realized in P^2 form a very special class (e.g., see the restrictions on k below). However there are plenty of examples of 3-nets in P^2. The simplest nontrivial one is given by all the reflection lines of the Coxeter group of type A_3. In appropriate coordinates the blocks can be described by the equation

$$[(x^2 - y^2)][(x^2 - z^2)][(y^2 - z^2)] = 0.$$

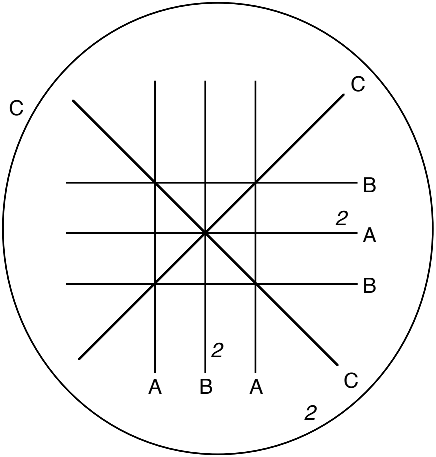
This is essentially the only example of a $(3,2)$-net.

As a classical example of a $(3,3)$-net one can use the famous generic Pappus configuration taking for $\mathcal{X}$ all the triple points.

A (3,2)-net on Coxeter A_3



A (3,4)-multinet on Coxeter B_3



Multinets and pencils

Viewing the picture of the projectivized Coxeter arrangement of type B_3 we see that it is not a net but becomes a kind of a net if we assign some multiplicities to the lines and points (this is another case were the multiplicities come naturally). Such structures we call multinets.

Now we need to recall pencils of plane curves. We will identify a homogeneous polynomial in three variables with the projective plane curve. A pencil of plane curves is a line in the projective space of homogeneous polynomials of some fixed degree. Thus for every its curves C_1, C_2 an arbitrary curve in the pencil (called a fiber) is $aC_1 + bC_2,\ [a:b] \in \mathbb{P}^1$. We consider only pencils whose fibers do not have a common component. Also curve of the form $\prod_{i=1}^{q} \alpha_i^{m_i}$, where α_i are different linear forms will be called completely reducible.

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Characterization of R^1

Now we can characterize the resonance variety R^1.

Theorem

Let $(\mathcal{A}, m)$ be a multi-arrangement of lines in $\mathbb{P}^2$ and, as usual, fix for each line H a defining linear polynomial α_H. The following are equivalent:

(i) There exists a partition on $(\mathcal{A}, m)$ that gives a (k, d)-multinet;

(ii) $\mathcal{A}$ is the union of all the factors of k completely reducible fibers of a pencil with irreducible generic fiber of degree d;

(iii) There is an irreducible component of R^1 for $\mathcal{A}$ of dimension $k - 1$.

Details for the theorem

More precisely
(i)$\Leftrightarrow$(ii) For the partition from (i) let $\mathcal{A}_1, \ldots \mathcal{A}_k$ be its blocks. Then the curves $C_i = \prod_{\ell \in \mathcal{A}_i} \alpha_\ell^{m(\ell)}$ are fibers of the pencil from (ii) (generated by any two of them);

(ii)$\Rightarrow$(iii) The cohomology classes in A^1 of the logarithmic forms $\frac{dC_i}{C_i} - \frac{dC_1}{C_1}$, $i = 2, 3, \ldots, k$ (see above), form a basis of the component of R^1 from (iii).

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Upper bound on k

Using the previous theorem it is possible to find a strong restriction on the dimension of R^1.

Theorem

A non-local component of R^1 has dimension either 2 or 3.

While there are plenty of examples with $\dim R^1 = 2$, there is only one known example of dimension 3. It corresponds to the Hesse pencil generated by $x^3 + y^3 + z^3$ and xyz that has 4 completely reducible fibers. The only other possible examples would correspond to (still unknown) nets with $d \geq 7$.

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Fundamental group of M

We will consider central arrangements only. Using as before the Lefschetz hyperplane section theorem it suffices to consider arrangements of planes in $\mathbb{C}^3$.

An important result about $\pi_1(M)$ is negative - this group is not determined by the lattice L. The example (by G. Rybnikov in 1994) consists of two arrangements of 13 hyperplanes each with $\operatorname{rk} L = 4$. It is still not very well understood, in particular no general group invariant is known that distinguishes π_1 for the two arrangemens in the example.

π_1 is generated by a set $\{z_1, z_2, \ldots, z_n\}$ which is in correspondence with $\mathcal{A}$. There are several known presentations of $\pi_1 = \pi_1(M)$ using this set. None of them is sufficiently simple to describe it in general in the introduction. We give several examples.

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Examples

Examples. (i) First we define generic arrangements. $\mathcal{A}$ in $\mathbb{C}^3$ is generic if every subset of it of rank 2 is independent. For a generic arrangement $\pi_1(M) = \mathbb{Z}^n$. Moreover $\pi_p(M) = 0$ for $1 < p < \ell$.

(ii) Let $\mathcal{A}$ be a Coxeter arrangement corresponding to a Coxeter group G. Then π_1 is the pure Artin group corersponding to G. (Moreover M is $K[\pi, 1]$.)

(iii) Let $\mathcal{A}$ be given by the polynomial $Q = x(x - y)(x + y)(2x - y + z)$. Then π_1 is given by presentation

$$\Big\langle x_1, x_2, x_3, x_4 \Big| x_1 x_2 x_4 = x_4 x_1 x_2 = x_2 x_4 x_1,$$
$$[x_1, x_3] = [x_2, x_3] = [x_4, x_3] = 1 \Big\rangle.$$

It gives $\pi_1 \simeq F_2 \times \mathbb{Z}^2$ where F_2 is the free group on two generators.

Open problems

In spite of known presentations the following questions about π_1 are open in general.
(1) Is it torsion-free?
(2) Is it residually nilpotent?
(3) Is it residually finite?
(4) Find a group invariant that distinguishes two groups in the Rybnikov example.

Higher homotopy groups

Although very little is known about $\pi_p(M)$ for all arrangements there is an interesting class of arrangements to which the result for generic arrangements can be generalized.

Let $\mathcal{A}$ be a projective arrangement in a space $\mathbb{P}^{\ell-1}$, $\mathcal{B}$ its proper subarrangement , and $\bar{\mathcal{B}} = \mathcal{A} \setminus \mathcal{B}$. Then $(\mathcal{A}, \mathcal{B})$ is a solvable extension if the following conditions are satisfied:

(i) No $H \in \bar{\mathcal{B}}$ form a dependent triple with $H_1, H_2 \in \mathcal{B}$ ($\mathcal{B}$ is closed in $\mathcal{A}$);

(ii) For every distinct $H_1, H_2 \in \bar{\mathcal{B}}$ there exists $H \in \mathcal{B}$ making a dependent triple with them; H is unique due to (i) and denoted by $h(H_1, H_2)$;

(iii) For every distinct $H_1, H_2, H_3 \in \bar{\mathcal{B}}$ the three hyperplanes $h(H_i, H_j)$ are dependent.

Note that $\operatorname{rk} \mathcal{A} - \operatorname{rk} \mathcal{B} \leq 1$.

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Hypersolvable arrangements

Definition An arrangement is hypersolvable if it has an ascending chain of subarrangements

$$\mathcal{A}_1 \subset \cdots \subset \mathcal{A}_\ell$$

with ranks strictly increasing and with each extension $(\mathcal{A}_{i+1}, \mathcal{A}_i)$ being solvable.

Theorem

Let $\mathcal{A}$ be a hypersolvable arrangement. Then
(1) M is a $K[\pi, 1]$ if and only if $\mathcal{A}$ is supersolvable;
(2) The first non-trivial higher homotopy group of M is π_{p-1} where p is the smallest integer such that
$\dim H^p(M, \mathbb{Q}) \neq \dim H^p(K[\pi_1, 1], \mathbb{Q})$.

Hypersolvble arrangements have a surprising algebraic property.

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Hypersolvble arrangements have a surprising algebraic property.

Quadratic and Koszul algebras

A distinguished class of graded algebras is formed by Koszul algebras. One of many equivalent definitions of this class is as follows. A graded connected algebra over an arbitrary field k is Koszul if the minimal free graded resolution of its trivial module k is linear, i.e., the matrices of all mappings in it have all their entries of degree one. If an algebra is Koszul then it is generated in degree one and the ideal of relations among generators is generated in degree two. An algebra with these two properties is a quadratic algebra.

A quadratic algebra $A = \oplus A_p$ $(A_0 = k)$ can be represented as $A = T(A_1)/J$ where $T(A_1)$ is the tensor algebra on the space A_1 in degree one and J is the graded ideal of relations. Then the quadratic algebra $A^! = T(A_1^*)/J^*$ where A_1^* is the dual linear space of A_1 and J^* is the annihilator of J is called the quadratic dual ("shriek") of A.

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Properties of Koszul algebras

The following implications are very well-known.

(1) A quadratic algebra A is Koszul if and only if $A^!$ is Koszul.

(2) If A is Koszul then the following relation between the Hilbert series holds

$$H(A,t)\cdot H(A^!,-t)=1.$$

The converse of (2) is false in general.

For the listeners who know what a Gröbner basis is we can give also a sufficient condition (for a commutative or graded commutative) algebras to be Koszul.

(3) If J has a quadratic (i.e., of degree 2) Gröbner basis then A is Koszul.

If an algebra A is not quadratic then one can work with its quadratic closure $\bar{A}$ where $\bar{A}=T(A_1)/(J_2)$ where (J_2) is the ideal generated by the elements of degree 2 of J.

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Koszul and quadratic OS algebras

Recall that an OS-algebra A is graded commutative and generated in degree one.

(1) A is not necessarily quadratic. There are several necessary conditions on L but no nice equivalent condition is known.

(2) On the other hand, if $\mathcal{A}$ is supersolvable then A is Koszul whence also quadratic. Moreover $\mathcal{A}$ is supresolvable if and only if the defining ideal of A has a quadratic Gröbner basis.

(3) If $\mathcal{A}$ is hypersolvable then $\mathcal{A}$ is not necessarily quadratic but $\bar{A}$ is Koszul.

(4) A being Koszul is equivalent to a topological property of the complement M. A space with this property is called a rational $K[\pi, 1]$ and can be defined using the rational model of the space. The definition may be given in Kohno's lectrure.

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One corollary of A being Koszul (equiv., M being a rational $K[\pi, 1]$) is a connection between $G = \pi_1(M)$ and A.

More precisely assume A is quadratic and let $\Gamma_1 = G \supset \Gamma_2 \supset \cdots \supset \Gamma_p \supset \cdots$ be the lower central series of G (i.e., $\Gamma_p = [G, \Gamma_{p-1}]$). The Abelian group $G^* = (\oplus_p \Gamma_P/\Gamma_{p-1}) \otimes \mathbb{Q}$ has the natural structure of a graded Lie algebra induced by taken commutators in G. If U is the universal enveloping algebra of G^* then $U \simeq A^!$ and the Hilbert series of U is

$$H(U, t) = \prod_{p \geq 1} (1 - t^n)^{-\phi_p}$$

where $\phi_p = \operatorname{rk} \Gamma_p/\Gamma_{p+1}$.
Now if A is Koszul then

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Open problems

The interesting open problems in that circle of questions are as follows.

Which of the following implications can be inverted in the realm of OS algebras:

The defining ideal of A has a quadratic Gröbner basis $\Longrightarrow$ A is Koszul $\Longrightarrow$ $H(A,t)\cdot H(A^!,-t)=1$.

Local systems

As above we consider arrangements of projective hyperplanes in $\mathbb{P}^{\ell-1}$. Every $x \in A_1$ defines a local system $\mathcal{L}(x)$ on M associated with the one-dimensional representation of $\pi_1(M)$ sending its generator corresponding to H_i to $\exp(2\pi\sqrt{-1}x_i)$. In other words, the 1-form $\omega_x = \sum_i x_i \frac{d\alpha_i}{\alpha_i}$ on M defines the integrable connection $\nabla_x = d + \omega_x$ on the (topologically trivial) vector bundle of rank 1 (here d is the exterior differential).
The sheaf $\mathcal{L}(x)$ is formed by the flat sections with respect to this connection. The cohomology $H^*(M, \mathcal{L}(x))$ can be defined as the cohomology of the complex

$$\Omega = (0 \to \Omega^* \to \Omega^1 \to \cdots \to \Omega^{\ell-1} \to 0)$$

which is the global de Rham complex on M provided with the twisted differential $d + \omega_x \wedge$.
The first part of the theorem about $H^*(M, \mathbb{C})$ gives the embedding ϵ of the complexes $\epsilon : (A, x) \to \Omega$.

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Comparison results

Theorem

Suppose that $x \in A_1$ and for all $X \in L$ (such that $\mathcal{A}_X$ is irreducible), the sum $\sum_{i \in X} x_i$ is not a positive integer. Then the embedding ϵ is a quasiisomorphism.

Corollary

For every $a \in \bar{A}_1$ and every p

$$\dim H^p(M, \mathcal{L}(x)) \geq \dim H^p(A, x).$$

Characteristic varieties

The set of all rank 1 local systems on M can be identified with the torus $H^1(M, \mathbb{C}^*) = \mathrm{Hom}(\pi_1(M), \mathbb{C}^*) \cong (\mathbb{C}^*)^n$. For an integer $p \geq 1$ the characteristic variety Σ^p of M is the algebraic subvariety of the torus consisting of those local systems $\mathcal{L}$ for which $H^p(M, \mathcal{L}) \neq 0$. The characteristic and resonance varieties relate as follows.

Theorem

(i) The tangent cone of Σ^1 at the torus identity $\mathbf{1}$ coincides with R^1.
(ii) Let V be an irreducible component of the resonance variety R^1. Then the image of V under the map $x \mapsto \mathcal{L}(x)$ is a universal covering of an irreducible component (passing through $\mathbf{1}$) of Σ^1. Any positive dimensional component of Σ^1 containing $\mathbf{1}$ is covered in this sense by a component of R^1.

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Classical hypergeometric integral

Let $\lambda = (\lambda_1 \lambda_2, \lambda_3) \in \mathbb{C}^3$ be a triple of parameters and fix $x \in \mathbb{C} \setminus \{0, 1\}$. Then

$$\Phi(u, \lambda, x) = (1 - u)^{\lambda_1} u^{\lambda_2} (1 - xu)^{\lambda_3}$$

is a multivalued holomorphic function on the complement M in $\mathbb{C}$ to the set $1, 0, x^{-1}$. The logarithmic 1-form

$$\omega_\lambda = \frac{d\Phi}{\Phi} = -\lambda_1 \frac{du}{1 - u} + \lambda_2 \frac{du}{u} - \lambda_3 \frac{xdu}{1 - xu}$$

is well-defined and holomorphic on M.

$\pi_1(M)$ is a free group with 3 generators $\gamma_1,\ \gamma_2$ and γ_3 that can be represented as loops starting at a fixed point $P \in \mathbb{C}$ and going once around respectively $1,\ 0$, and x^{-1}. Thus putting $\rho(\gamma_i) = exp(-2\pi\sqrt{-1}\lambda_i)$ defines a representation $\pi_1 \to \mathbb{C}^*$ whence a rank one dual local systems $\mathcal{L} = \mathcal{L}_\lambda$ and $\mathcal{L}^\vee = \mathcal{L}_{-\lambda}$ on M.

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Generalization to several variables

Now let $\mathcal{A}$ be an arbitrary hyperplane arrangement and $\lambda : \mathcal{A} \to \mathbb{C}$ (complex weights). Then we define a multivalued function as

$$\Phi = \prod_{H \in \mathcal{A}} \alpha_H^{\lambda(H)}$$

and define the local systems $\mathcal{L}$ and $\mathcal{L}^\vee$ exactly as above. Recall that $\mathcal{L}$ can be defined from the exact sequence of sheaves

$$0 \to \mathcal{L} \to \Omega^0 \to \Omega^1 \to \cdots \to \Omega^\ell \to 0$$

where the differential is twisted: $\nabla = d + \omega_\lambda \wedge$.
The local system $\mathcal{L}$ ($\mathcal{L}^\vee$) determines the cohomology $H^*(M, \mathcal{L})$ (homology $H_*(M, \mathcal{L}^\vee)$) and a non-degenerate pairing $H^p(M, \mathcal{L}) \times H_p(M, \mathcal{L}^\vee) \to \mathbb{C}$.

This pairing can be defined by the integral

$$\langle \omega, \sigma \rangle = \int_\sigma \Phi\omega$$

where $\omega \in \Omega^p(M, \mathcal{L})$ is a closed form and σ is a locally smooth cycle with values in $\mathcal{L}^\vee$. This formula is well-defined due to the 'twisted Stokes theorem':

$$\langle \omega, \partial\sigma \rangle = \langle \nabla(\omega), \sigma \rangle.$$

In particular in the classical case we recover the classical hypergeometric integral

$$\int_\gamma (1-u)^{\lambda_1} u^{\lambda_2} (1-xu)^{\lambda_3} f(u) du$$

where γ is a twisted cycle in the complement of $\mathbb{C}$ to three points.

Families of arrangements

In the classical theory, it is not assumed that x is fixed. This corresponds in higher dimension to considering a family of arrangements depending on parameters whose combinatorial structures are isomorphic. In this case even to describe this moduli space of arrangements with a fixed combinatorial type is a hard problem in general.

Assuming that this structure is known there is another simplifying assumption that the weight function λ is in general position. One can choose the general position so that the cohomology $H^p(M, \mathcal{L})$ vanishes unless $p = \ell$. Under these assumptions there is a tool to deal with a family of arrangements called the Gauss-Manin connection that can be computed in some cases. For examples we refer the listeners to Varchenko's and Kohno's lectures.

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Motion planning

From several topics of applied arrangement theory I will talk only about topological robotics.

Let X be a topological space, thought of as the configuration space of a mechanical system. Given two points $A, B \in X$, one wants to connect them by a path in X; this path represents a continuous motion of the system from one configuration to the other. A solution to this motion planning problem is a rule (algorithm) that takes $(A, B) \in X \times X$ as an input and produces a path from A to B as an output.

Let PX denote the space of all continuous paths $\gamma : [0, 1] \to X$, equipped with the compact-open topology, and let $f : PX \to X \times X$ be the map assigning the end points to a path: $f(\gamma) = (\gamma(0), \gamma(1))$. The map f is a fibration whose fiber is the based loop space ΩX. The motion planning problem consists of finding a section s of this fibration.

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Topological complexity

The section s cannot be continuous, unless X is contractible. M.Farber has defined $\mathbf{TC}(X)$, the topological complexity of X, as the smallest number k such that $X \times X$ can be covered by open sets $U_1, \dots, U_k$, so that for every $i = 1, \dots, k$ there exists a continuous section $s_i : U_i \to PX, f \circ s_i = \mathrm{id}$.

Farber's topological complexity has various properties allowing one to obtain several lower and upper bounds for it in terms of other invariants. However precise computation of $\mathbf{TC}(X)$ for concrete X is often a challenging problem.

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TC(M)

Arrangement complements (as well as configuration spaces of points in $\mathbb{R}^n$) can be very naturally viewed as configuration spaces of mechanical systems. For instance, for a braid arrangement the complement M appears for the system of several robots on a large plane. The complement of an arbitrary arrangement in $\mathbb{C}^\ell$ would appear if a robot has 2ℓ parameters and the hyperplanes represent linear obstructions.

The known results about **TC**(M) are as follows:
(1) If $\mathcal{A}$ is an arrangement of n hyperplanes in general position in $\mathbb{C}^\ell$ then $\mathbf{TC}(M(\mathcal{A})) = \min\{n+1, 2\ell+1\}$.

(2) Let $\mathcal{A}$ be a Coxeter arrangement of classical types (A,B, or D). Then $\mathbf{TC}(M(\mathcal{A}) = 2\,\mathrm{rk}(\mathcal{A})$.

Probably more details will be given in Dan Cohen's talk.

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