Arrangements

and

Algebraic Geometry

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Arrangements of Hyperplanes
Sapporo, 1–13, Aug. 2009

-1 Welcome!

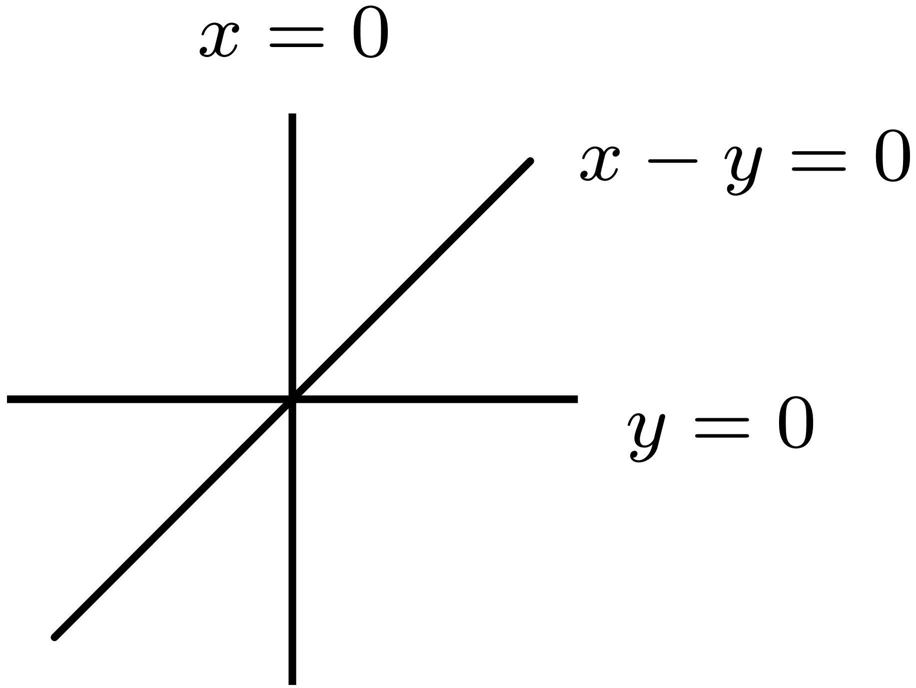
- Focus: Log bundle of arrangements (Sheaf of logarithmic vector fields / forms).

$$D(\mathcal{A}, \mathbf{m}), \Omega^1(\mathcal{A}, \mathbf{m})$$

- Theme: Arrangements and AG.
  Two directions.
  - AG helps understanding Arrangements.
  - "Arrangements" as special varieties.
    (Arrangements cause AG problems.)

0 Contents

1 Introduction



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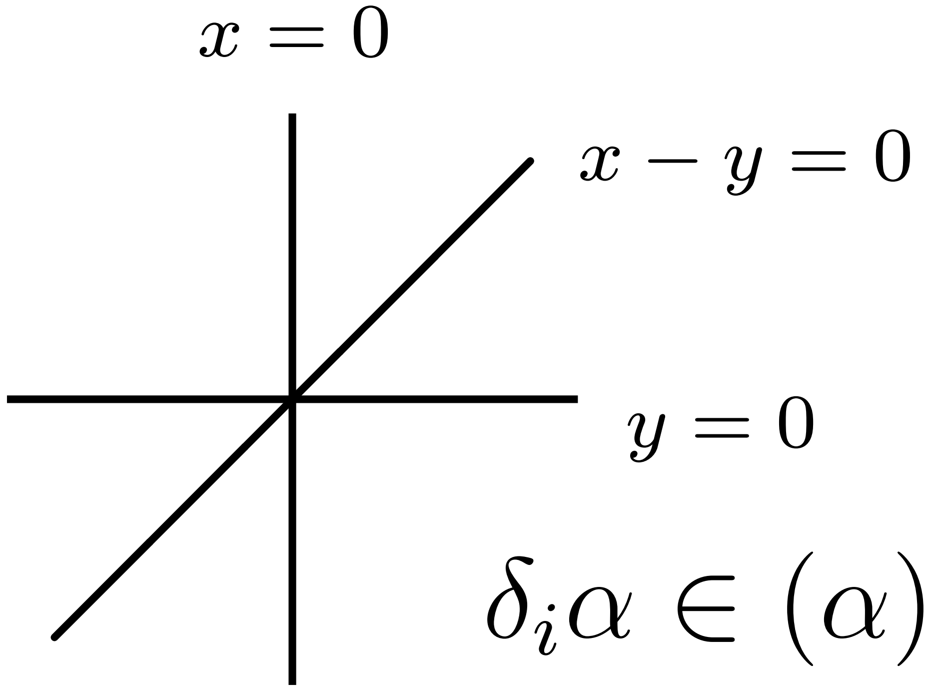

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$$\delta_i \alpha \in (\alpha), \text{ for } \alpha = x, y, x - y$$

e.g. $\delta_2(x - y) = x^2 - y^2 = (x - y)(x + y)$

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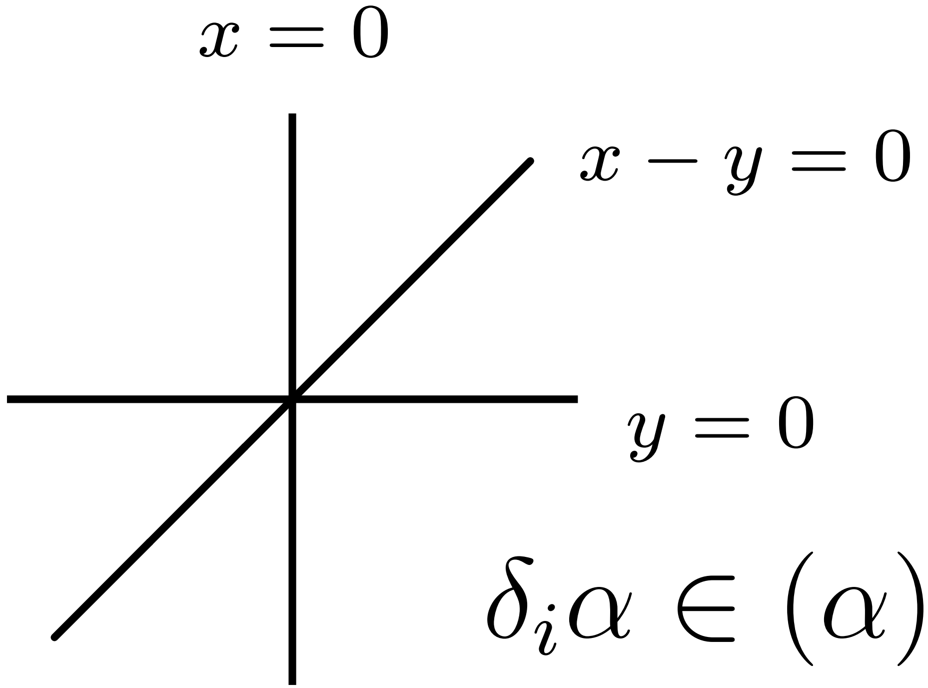

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$$\det \begin{pmatrix} x & y \\ x^2 & y^2 \end{pmatrix} = xy^2 - x^2y = -xy(x - y)$$

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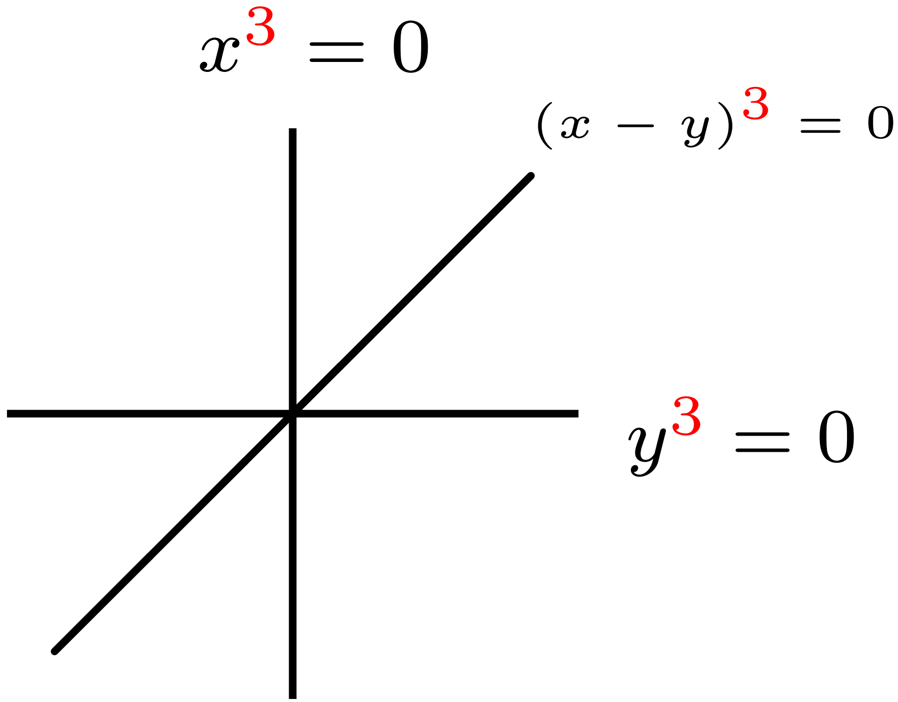

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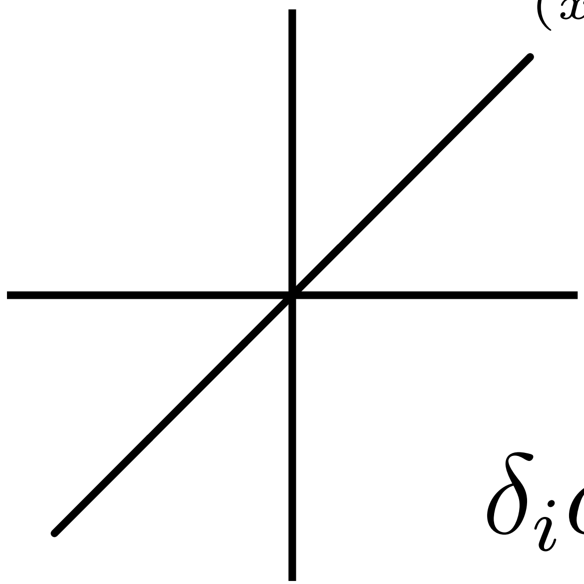

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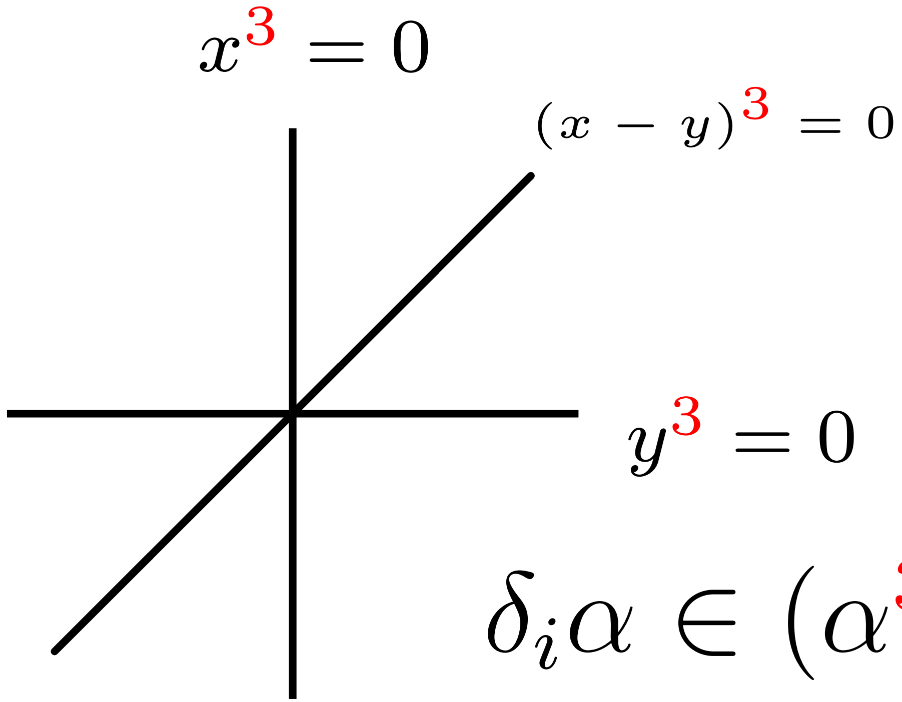

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$$\det \begin{pmatrix} x^3(x-2y) & y^3(-2x+y) \\ x^4(3x-5y) & y^4(-5x+3y) \end{pmatrix} = 6x^3y^3(x-y)^3$$

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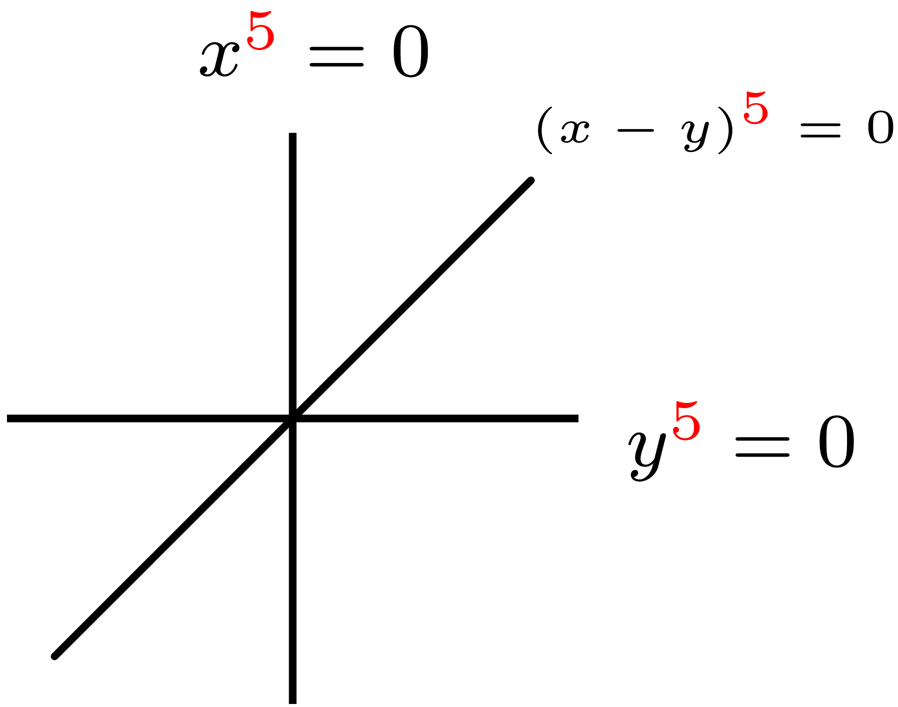

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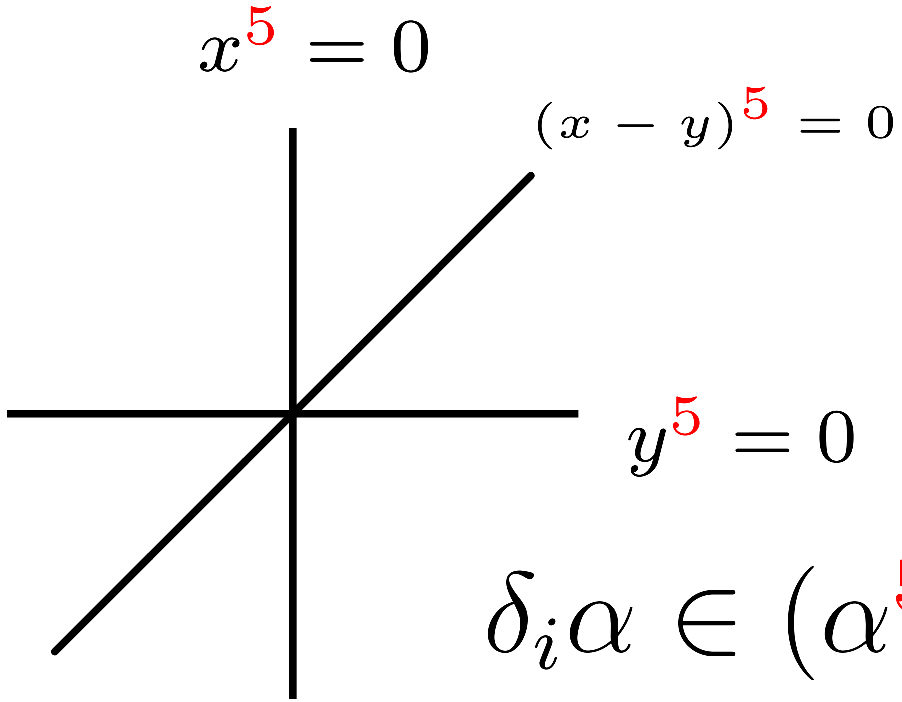

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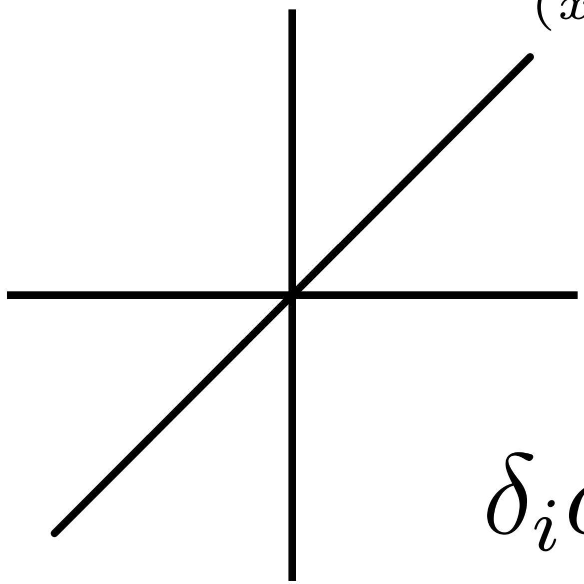
$x^5 = 0$

$(x-y)^5 = 0$



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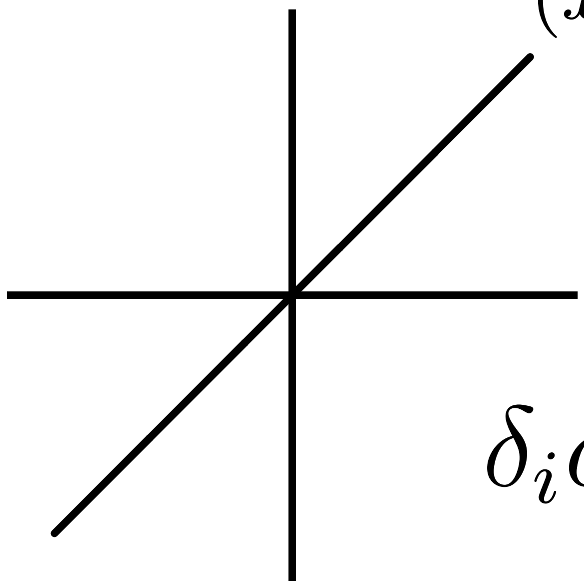

$$\exists \delta_1 = f_1 \partial_x + f_2 \partial_y$$

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s.t.

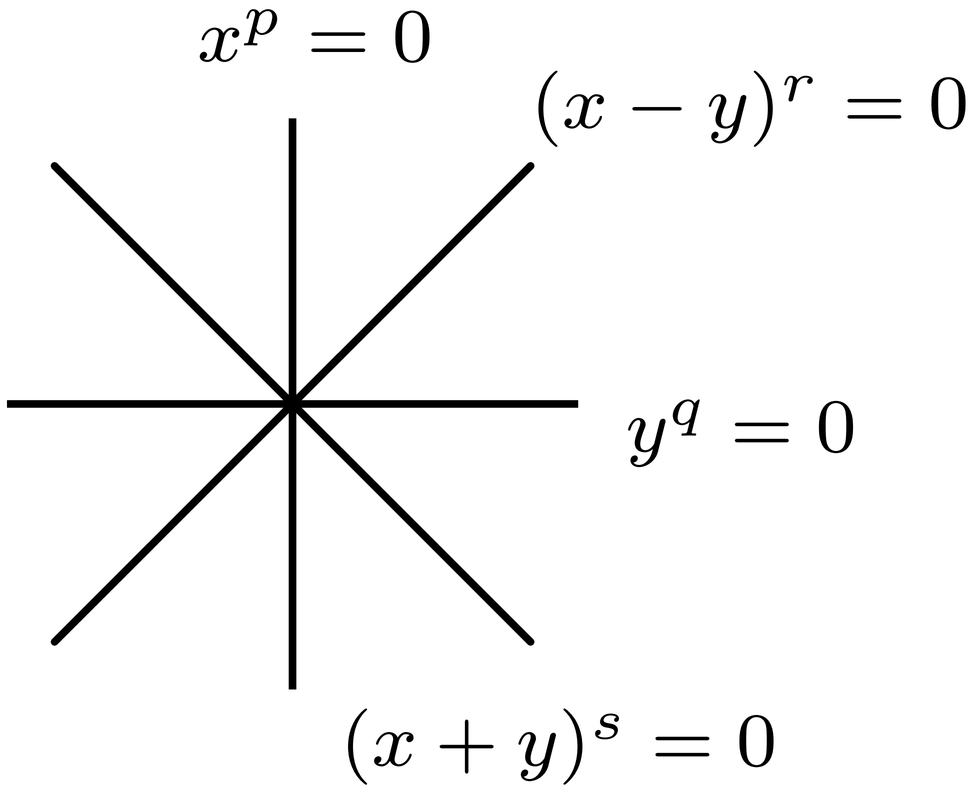
$$\delta_i \alpha \in (\alpha^n), \text{ for } \alpha = x, y, x-y, \quad n = p, q, r,$$

$$\det \begin{pmatrix} f_1 & f_2 \\ g_1 & g_2 \end{pmatrix} = x^p y^q (x-y)^r.$$

See Wakamiko (2007), for explicit formula

(Using Schur functions)

1 Introduction



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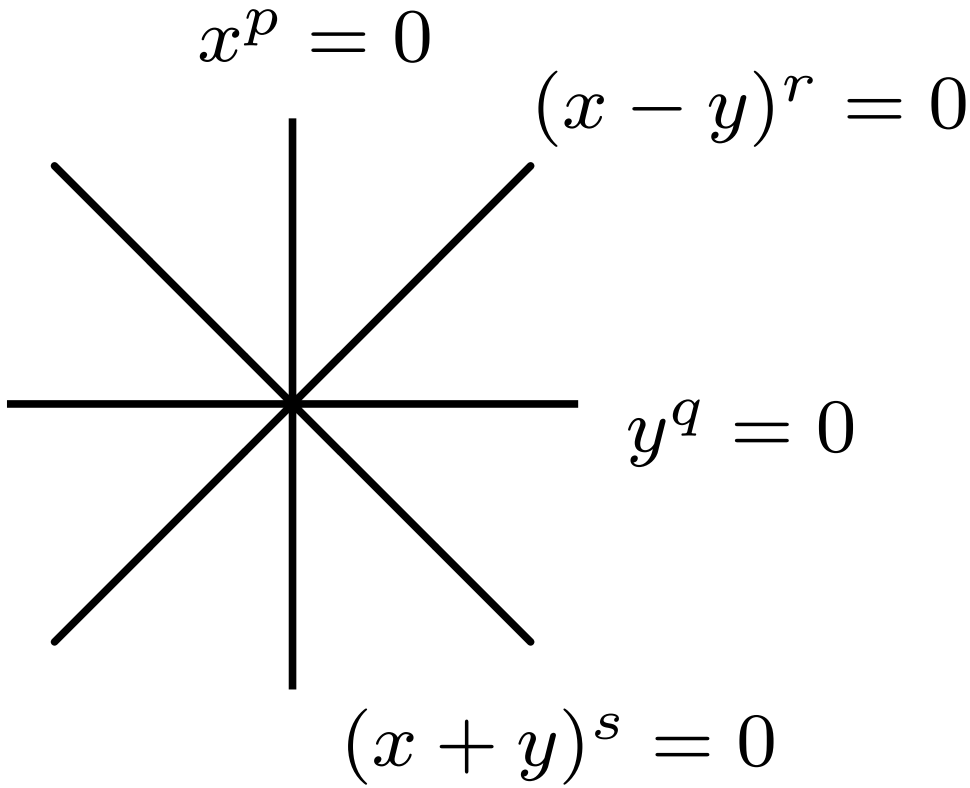

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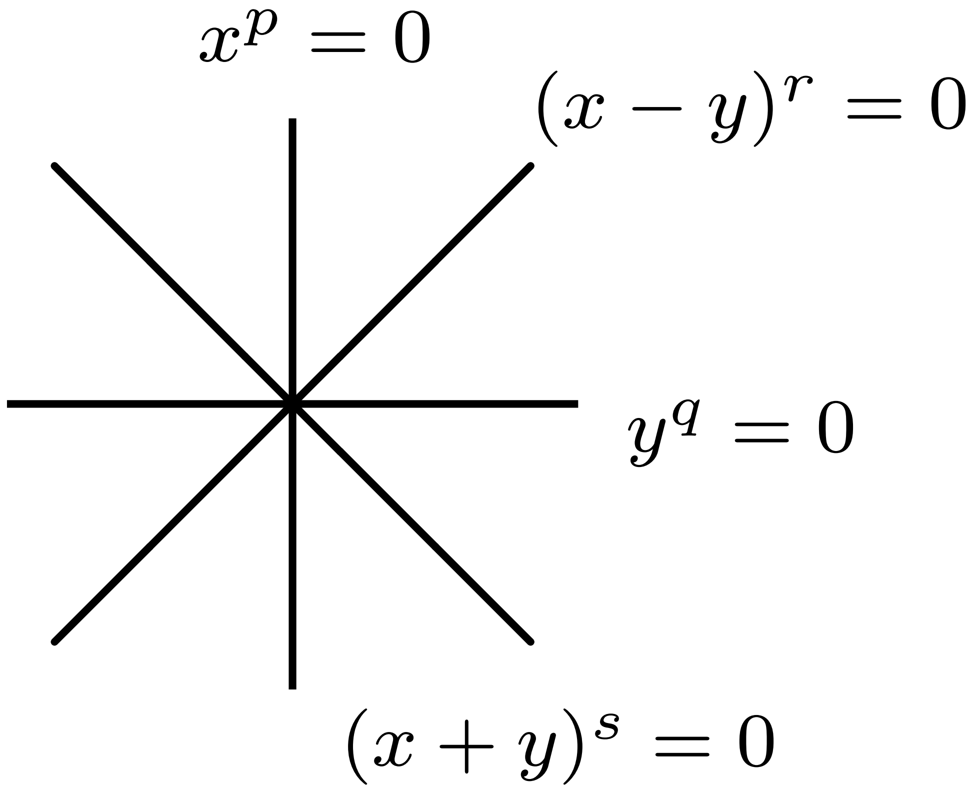

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But....

Explicit formula is NOT KNOWN!

(Even $\deg \delta_i$ is unclear.)

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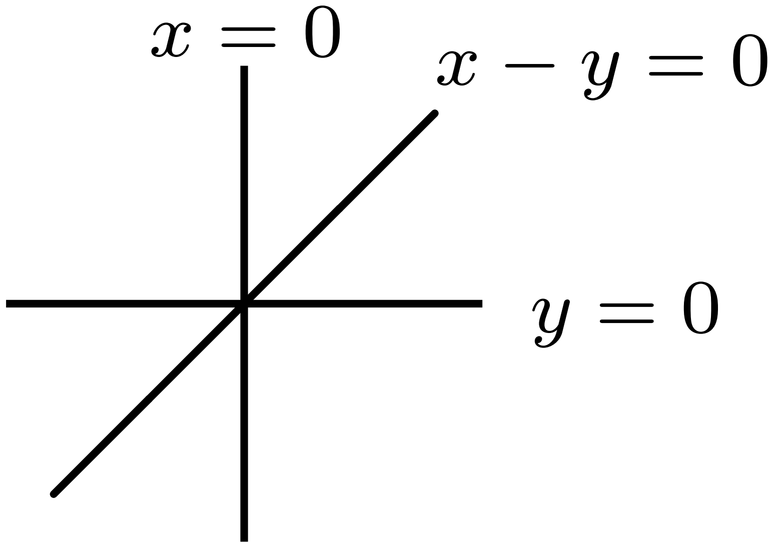

$$\delta_1 = x\partial_x + y\partial_y$$
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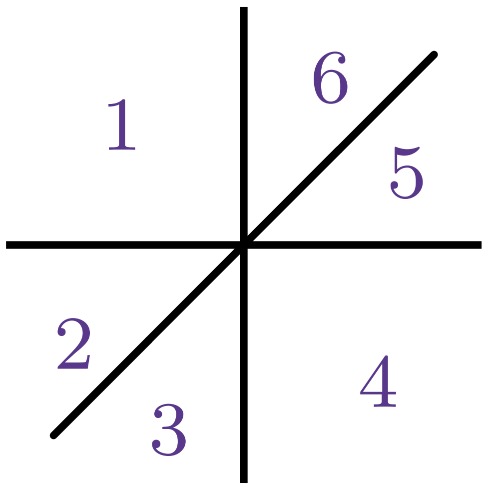
Exponents:

$$\exp = (\deg \delta_1, \deg \delta_2) = (1, 2)$$

$\Updownarrow$ Compare



$\sharp$ bdd. chamber $= 0 = (1-1)(2-1)$

$\sharp$ chamber $= 6 = (1+1)(2+1)$

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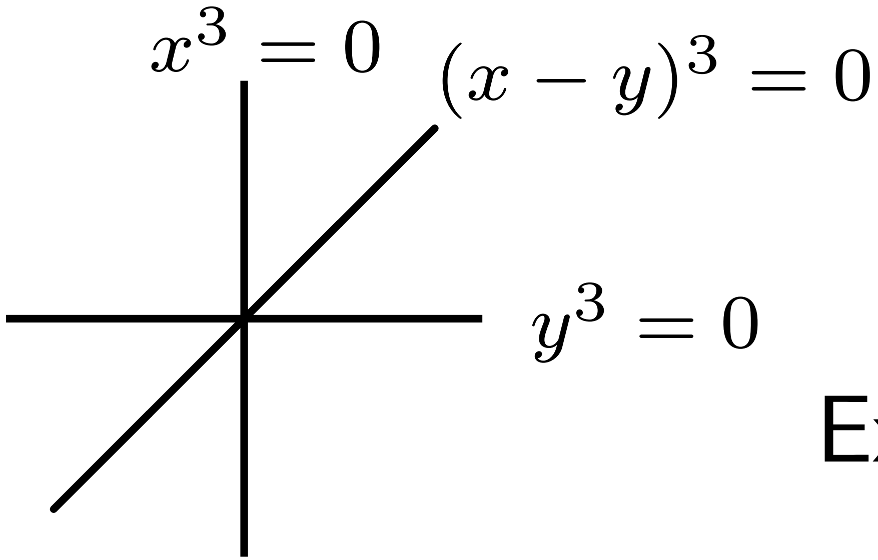

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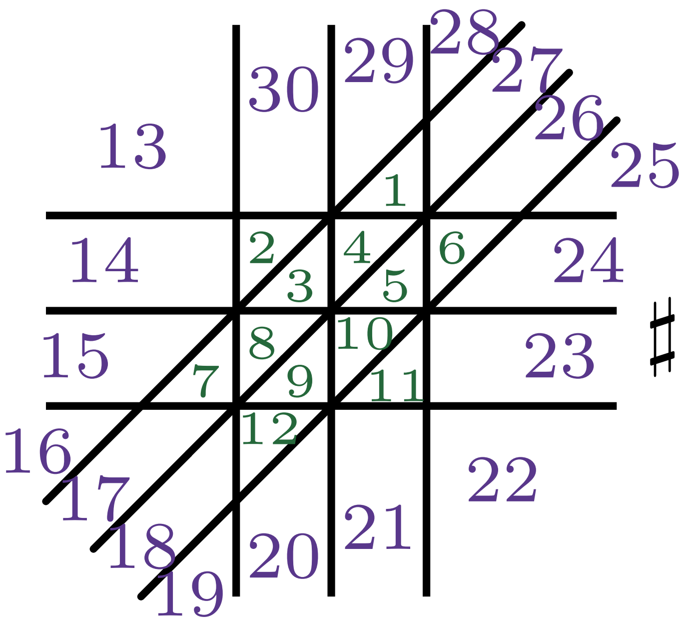

$\sharp$ bdd. chamber $= 12 = (4-1)(5-1)$

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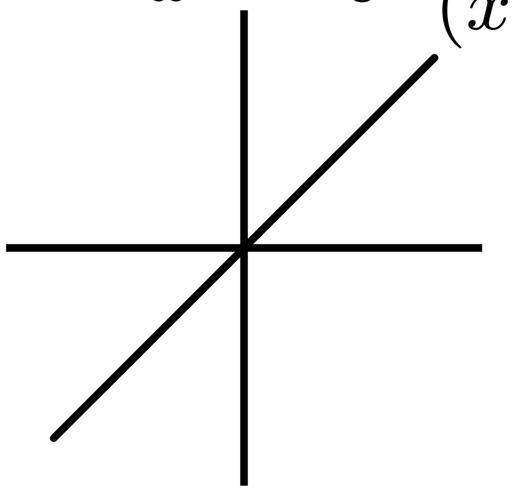

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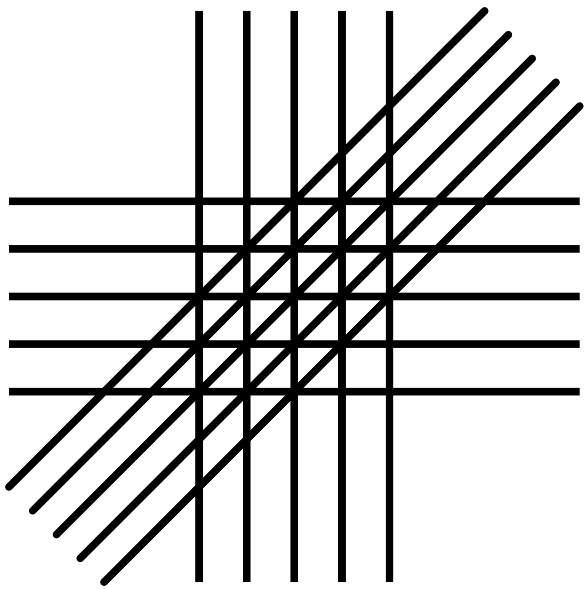

$\sharp$ bdd. chamber $= 42 = (7-1)(8-1)$

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1 Introduction

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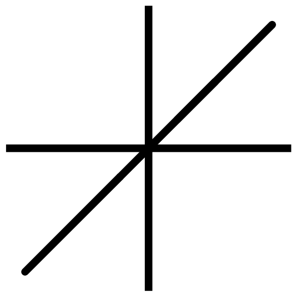
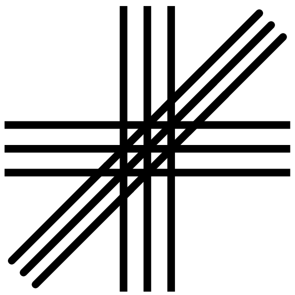
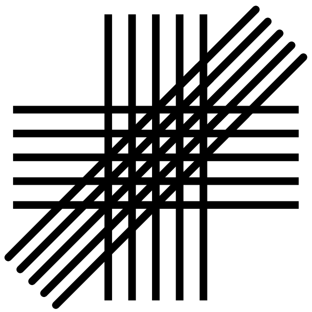
	$(1, 2)$	$0 = (1-1)(2-1)$ $6 = (1+1)(2+1)$
	$(4, 5)$	$12 = (4-1)(5-1)$ $30 = (4+1)(5+1)$
	$(7, 8)$	$42 = (7-1)(8-1)$ $72 = (7+1)(8+1)$

1 Introduction

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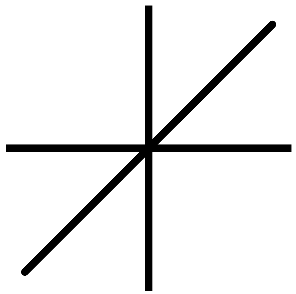
bdd. chamber

chamber



$$(1, 2)$$

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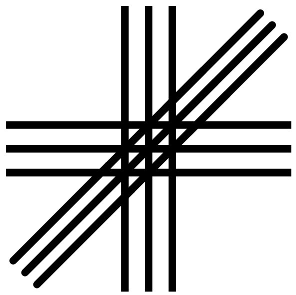
$$\downarrow +3$$



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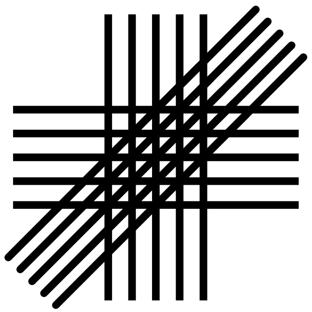

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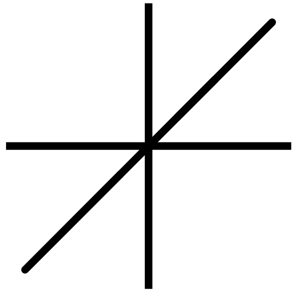

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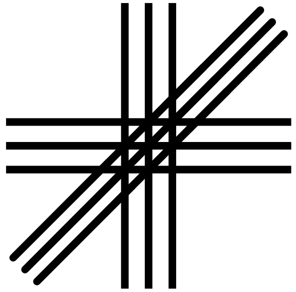

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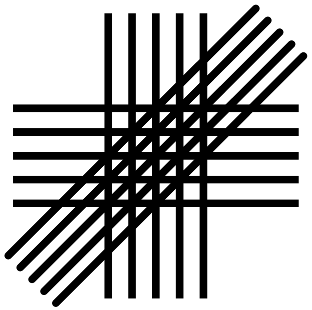

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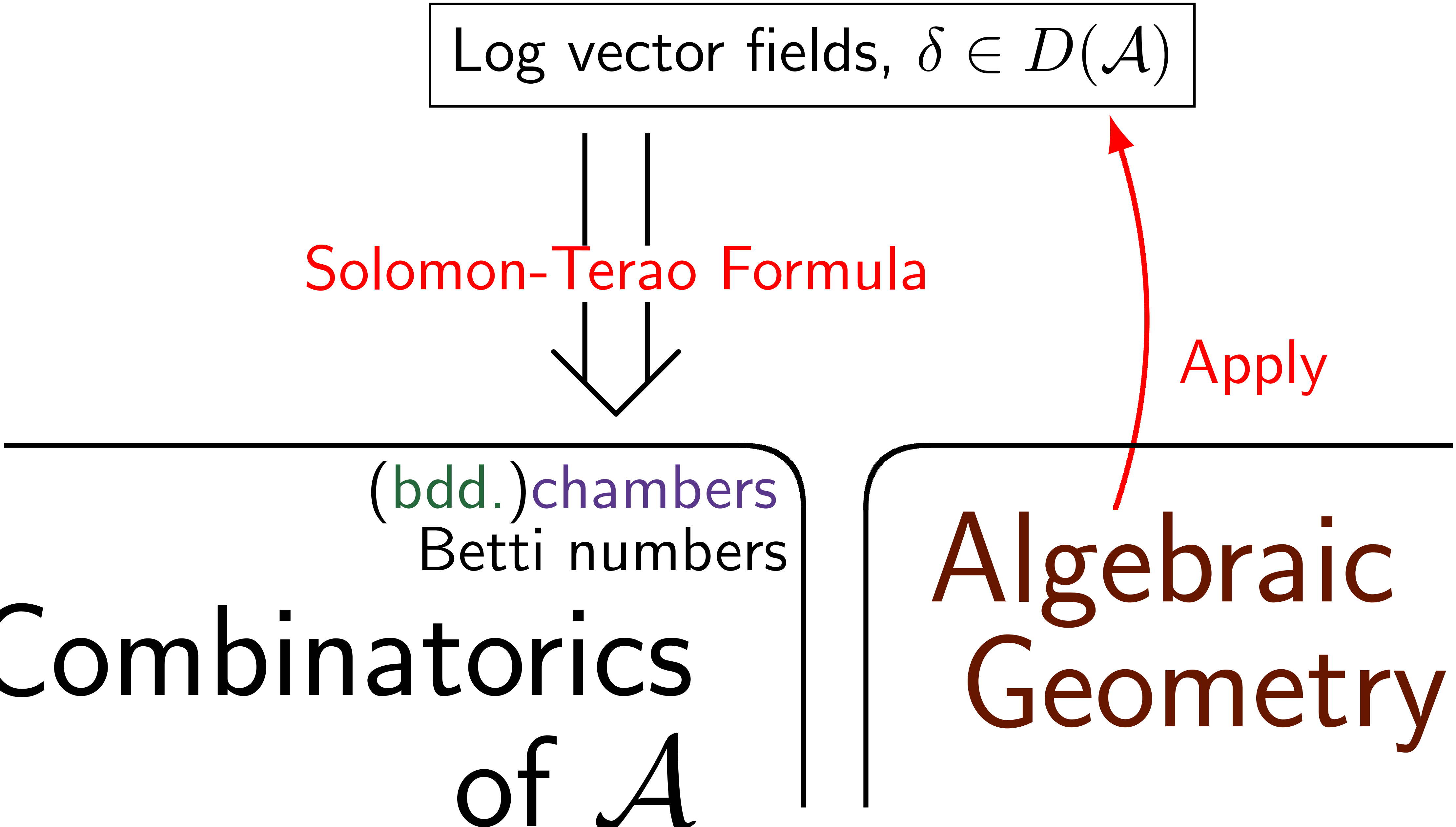
Are they accidental? No, they are related!

1 Introduction

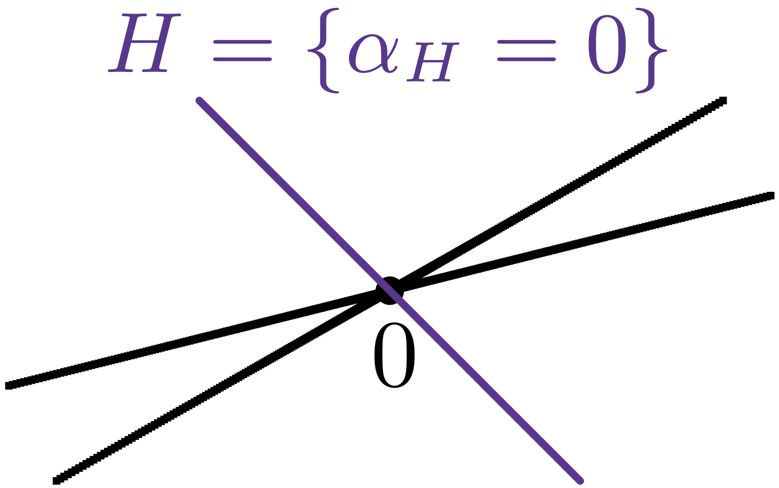


2 Origin

Def. $\mathcal{A} = \{H_1, \ldots, H_n\}$,

$0 \in H_i \subset \mathbb{C}^\ell, H_i = \alpha_H^{-1}(0)$

α_H is a linear form.



A *multiplicity* is a map

$$\mathbf{m} : \mathcal{A} \longrightarrow \mathbb{Z}_{\geq 0}.$$

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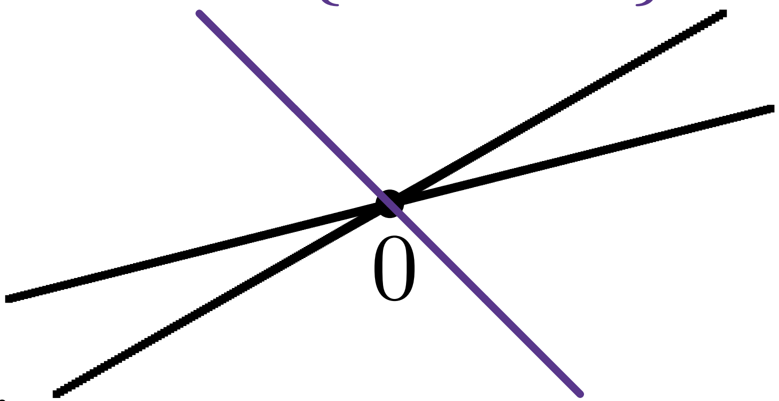

$$S = \mathbb{C}[x_1, \ldots, x_\ell]$$

$$\delta = \sum_{i=1}^{\ell} f_i \partial_{x_i} \in \mathrm{Der}_S = \bigoplus_{i=1}^{\ell} S \cdot \partial_{x_i}$$

is called *derivation* or *vector fields*.

$$D(\mathcal{A}, \mathbf{m}) = \{\delta \in \mathrm{Der}_S \mid \delta\alpha_H \in (\alpha_H^{\mathbf{m}(H)}), \forall H\}$$

is *logarithmic vector fields* (with mult. $\mathbf{m}$).

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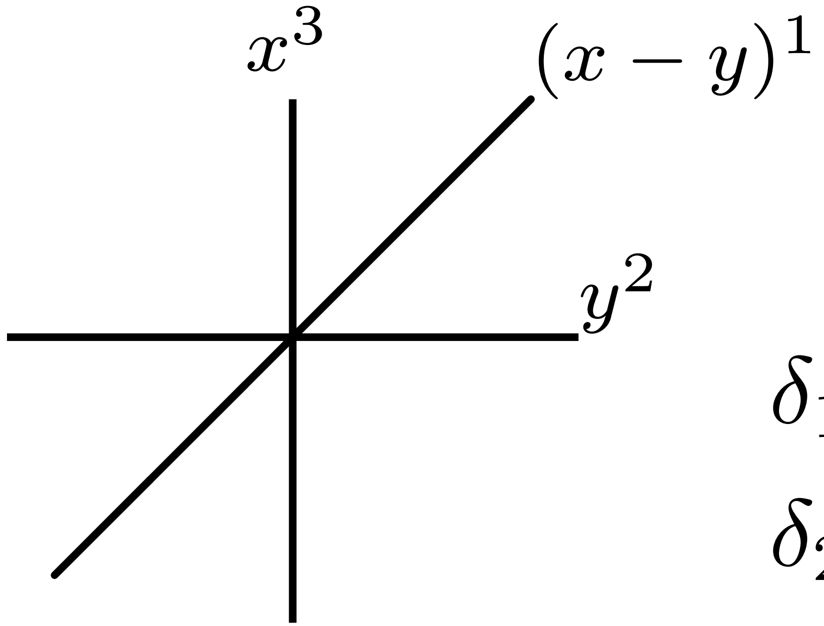
$$D(\mathcal{A},\mathbf{m}) = \{\delta \in \mathrm{Der}_S \mid \delta\alpha_H \in (\alpha_H^{\mathbf{m}(H)}), \forall H\}$$



x^3 ← multiplicity (3); ← linear form (x)

$\delta_1 = x^3\partial_x + y^3\partial_y$, and

$\delta_2 = y^2(x-y)\partial_y$

are in $D(\mathcal{A},\mathbf{m})$

$$\det\begin{pmatrix} x^3 & y^3 \\ 0 & y^2(x-y)\end{pmatrix} = \prod \alpha_H^{\mathbf{m}(H)} \Longrightarrow\Longrightarrow \left\{ \begin{array}{l} D(\mathcal{A},\mathbf{m}) \text{ is free,} \\ \delta_1, \delta_2 \text{ is a basis} \end{array}\right.$$

Saito's criterion

2 Origin

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Where do they come from?

(Geometry? Singularity?)

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Example. $f : X_1 = \mathbb{C} \to X_2 = \mathbb{C}, (x \mapsto x^2 = t)$

When vector field $\delta = f(t)\frac{d}{dt}$ liftable?

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δ is liftable $\Longleftrightarrow \delta \in D(\mathcal{A},\mathbf{1})$

2 Origin

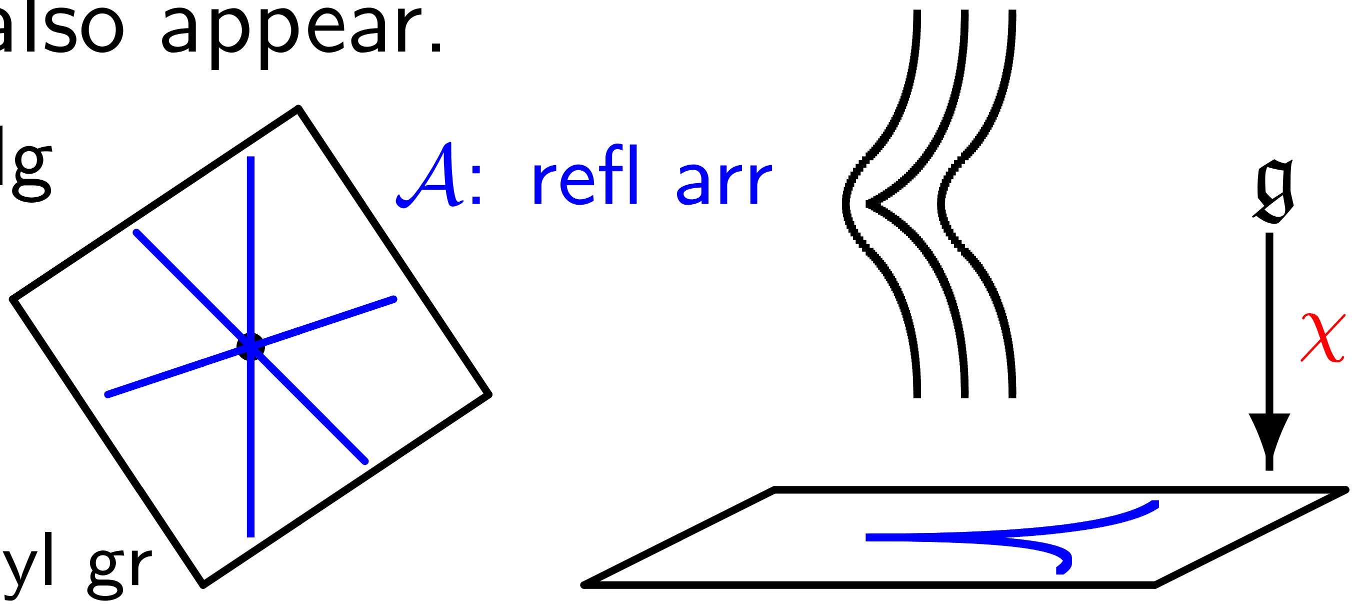
In more complicated promlems, multiplicities also appear.

$\mathfrak{g}$: simple Lie alg

G: Adjoint group

$\mathfrak{h} \curvearrowleft W$: Cartan and Weyl gr

$\chi : \mathfrak{g} \to \mathfrak{h}/W$ Adjoint quotient map



$$\mathfrak{h} \xrightarrow{\pi} S = \mathfrak{h}/W = \mathfrak{g}//G$$

<u>Thm.</u> If $\mathfrak{g}$ is ADE, ω is the Kostant-Kirillov form, then

$$\nabla_{\bullet}\omega : D(\mathcal{A}, \mathbf{3})^W \xrightarrow{\simeq} \mathbb{R}^2\chi_*\Omega^{\cdot}_{\mathfrak{g}/S}, (\delta \longmapsto \nabla_\delta\omega) \text{ is isom.}$$

2.1 Solomon-Terao Formula

For $(\mathcal{A}, \mathbf{m})$, define $Q = \prod_H \alpha_H^{\mathbf{m}(H)}$.

Def.

$$\Omega^p(\mathcal{A}, \mathbf{m}) = \left\{ \omega \in \frac{1}{Q}\Omega^p \mid d\alpha_H \wedge \omega \text{ has no pole along } H \right\}$$

Thm (S-T)

Hilbert series

$$\chi(\mathcal{A}, t) = \lim_{x \to 1} \sum_{p=0}^{\ell} HS(\Omega^p(\mathcal{A}), x)(t(1-x)-1)^p$$

Thm (Orlik-Solomon, Zaslavsky)

"of deconing"

$$\lim_{t\to 1} |\chi(\mathcal{A}, t)/(t-1)| = \sharp \text{ bdd. chambers}$$

$$\lim_{t\to -1} |\chi(\mathcal{A}, t)/(t-1)| = \sharp \text{ chambers}$$

2.2 Summary/Comments

- $D(\mathcal{A}, \mathbf{m}) = \{\delta \in \mathrm{Der}_S \mid \delta\alpha_H \in (\alpha_H^{\mathbf{m}(H)})\}$ introduced by Ziegler, appeared geom problems of singularity.
- $\ell = 2 \Longrightarrow D(\mathcal{A}, \mathbf{m})$ is free, but difficult to find the basis.
- $\ell = 2$, generically stable i.e., $|\deg \delta_1 - \deg \delta_2| \leq 1$ (Yuzvinsky-Wakefield).
- $\ell \geq 2$, $\exists$ many techniques (addition-deletion, characteristic poly) to study freeness (Abe-Terao-Wakefield)

3 Coxeter arrangements.

3.1 Basic Techniques

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$D(\mathcal{A}) := D(\mathcal{A}, \mathbf{1}) = \{\delta \mid \delta\alpha_H \in (\alpha_H), \forall H\}$

$\theta_E = \sum_i x_i \partial_i$: Euler vect field. $\in D(\mathcal{A})$.

<u>Def</u>. Let $H_0 \in \mathcal{A}$.

$$D_{H_0}(\mathcal{A}) := \{\delta \in D(\mathcal{A}) \mid \delta\alpha_{H_0} = 0\}$$

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Prop. $D(\mathcal{A}) = \langle \theta_E \rangle \oplus D_{H_0}(\mathcal{A})$.

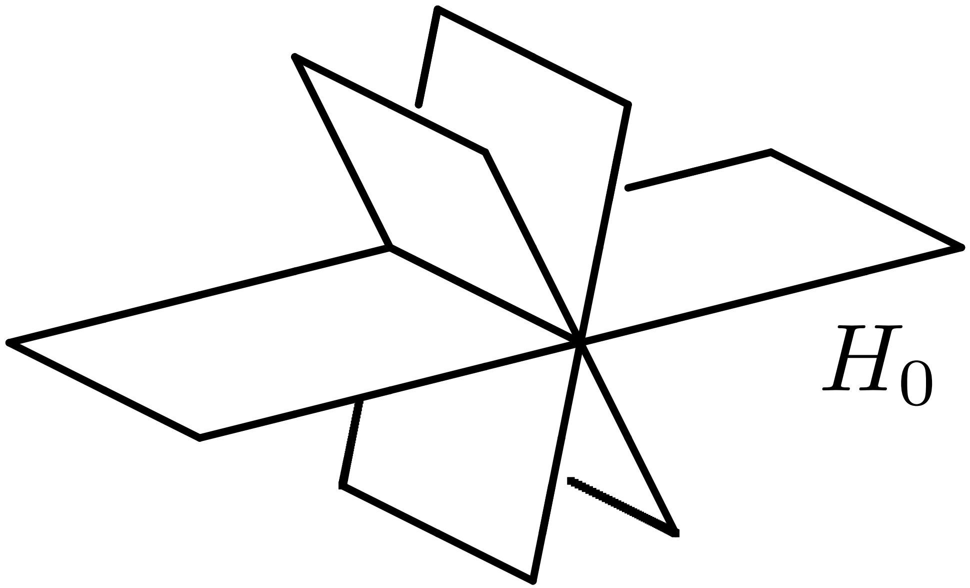
Proof.

$$\delta = \frac{\delta\alpha_0}{\alpha_0}\theta_E + \left(\delta - \frac{\delta\alpha_0}{\alpha_0}\theta_E\right)$$

Since $\theta_E \alpha_0 = \alpha_0$, we have $\left(\delta - \frac{\delta\alpha_0}{\alpha_0}\theta_E\right)\alpha_0 = 0$

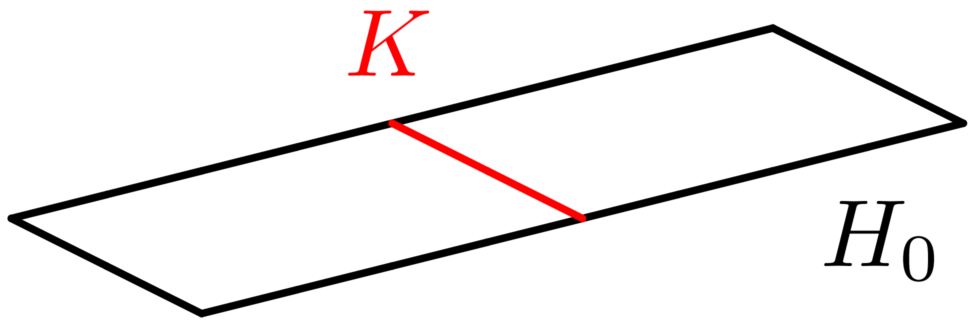
3.1 Basic Techniques

$\dim = \ell$



$\mathcal{A}$

$\dim = \ell - 1$



$\mathbf{m}_{H_0}(K) = 2$

$(\mathcal{A}^{H_0}, \mathbf{m}_{H_0})$

3.1 Basic Techniques

Thm. (Ziegler)

$$\delta \in D_{H_0}(\mathcal{A}) \Longrightarrow \delta|_{H_0} \in D(\mathcal{A}^{H_0}, \mathbf{m}_{H_0}).$$

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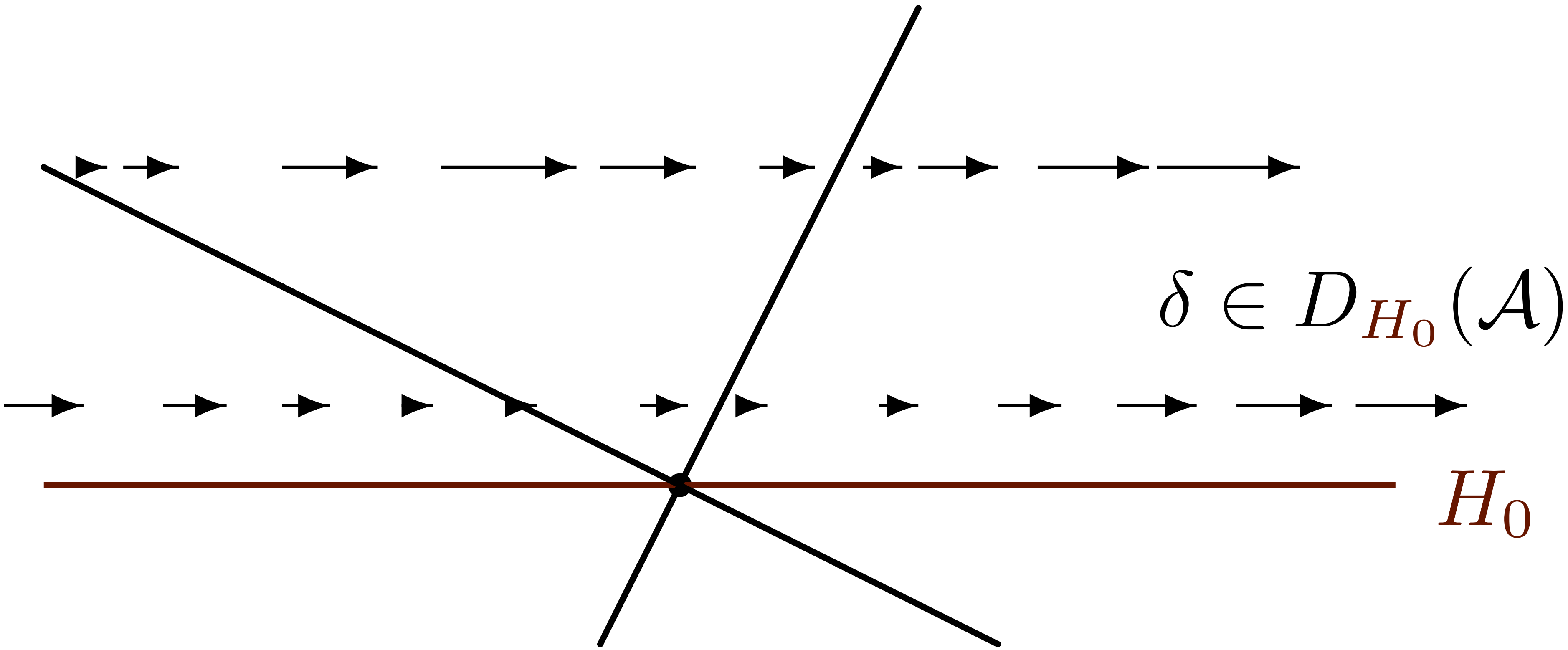

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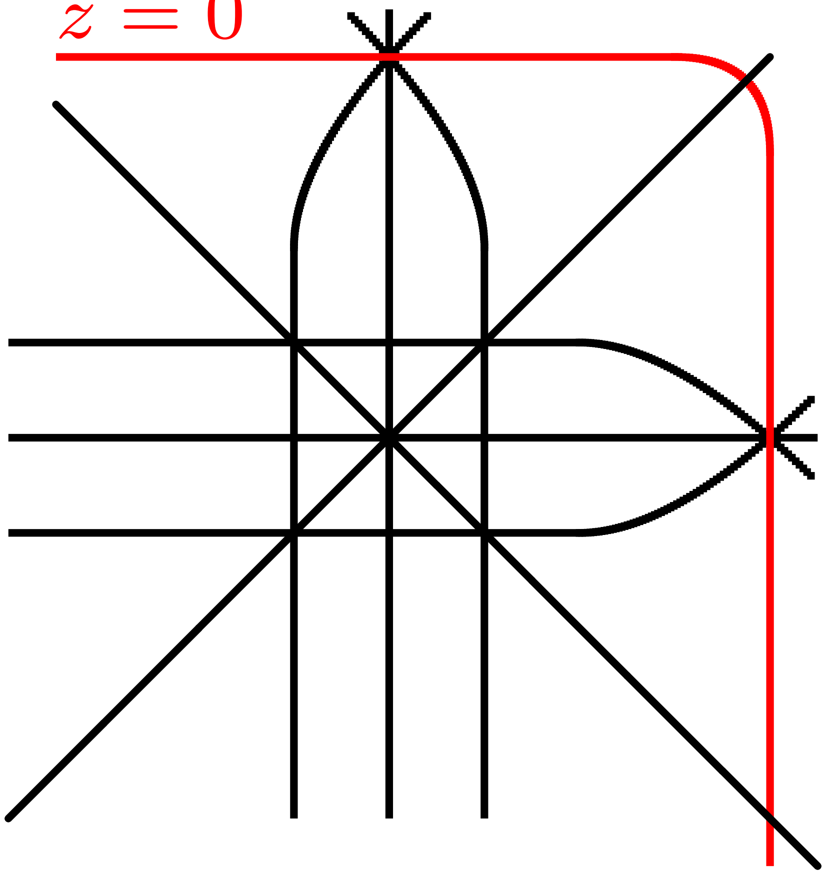

$$Q = xyz(x^2 - y^2) \times (x^2 - z^2)(y^2 - z^2)$$

$$\theta_E = x\partial_x + y\partial_y + z\partial_z$$
$$\delta_1 = x(x^2 - z^2)\partial_x + y(y^2 - z^2)\partial_y$$
$$\delta_2 = x(x^4 - z^4)\partial_x + y(y^4 - z^4)\partial_y$$

$$\delta_1, \delta_2 \in D_{H_0}(\mathcal{A})$$

$$\delta_1|_{z=0} = x^3\partial_x + y^3\partial_y$$
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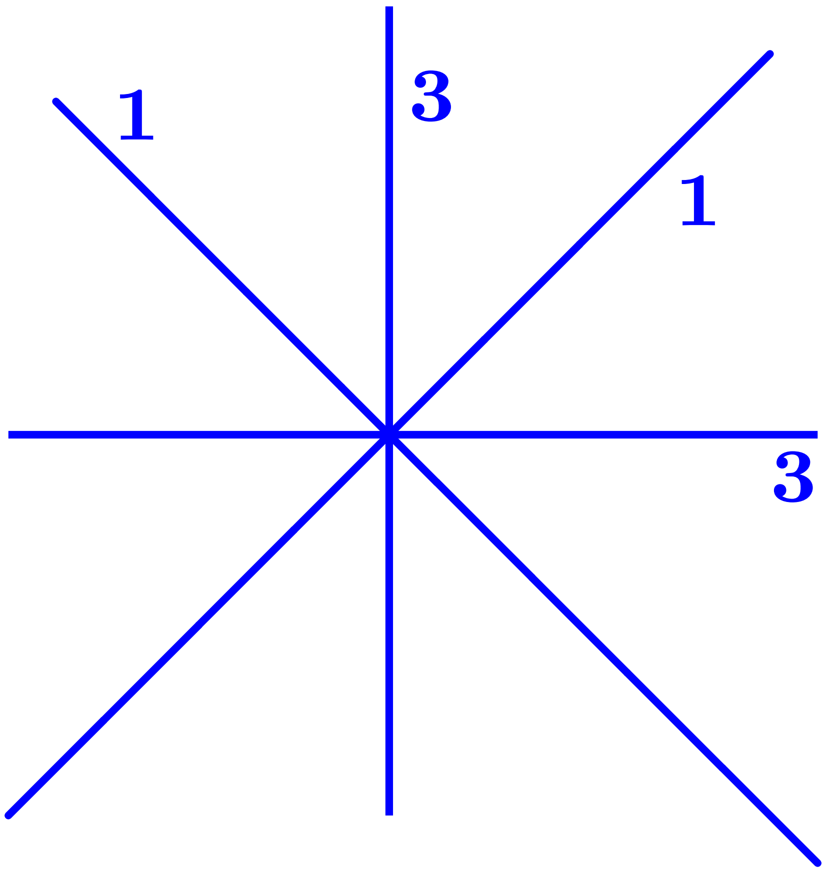

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3.1 Basic Techniques

Thm. (Ziegler)

$$\delta \in D_{H_0}(\mathcal{A}) \Longrightarrow \delta|_{H_0} \in D(\mathcal{A}^{H_0}, \mathbf{m}_{H_0}).$$

Thm. (Z)

$\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1, d_2, \ldots, d_\ell)$,

$\Rightarrow (\mathcal{A}^{H_0}, \mathbf{m}_{H_0})$ is free with $\exp = (d_2, \ldots, d_\ell)$

Cor. $\mathcal{A}$ is free iff $D(\mathcal{A}^{H_0}, \mathbf{m}_{H_0})$ is free, and

$D_{H_0}(\mathcal{A}) \to D(\mathcal{A}^{H_0}, \mathbf{m}_{H_0})$ is surjective.

3.2 AG aspects

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$S = \mathbb{C}[x_1, \dots, x_\ell]$

"M: a graded S-module"

⟷

Corresp.

"$\widetilde{M}$: a sheaf on $\mathbb{P}^{\ell-1}$"

s.t. $H^0(\mathbb{P}^{\ell-1}, \widetilde{M} \otimes \mathcal{O}(d)) = M_d$.

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s.t. $H^0(\mathbb{P}^{\ell-1}, \widetilde{M} \otimes \mathcal{O}(d)) = M_d$.

Using this correspondence,

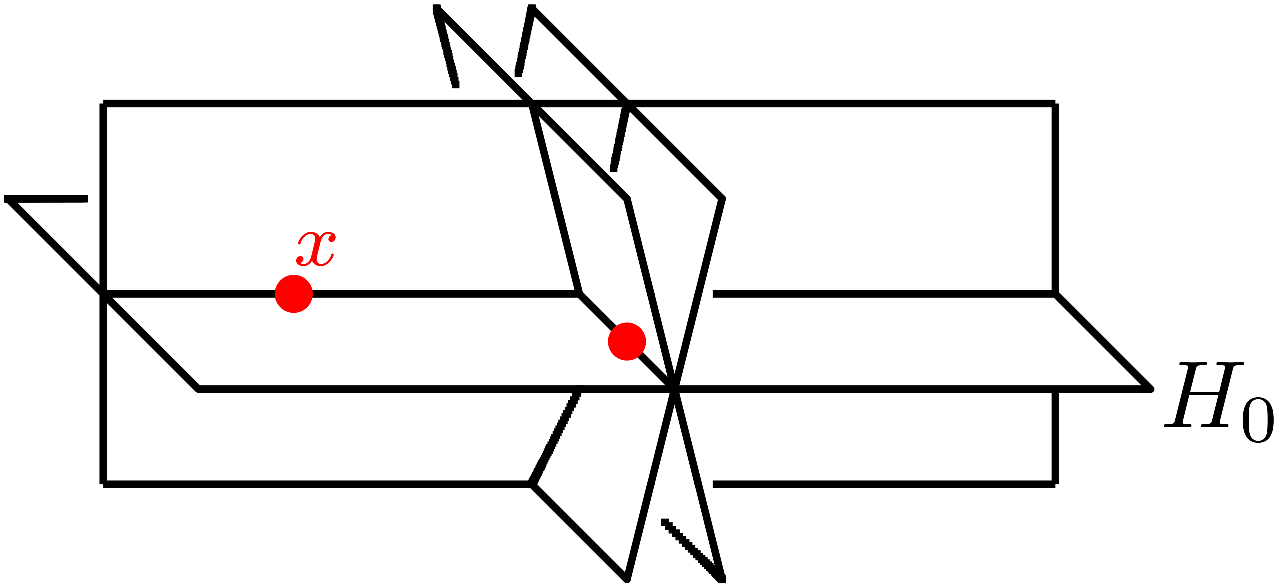
Free with $\exp = (d_1, \ldots, d_\ell)$ $\iff \widetilde{D(\mathcal{A})} = \bigoplus_{i=1}^{\ell} \mathcal{O}(-d_i)$

Freeness $\iff$ Splitting

3.3 Local version of Ziegler

Def. $\mathcal{A}$ is *locally free along* H_0 if $\forall x \in H_0 \setminus \{0\}$,

$\mathcal{A}_x = \{H \in \mathcal{A} \mid H \ni x\}$ is free.



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Under this assumption, (put $\mathcal{F} = \widetilde{D_{H_0}(\mathcal{A})}$)

$$0 \longrightarrow \mathcal{F}(-1) \longrightarrow \mathcal{F} \longrightarrow D(\widetilde{\mathcal{A}^{H_0}, \mathbf{m}_{H_0}}) \longrightarrow 0$$

is exact (of sheaves).

$$\therefore \quad D(\widetilde{\mathcal{A}^{H_0}, \mathbf{m}_{H_0}}) = \mathcal{F}|_{\mathbb{P}(H_0)}$$

3.4 Splitting criterion

Thm. (Horrocks) $H \subset \mathbb{P}^m, (m \geq 3)$: a hyperplane
E is a holomorphic vector bundle on $\mathbb{P}^m$.
Then E splits iff $E|_H$ splits.

Rem. The above is true for E: reflexive (Abe, -)

3.5 Characterizing freeness

Thm. $\mathcal{A}$: central arr in $\mathbb{C}^\ell$, $H_0 \in \mathcal{A}$.

(1) When $\ell \geq 4$, $\mathcal{A}$ is free iff

— $\mathcal{A}$ is locally free along H_0, and

— $(\mathcal{A}^{H_0}, \mathbf{m}_{H_0})$ is free.

(2) When $\ell = 3$, $\mathcal{A}$ is free iff

— $\chi(\mathcal{A}, t) = (t-1)(t-d_2)(t-d_3)$,

— $(\mathcal{A}^{H_0}, \mathbf{m}_{H_0})$ is free with $\exp = (d_2, d_3)$.

Rem. Freeness is characterized by informations around H_0 only.

3.6 Applications to affine Weyl arrangements

$V = \mathbb{R}^\ell$: Euclidean space,

$\Phi \subset V^*$: a root system, (irred, reduced)

$\Phi_+ = \{\alpha_1, \ldots, \alpha_n\} \subset \Phi$: fix a positive system,

$e_1, \ldots, e_\ell$: exponents, and

h: the Coxeter number.

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$e_1, \dots, e_\ell$: exponents, and

h: the Coxeter number.

Def. Let $a, b \in \mathbb{Z}, a \leq b$.

$$\mathcal{A}_\Phi^{[a,b]} := \{\alpha^{-1}(k) \mid \alpha \in \Phi_+, k \in \mathbb{Z}, a \leq k \leq b\}$$

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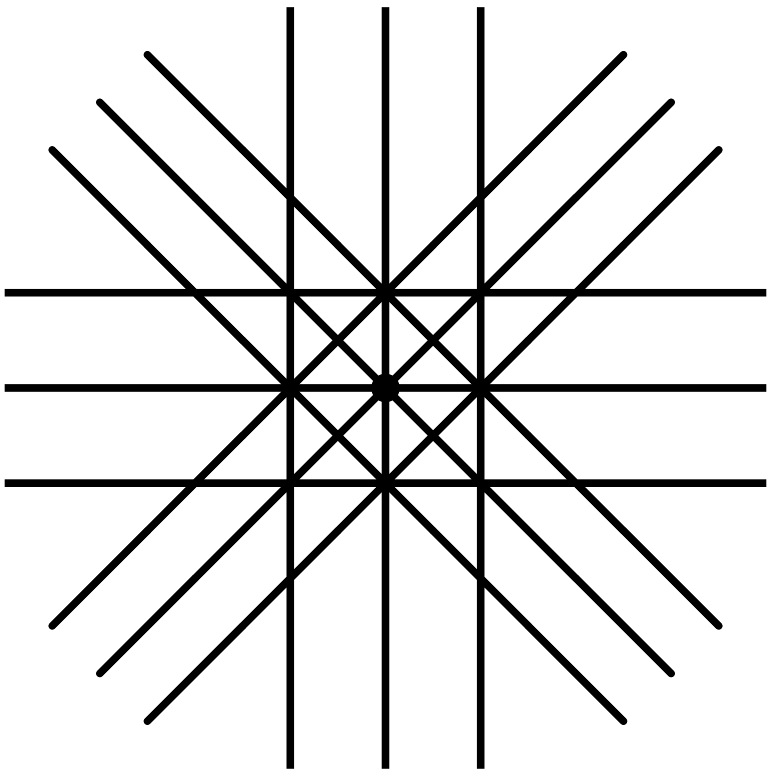
Type B_2

$\Phi_+ = \{x, y, x+y, x-y\}$



$\mathcal{A}_{\Phi}^{[-1,1]}$

3.6 Applications to affine Weyl arrangements

$$\mathcal{A}_\Phi^{[a,b]} := \{\alpha^{-1}(k) \mid \alpha \in \Phi_+, k \in \mathbb{Z}, a \leq k \leq b\}$$

There are many works on $\chi(\mathcal{A}_\Phi^{[a,b]}, t)$.

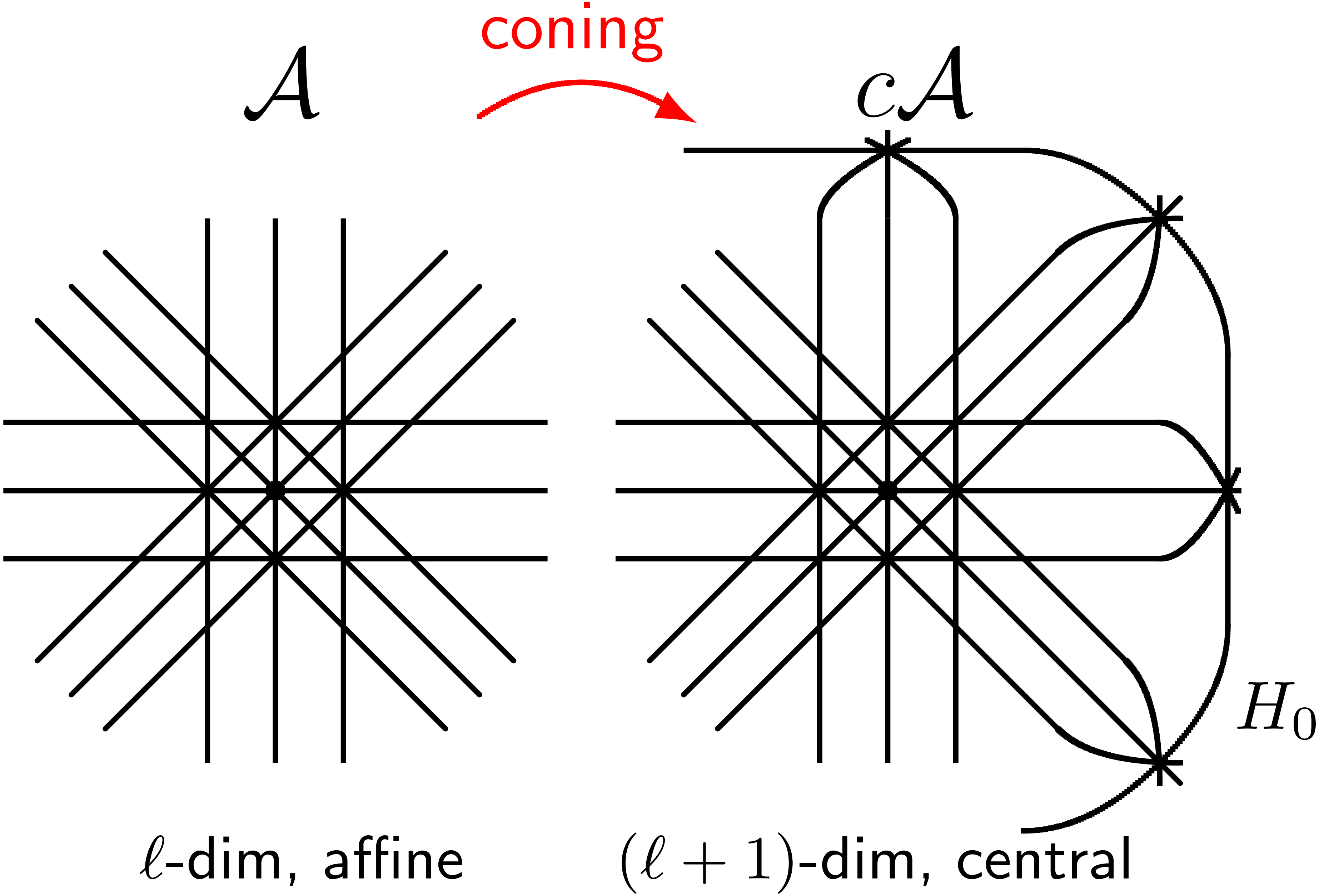
Thm. (Athanasiadis) $m \geq 0$, then

$$\chi(\mathcal{A}_\Phi^{[-m,m]}, t) = \prod_{i=1}^{\ell} (t - e_i - mh).$$

Conj. (Postnikov-Stanley) $m > 0$, then

The real part of zeros of $\chi(\mathcal{A}_\Phi^{[0,m]}, t) = 0$ is $(m+1)h/2$.

3.6 Applications to affine Weyl arrangements



3.6 Applications to affine Weyl arrangements

Thm. Let $m \geq 0$

- $c\mathcal{A}^{[-m,m]}$ is free with
$$\exp = (1, e_1 + mh, \ldots, e_\ell + mh).$$
- $c\mathcal{A}^{[1-m,m]}$ is free with
$$\exp = (1, mh, mh, \ldots, mh).$$

Cor.

- $\chi(\mathcal{A}^{[-m,m]}, t) = \prod_i (t - e_i - mh)$.
- $\chi(\mathcal{A}^{[1-m,m]}, t) = (t - mh)^\ell$

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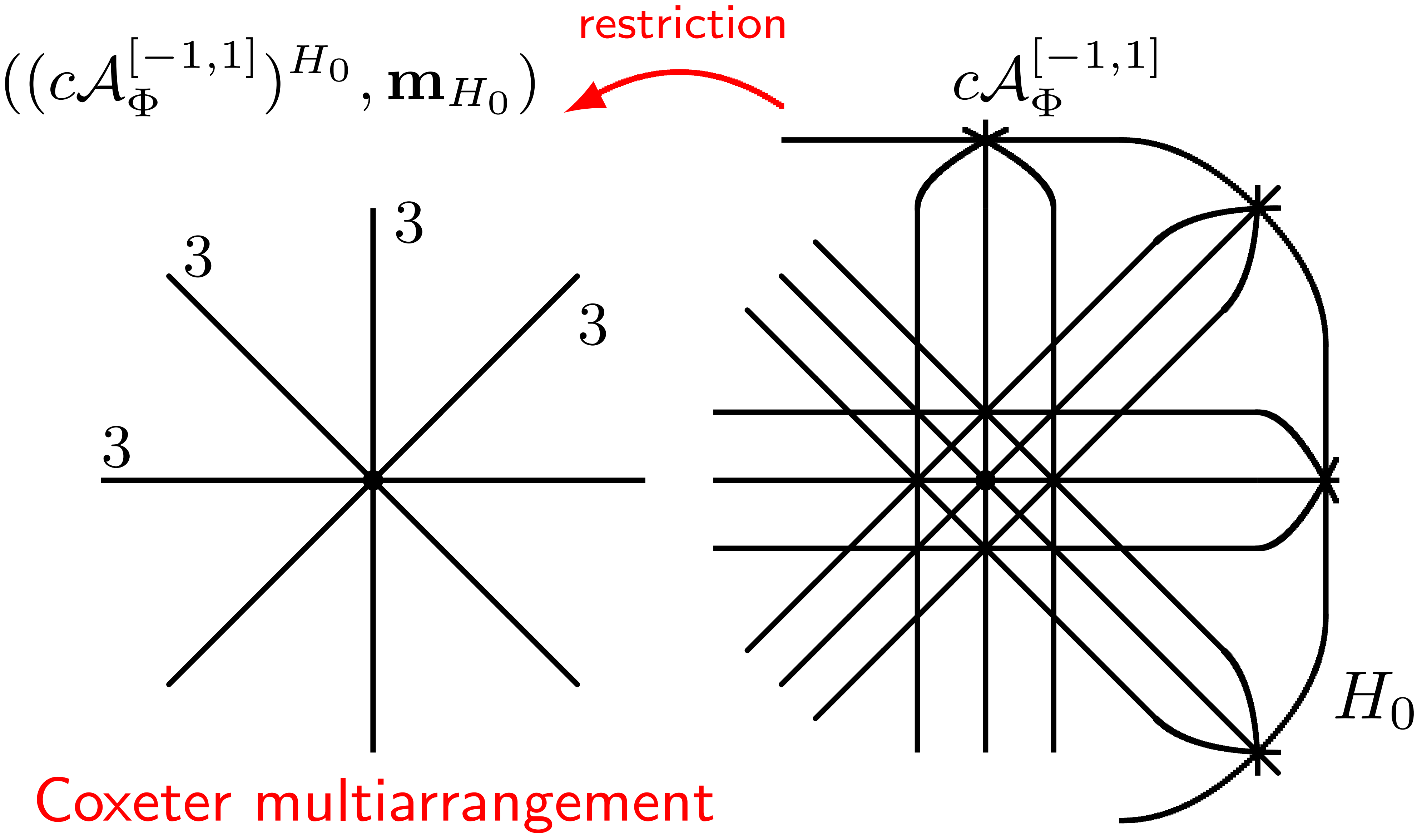
(*Sketch of proof*).

Thm. $\mathcal{A}$: central arr in $\mathbb{C}^\ell$, $H_0 \in \mathcal{A}$. (1) When $\ell \geq 4$, $\mathcal{A}$ is free iff — $\mathcal{A}$ is locally free along H_0, and — $(\mathcal{A}^{H_0}, \mathbf{m}_{H_0})$ is free.

Check!: (i) $c\mathcal{A}_\Phi^{[a,b]}$ is loc free along $H_0 \Leftarrow$ lower rank cases.
(ii) $((c\mathcal{A}_\Phi^{[a,b]})^{H_0}, m_{H_0})$ is free. $\Leftarrow$ next page.

3.6 Applications to affine Weyl arrangements



3.7 Coxeter multiarrangements

Thm. (Terao) $\mathcal{A}$: Coxeter arrangement, $k \geq 0$

- $(\mathcal{A}, 2k+1)$ is free with
$$\exp = (e_1 + mh, \ldots, e_\ell + mh).$$
- $(\mathcal{A}, 2k)$ is free with
$$\exp = (mh, mh, \ldots, mh).$$

This completes the proof!

There are several generalizations.

3.7 Coxeter multiarrangements

Thm. (Abe, –, accepted to J. Alg., July 30)

$\mathcal{A}$: Coxeter arrangement, $\mathbf{m} : \mathcal{A} \longrightarrow \mathbb{Z}_{\geq 0}$ satisfies $\exists k$ s.t. $\mathbf{m}(\mathcal{A}) \subset \{k, k+1\}$, then

$$D(\mathcal{A}, \mathbf{m}+2) \cong D(\mathcal{A}, \mathbf{m})[-h].$$

Rem.

- Isomorphic even for non-free cases !
- Abe generalized further.

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(*brief sketch of proof*)

$\mathbb{C}[V]^W = \mathbb{C}[P_1, \ldots, P_\ell]$, P_i: basic invariant

Assume: $\deg P_1 \leq \cdots \leq \deg P_\ell$

$D := \frac{\partial}{\partial P_\ell}$ is called the primitive derivation.

$\mathbf{m} : \mathcal{A} \longrightarrow \{0, 1\}$

3.7 Coxeter multiarrangements

Rem. $D = \frac{\partial}{\partial P_\ell}$ is canonical, since $\deg P_i < \deg P_\ell$.

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(q.e.d.)

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This generate everything.
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This generate everything.
"Primitivity"

Rem.

- $\deg P_\ell = h$ explains h-shifting ($\mathbf{m} \in \{0, 1\}^{\mathcal{A}}$)
$D(\mathcal{A}, \mathbf{m} + 2) \cong D(\mathcal{A}, \mathbf{m})[-h]$

3.7 Coxeter multiarrangements

Remarks and Questions:

- $D(\mathcal{A}, \mathbf{m}+2) \cong D(\mathcal{A}, \mathbf{m})[-h]$ for $\mathbf{m} \in \{0,1\}^{\mathcal{A}}$ even they are not free.
- Athanasiadis proved even nonsplitting cases, $\chi(\mathcal{A}^{[a-1,b+1]}, t) = \chi(\mathcal{A}^{[a,b]}, t-h)$ holds.
- Will the primitive derivation $D = \frac{\partial}{\partial P_\ell}$ play further role? (e.g., in "RH", "functional equation" etc.)