Arrangements and Algebraic Geometry

Masahiko Yoshinaga
(Kyoto University)

Arrangements of Hyperplanes
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4 Vector bundles and plane curves

Today we are in $\ell = 3$, consider $\mathbb{C}^3$ or $\mathbb{P}^2$.

- Some conditions are automatically satisfied. (e.g., local freeness, freeness of restricted multiarrangement.)
- Some numerical invariants are easily computed.

4 Vector bundles and plane curves

Setting;

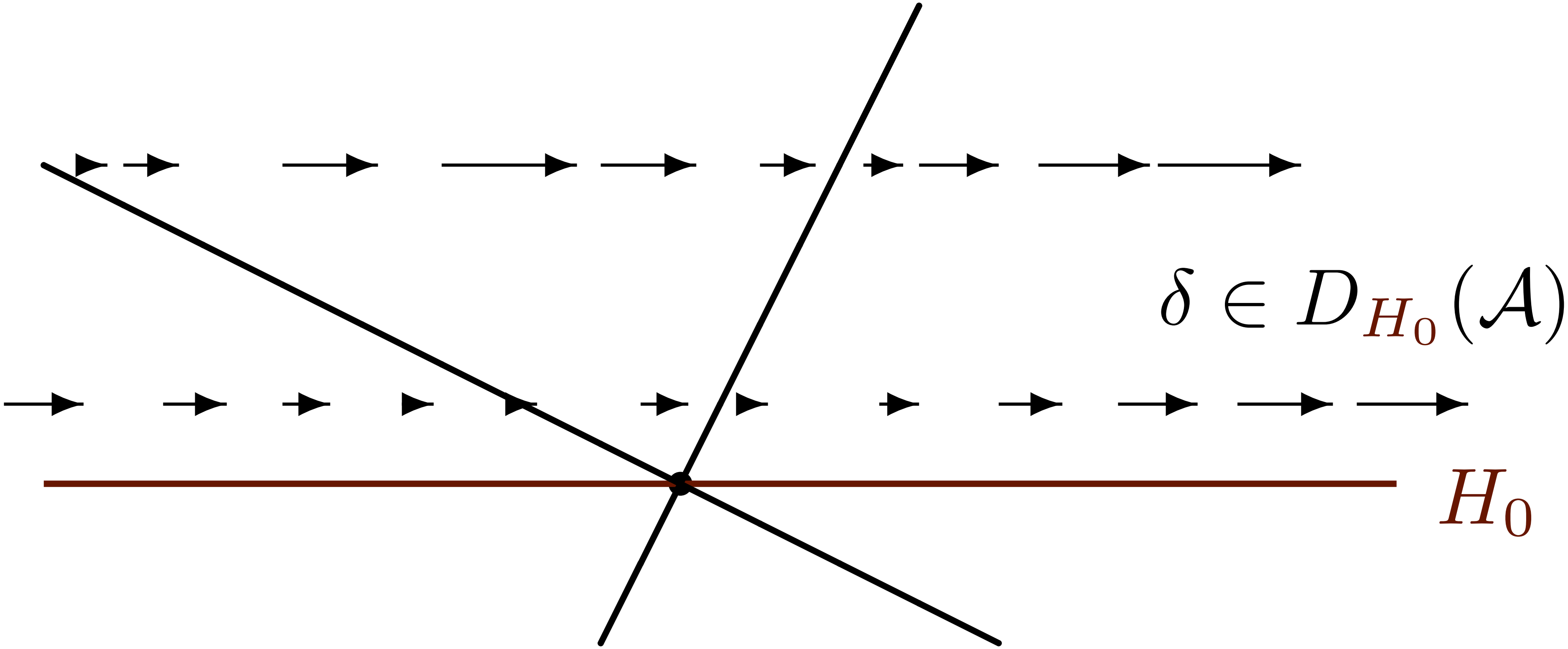
- $\mathcal{A}$ arrangement in $\mathbb{C}^3$, $\mathcal{A} = \{H_0, H_1, \ldots, H_n\}$ (Note $\sharp\mathcal{A} = n + 1$).
- H_0 sometimes plays as "Hyperplane at ∞"

Main object is rank 2 bundle:

$$\mathcal{E}(\mathcal{A}) := \widetilde{D_{H_0}(\mathcal{A})}.$$

4 Vector bundles and plane curves

$$\delta \in D_{H_0}(\mathcal{A}) = \{\delta \in D(\mathcal{A}) \mid \delta\alpha_{H_0} = 0\}$$



4.1 Chern classes

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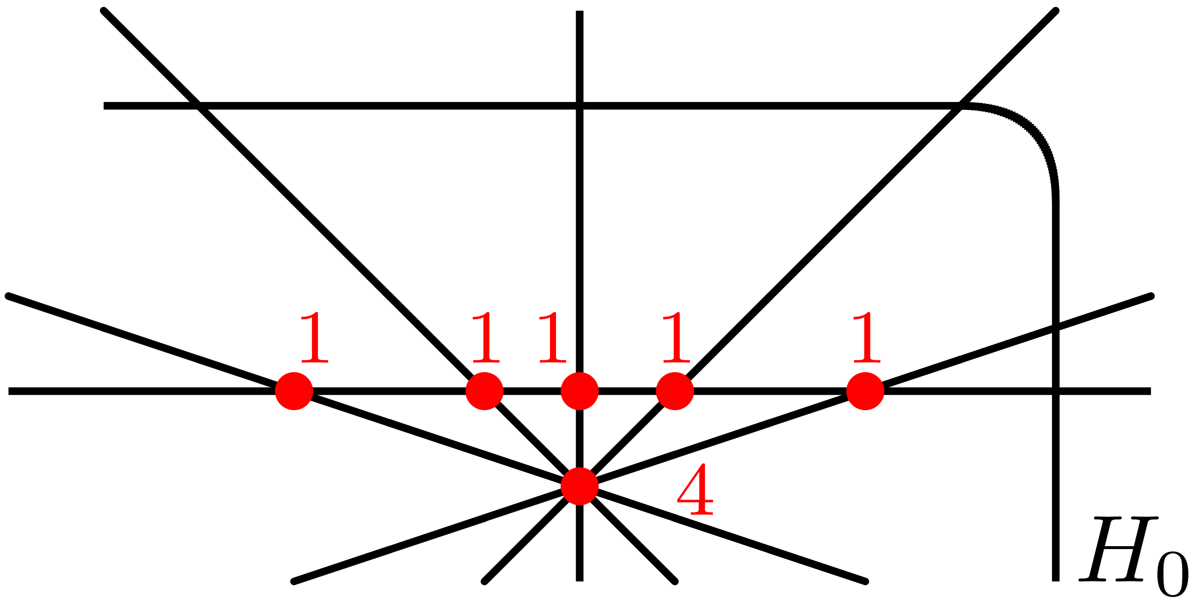
Thm. (Schenck)

- $c_1(\mathcal{E}(\mathcal{A})) = -n$,
- $c_2(\mathcal{E}(\mathcal{A})) = \sum_p \mu(p)$, where p runs intersections which is not on H_0



$c_1(\mathcal{E}(\mathcal{A})) = -6.$

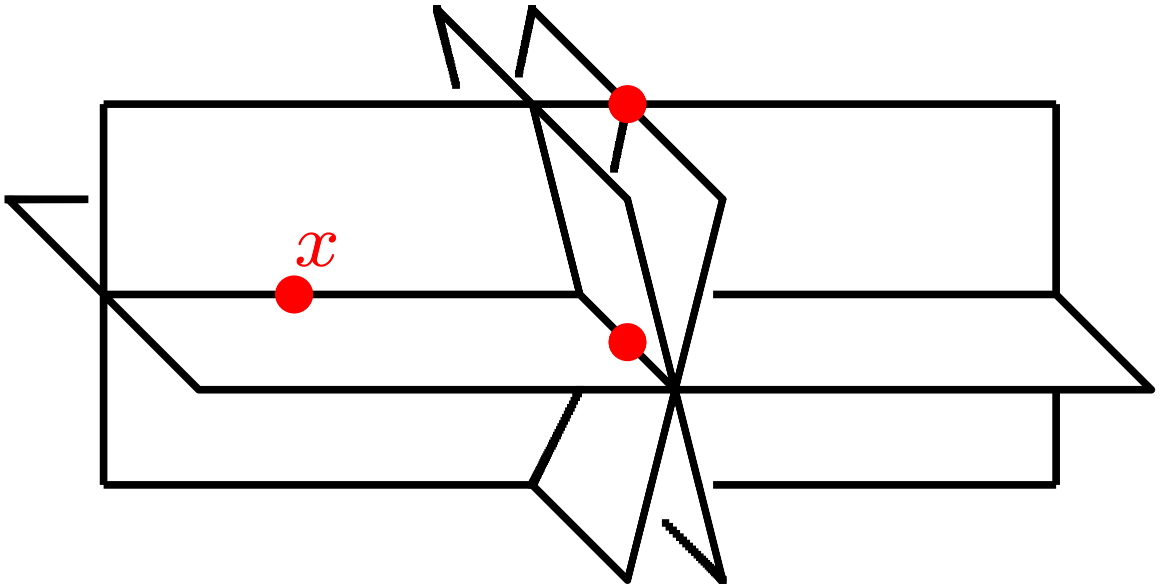
$c_2(\mathcal{E}(\mathcal{A})) = 9.$

$$c_t(\mathcal{E}(\mathcal{A})) = 1 - 6t + 9t^2 = (1-3t)^2.$$

4.1 Chern classes

Def. $\mathcal{A}$ is *locally free* if $\forall x \neq 0$,

$$\mathcal{A}_x = \{H \in \mathcal{A} \mid H \ni x\} \text{ is free.}$$



Rem. $\ell = 3 \Longrightarrow$ automatically locally free.

4.1 Chern classes

Thm. (Mustaţă-Schenck)

$\mathcal{A}$ is a central arrangement in $\mathbb{C}^{\ell+1}$.

(1) $\mathcal{E}(\mathcal{A}) = D_{H_0}(\mathcal{A})$ is a vector bundle iff $\mathcal{A}$ is locally free.

(2) Assume $\mathcal{A}$: loc free. Let $c_i := c_i(\mathcal{E})$ the i-th Chern number. Then

$$\chi(\mathcal{A}, t) = (t-1)(t^\ell - c_1 t^{\ell-1} + \cdots \pm c_\ell)$$

4.2 What makes arrangement free?

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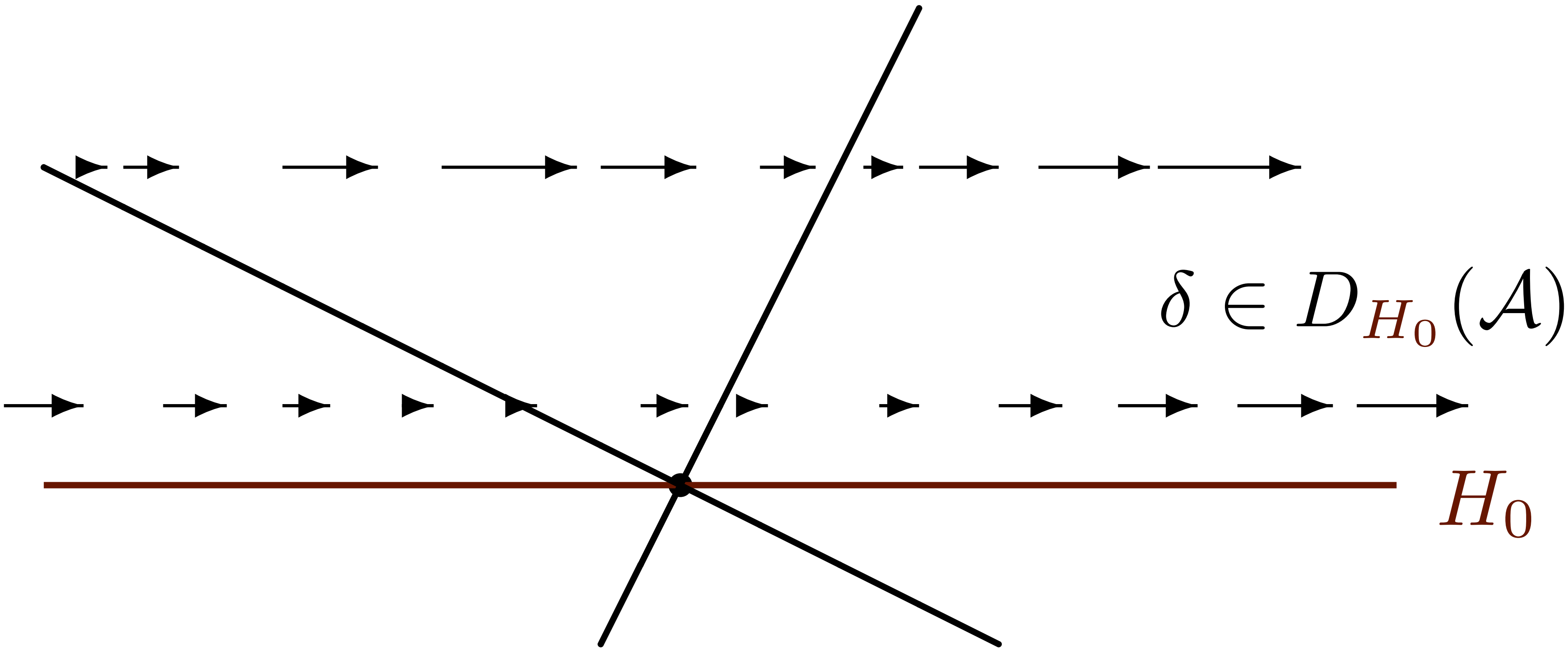
Thm. (Ziegler)

$$\delta \in D_{H_0}(\mathcal{A}) \Longrightarrow \delta|_{H_0} \in D(\mathcal{A}^{H_0}, \mathbf{m}_{H_0}).$$



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Thm. (Z)

$\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1, d_2, \ldots, d_\ell)$,

$\Rightarrow (\mathcal{A}^{H_0}, \mathbf{m}_{H_0})$ is free with $\exp = (d_2, \ldots, d_\ell)$

Cor. $\mathcal{A}$ is free iff $D(\mathcal{A}^{H_0}, \mathbf{m}_{H_0})$ is free, and $D_{H_0}(\mathcal{A}) \to D(\mathcal{A}^{H_0}, \mathbf{m}_{H_0})$ is surjective.

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automatically satisfied, when $\ell = 3$

4.2 What makes arrangement free?

$\ell = 3$, $\mathcal{A}$ is an arrangement in $\mathbb{C}^3$. $H_0 \in \mathcal{A}$.

<u>Prop.</u> $\mathcal{A}$ is free iff the restriction map

$$D_{H_0}(\mathcal{A}) \longrightarrow D(\mathcal{A}^{H_0}, \mathbf{m}_{H_0})$$

is surjective.

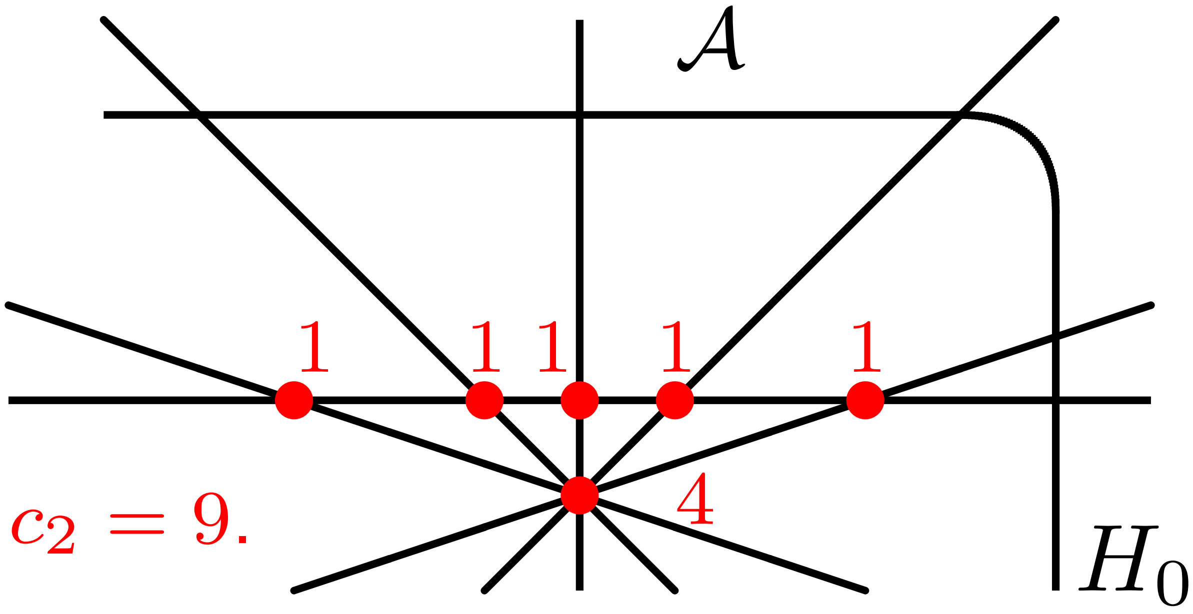
How far are they from surjective?

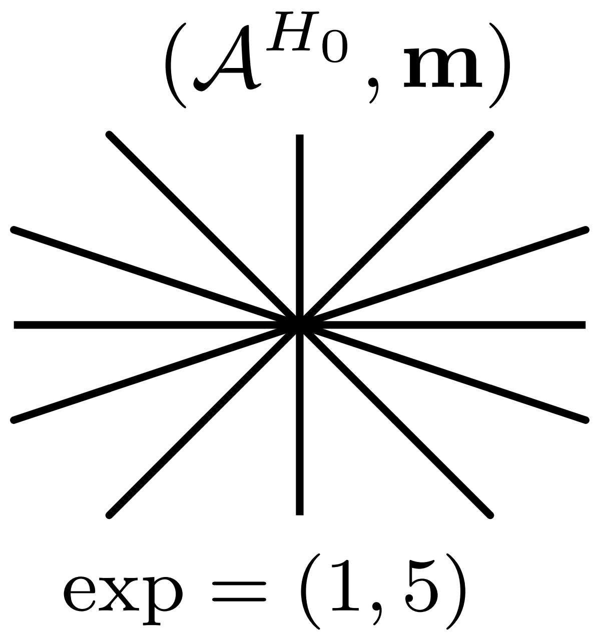
4.2 What makes arrangement free?

Thm. Put $\exp(\mathcal{A}^{H_0}, \mathbf{m}) = (d_2, d_3)$. Then

$$\dim \mathrm{Coker}(D_{H_0}(\mathcal{A}) \to D(\mathcal{A}^{H_0}, \mathbf{m})) = c_2 - d_2 d_3.$$

Cor. $\mathcal{A}$ is free iff $c_2 = d_2 d_3$.





$\dim \mathrm{Coker} = 9 - 5 = 4$ (Not Free)

4.2 What makes arrangement free?

Thm. Put $\exp(\mathcal{A}^{H_0}, \mathbf{m}) = (d_2, d_3)$. Then $\dim \mathrm{Coker}(D_{H_0}(\mathcal{A}) \to D(\mathcal{A}^{H_0}, \mathbf{m})) = c_2 - d_2 d_3$.

(*Proof*). Set $M \subset D(\mathcal{A}^{H_0}, \mathbf{m})$ is the image of the restriction map.

$$0 \longrightarrow D_{H_0}(\mathcal{A}) \xrightarrow{\alpha_0} D_{H_0}(\mathcal{A}) \longrightarrow M \longrightarrow 0$$

$\therefore HS(M, t) = (1-t) \cdot HS(D_{H_0}(\mathcal{A}), t)$

On the other hand, $HS(D_{H_0}(\mathcal{A}), t) \overset{\text{ST-formula}}{\Longrightarrow} c_2$

Then compare with $HS(D(\mathcal{A}^{H_0}, \mathbf{m}), t)$.

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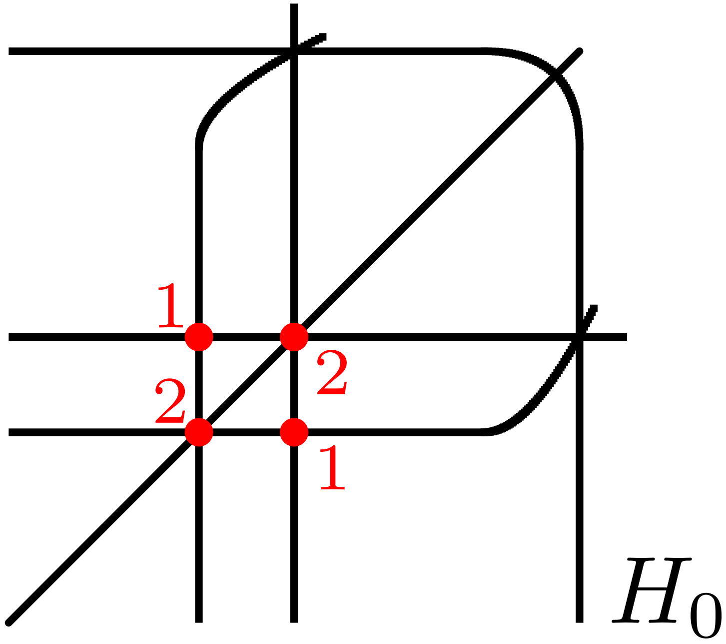
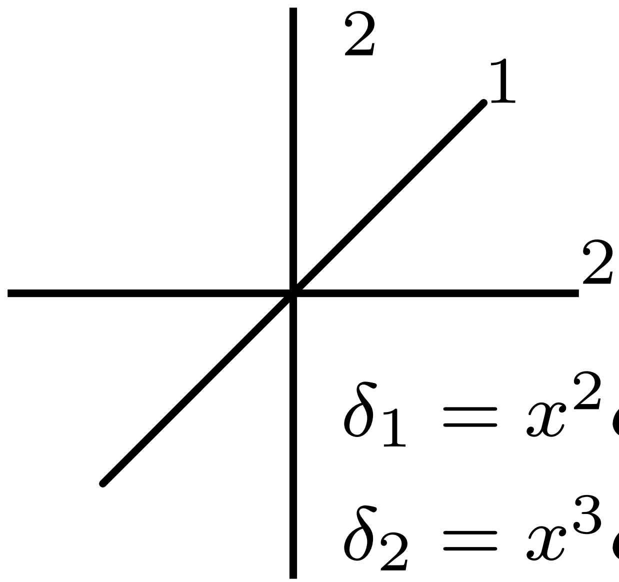
Ex. $\mathcal{A}$



$c_2 = 6$

(Free)

$(\mathcal{A}^{H_0}, \mathbf{m}_{H_0})$



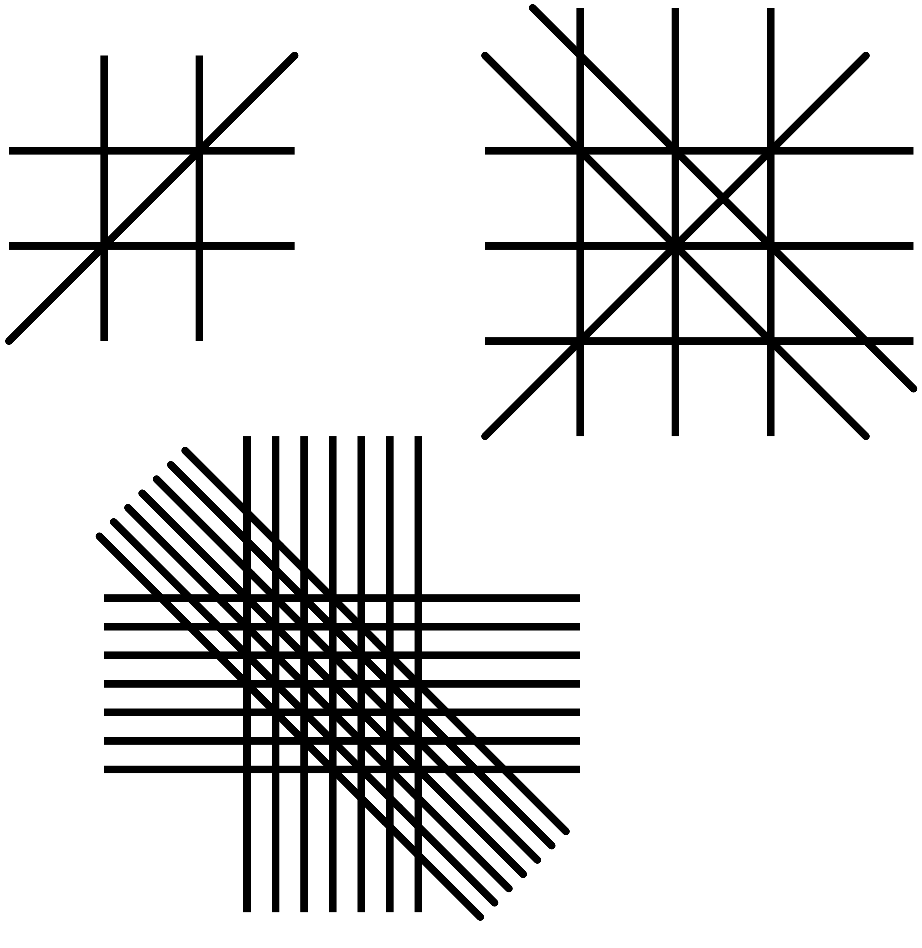
$\delta_1 = x^2\partial_x + y^2\partial_y$

$\delta_2 = x^3\partial_x + y^3\partial_y$

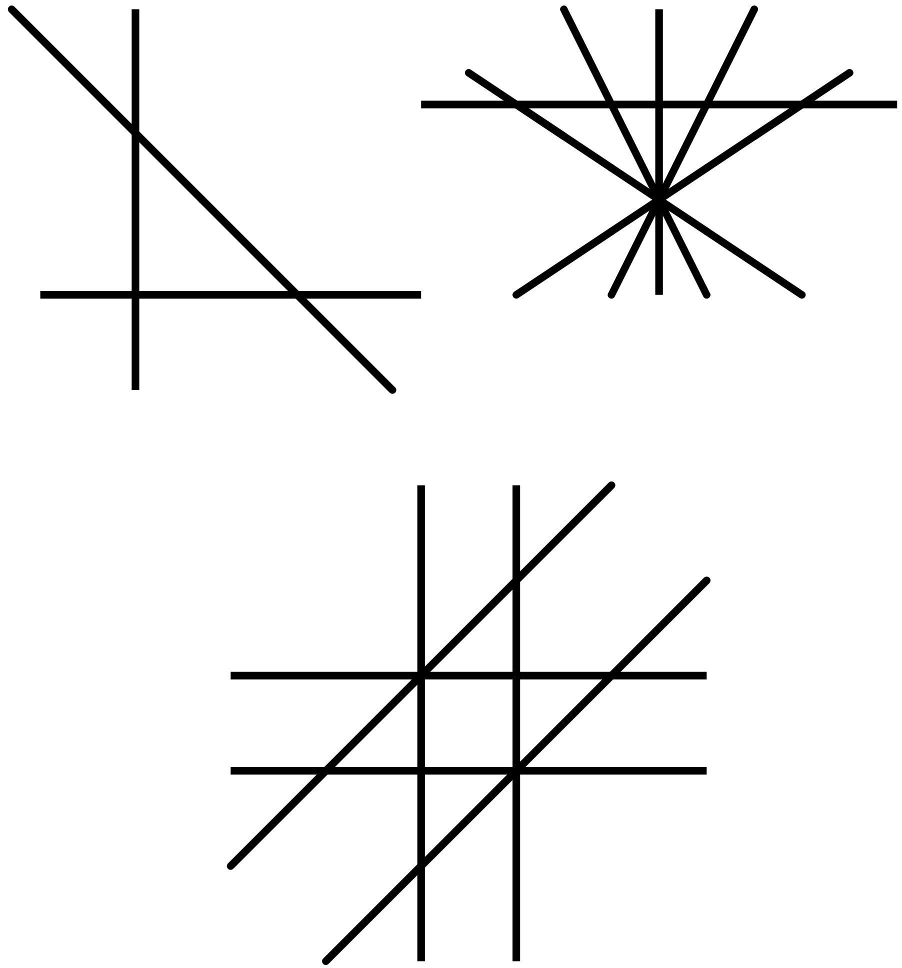
$\exp = (2, 3)$

4.2 What makes arrangement free?

Free



Not Free



4.2 What makes arrangement free?

Problems:

(Terao) $\mathcal{A}_1, \mathcal{A}_2$ arrangements in $\mathbb{C}^\ell$, $\mathcal{A}_1$ is free and $L(\mathcal{A}_1) \cong L(\mathcal{A}_2)$, then $\overset{???}{\Longrightarrow}$ Is $\mathcal{A}_2$ free?

(Saito) If $\mathcal{A}$ is free, then what is the homotopy type of the complement?

Are there further restrictions on $L(\mathcal{A})$ and the homotopy types?

4.3 Free arr's and plane curves

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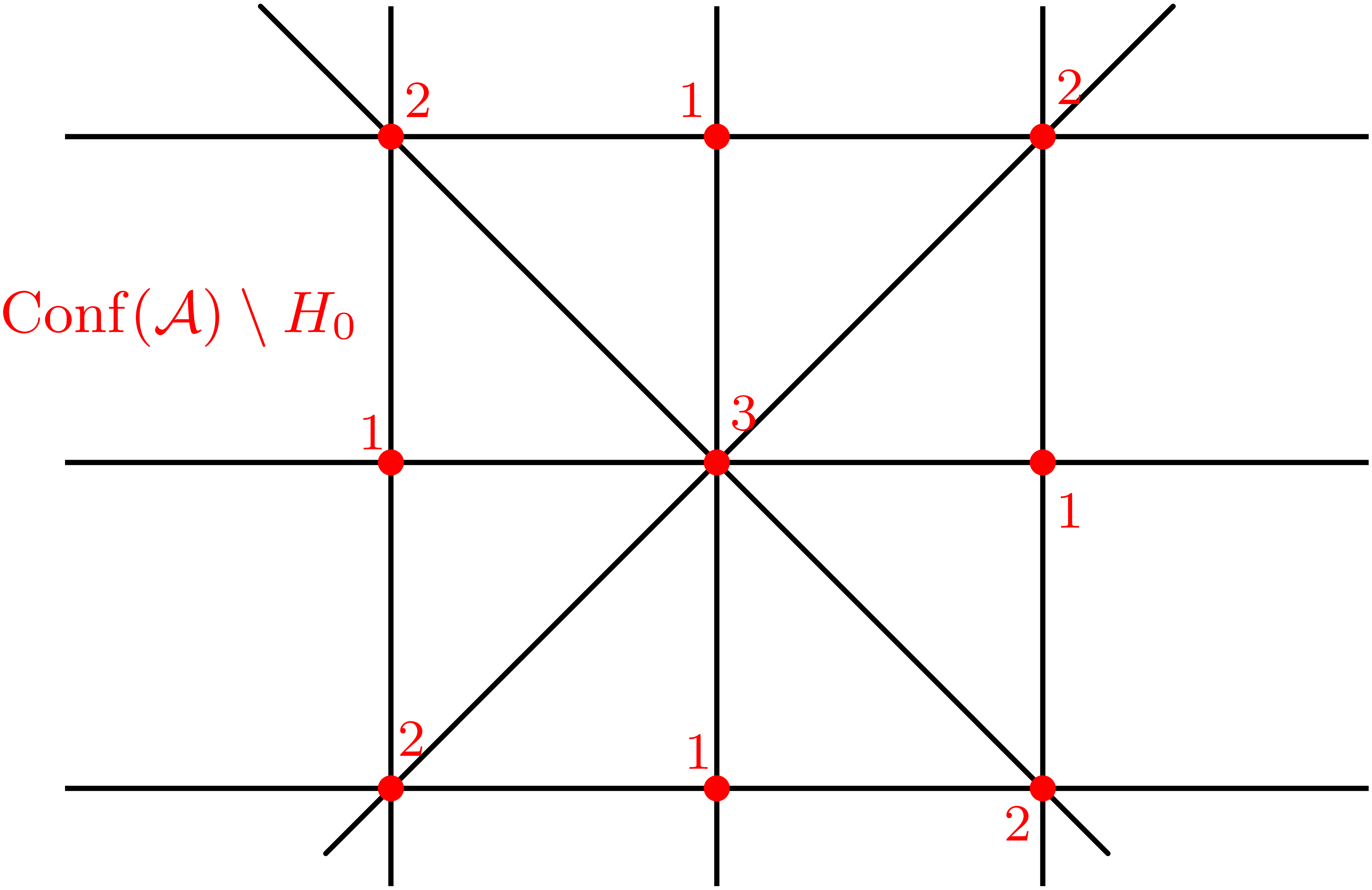
$\mathcal{A}$: an arr's in $\mathbb{C}^3$, $H_0 \in \mathcal{A}$.

Def.
$\mathrm{Conf}(\mathcal{A}) = \{(p, \mu(p)) \mid p$: intersection in $\mathbb{P}^2\}$, and $\mathrm{Conf}(\mathcal{A}) \setminus H_0 := \{(p, \mu(p)) \in \mathrm{Conf}(\mathcal{A}) \mid p \notin H_0\}$

Def. Let C_1, C_2 be curves in $\mathbb{P}^2$ which do not share components. Then
$C_1 \cdot C_2 := \{(p, \mathrm{mult}_p(C_1, C_2) \mid, p \in C_1 \cap C_2\}$

4.3 Free arr's and plane curves



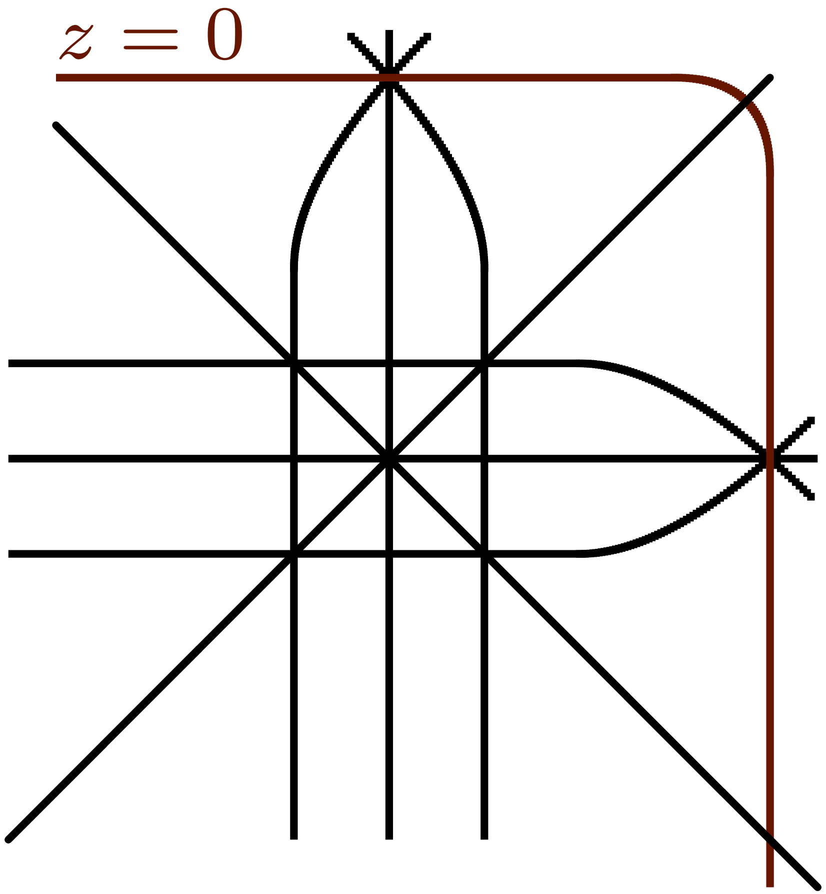
4.3 Free arr's and plane curves

<u>Thm</u>. If $\mathcal{A}$ is free with $\exp = (1, d_2, d_3)$, then there exist plane curves C_i of degree d_i such that

$$\mathrm{Conf}(\mathcal{A}) \setminus H_0 = C_1 \cdot C_2.$$

(*Proof*) $\delta_1, \delta_2 \in D_{H_0}(\mathcal{A})$ a basis. Take α be a generic linear form. Let us define a plane curve C_i by $\delta_i \alpha = 0$. The equality holds. (q.e.d.)
Look closely at...

4.3 Free arr's and plane curves



$$Q = xyz(x^2 - y^2) \times (x^2 - z^2)(y^2 - z^2)$$

$$\theta_E = x\partial_x + y\partial_y + z\partial_z$$

$$\delta_1 = x(x^2 - z^2)\partial_x + y(y^2 - z^2)\partial_y$$

$$\delta_2 = x(x^4 - z^4)\partial_x + y(y^4 - z^4)\partial_y$$

$$\delta_1, \delta_2 \in D_{H_0}(\mathcal{A})$$

Consider $\delta_i(\varepsilon x + y)$

$(0 < \varepsilon < 1)$

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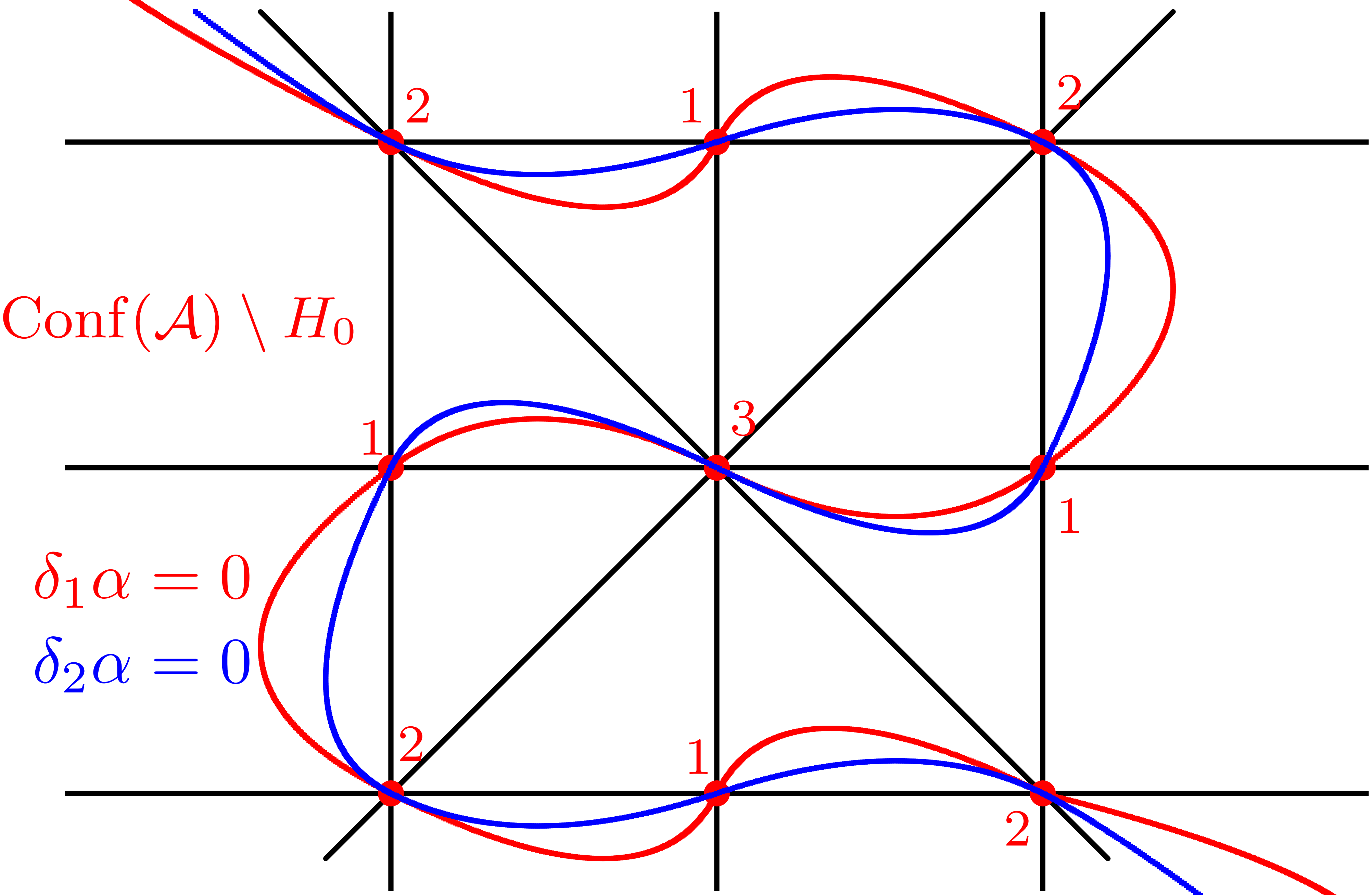

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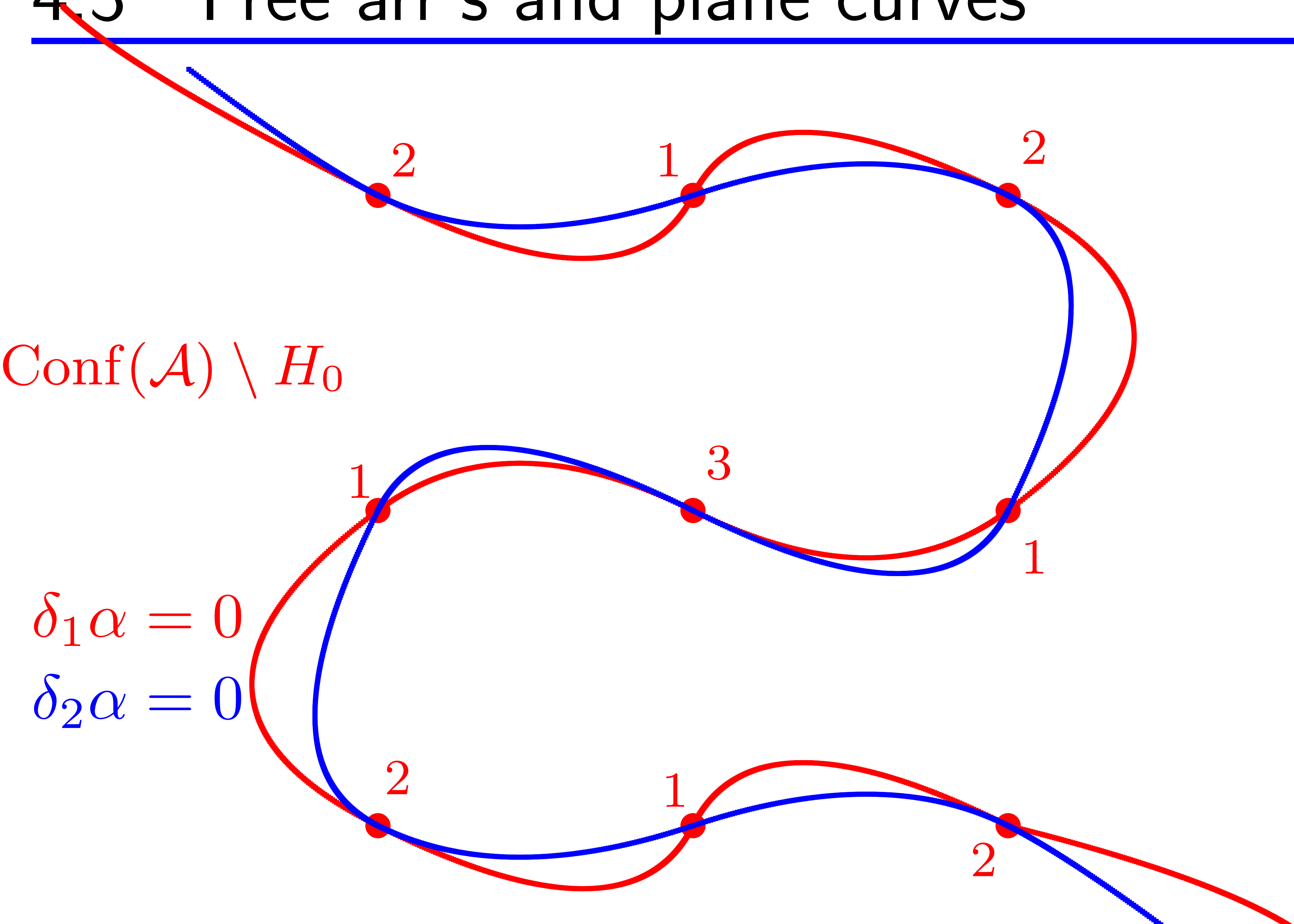

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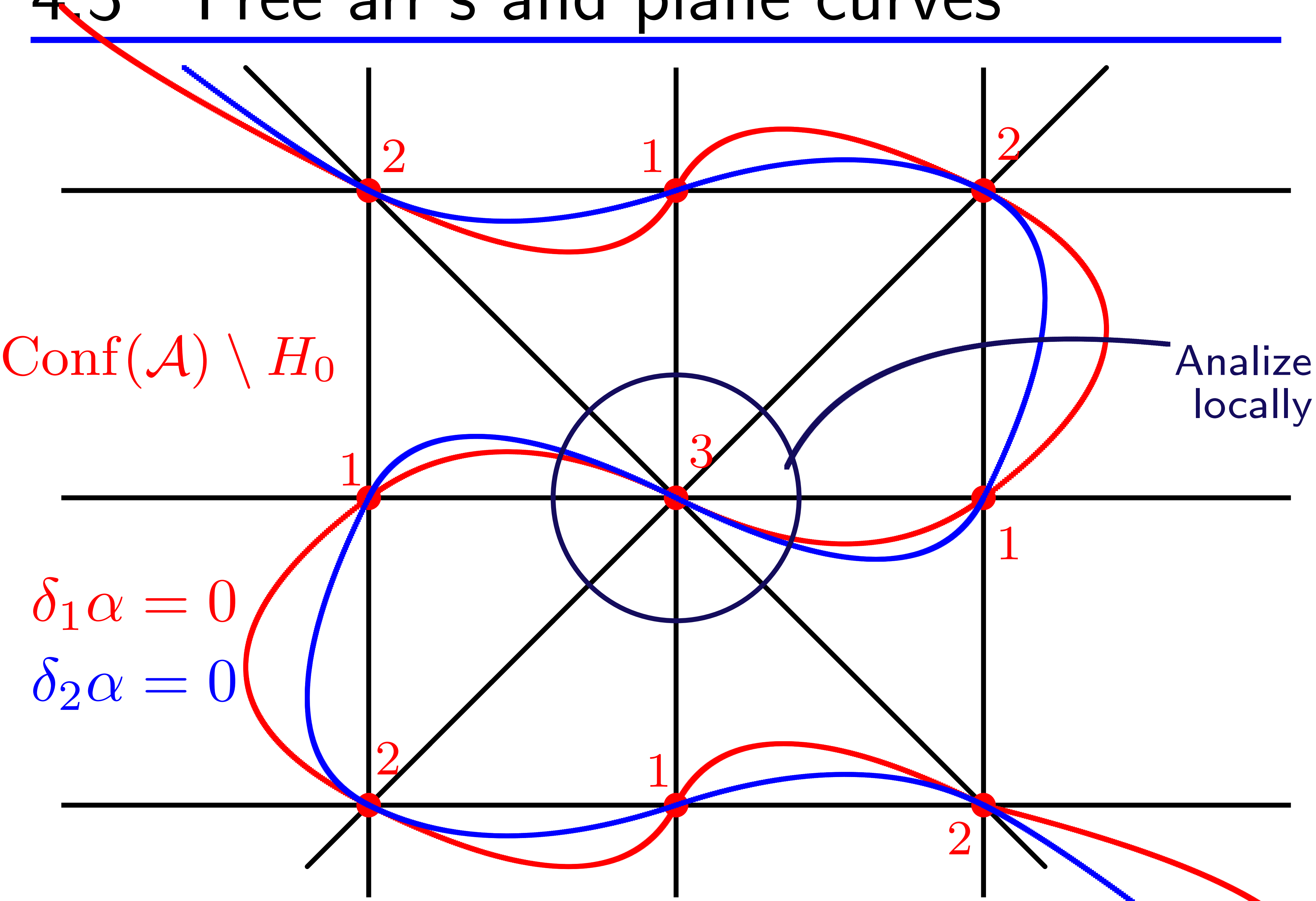

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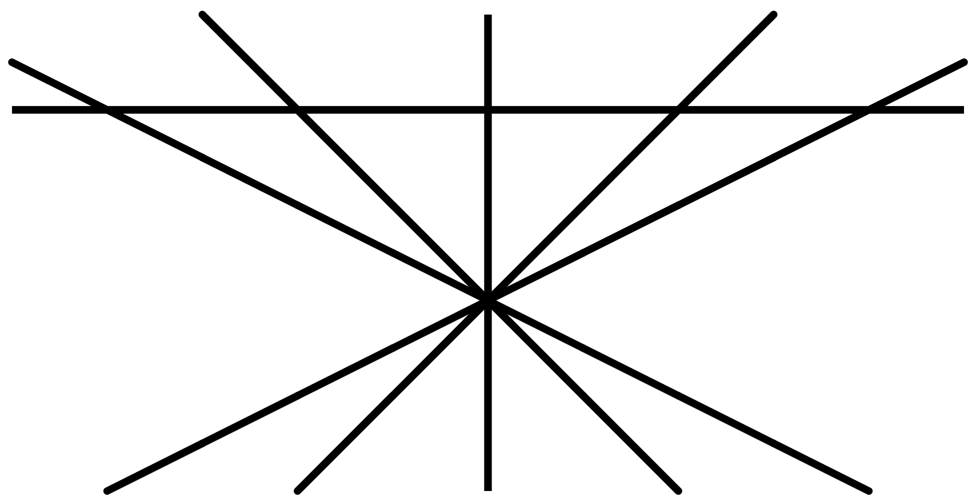

is not free.

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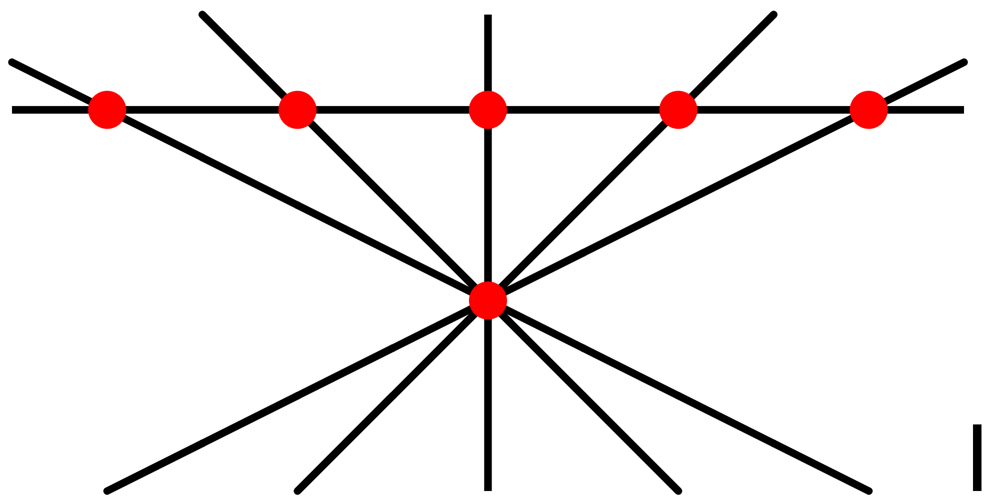

$$c_t(\mathcal{E}) = (1-3t)^2$$

If it is free, $\exp = (3,3)$

is not free.

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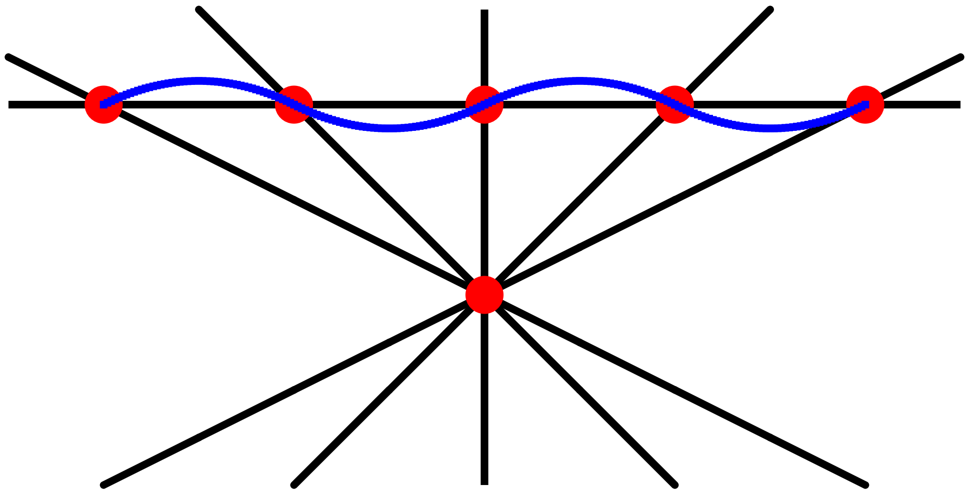

$$c_t(\mathcal{E}) = (1 - 3t)^2$$

If it is free, $\exp = (3, 3)$

Contradicts Bezout

is not free.

4.3 Free arr's and plane curves

Thm.(Again) If $\mathcal{A}$ is free, then plane curves $\exists C_i$ s.t.

$$\mathrm{Conf}(\mathcal{A}) \setminus H_0 = C_1 \cdot C_2.$$

Final Questions

(1) Is the converse true?

(2) By Serre construction, $\exists 0 \to \mathcal{O}(c_1) \to \mathcal{E}(\mathcal{A}) \to I_Z \to 0$, ($I_Z$: defining ideal of Conf). What is the extension class?

(3) $\exists$ any applications? (e.g. jumping lines etc.)

Thank you very much for your attention!