Formality of Subspace Arrangements

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Outline

- Subspace arrangements
- Relative atomic complex
- Edge colored hypergraphs
- Characteristic polynomials
- Formality
- Pascal arrangements

- Setting: V - a complex vector space of dimension ℓ

- Characters:
  - a subspace arrangement $\mathcal{A} = \{X_1, \ldots, X_k\}$ is a finite collection of linear subspaces in V
  - $L(\mathcal{A})$ is the intersection lattice of $\mathcal{A}$
  - $\chi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim(X)}$ is the characteristic polynomial of $\mathcal{A}$
  - $M(\mathcal{A}) = V - \bigcup_{X \in \mathcal{A}} X$ is the complement of $\mathcal{A}$
  - the braid arrangement $\mathcal{A}_\ell$ is the hyperplane arrangement defined by the linear forms $x_i - x_j$ where $1 \leq i < j \leq \ell$ and x_i is a basis for V^*
  - a hypergraph $\mathcal{H} = ([k], E)$ is a set of k vertices denoted $[k]$ and a set of subsets of $[k]$ called edges E

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Yuzvinsky's relative atomic complex

- For $\mathcal{A} = \{X_1, \ldots, X_n\}$ fix an order on the subspaces $X_1 < \cdots < X_k$, associate the integer s with the subspace X_s and let $\sigma = \{i_1, \ldots, i_s\} \subseteq [n]$

- let $D_{\mathcal{A}}$ be the d.g.a. generated by a_σ where

$$\deg(a_\sigma) = 2\mathrm{codim}(\bigvee \sigma) - |\sigma|$$

- the differential is

$$da_\sigma = \sum_{j : \bigvee \sigma \setminus i_j = \bigvee \sigma} (-1)^j a_{\sigma \setminus i_j}$$

- the products are defined by $a_\sigma a_\gamma = (-1)^{\epsilon(\sigma,\gamma)} a_{\sigma \cup \gamma}$ if $\mathrm{codim} \bigvee \sigma + \mathrm{codim} \bigvee \gamma = \mathrm{codim} \bigvee (\sigma \cup \gamma)$ and 0 otherwise

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$D_{\mathcal{A}}$ is a rational model

Theorem (Feichtner-Yuzvinsky)

$D_{\mathcal{A}}$ is quasi-isomorphic to the De Concini and Procesi wonderful model. Hence $D_{\mathcal{A}}$ is a rational model for the complement $M(\mathcal{A})$.

Theorem (Feichtner-Yuzvinsky)

If the intersection lattice $L(\mathcal{A})$ is geometric then $M(\mathcal{A})$ is formal.

Not all subspace arrangements are formal:
actually Denham-Suciu and Grbić-Theriault exhibited coordinate subspace arrangements that admit non-trivial Massey products. They constructed Massey products via Moment angle complexes based on the work of I.V. Baskakov.

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Edge Colored Hypergraph Arrangements

Let $\mathcal{A} = \{X_1, \dots, X_n\} \subseteq L(\mathcal{A}_\ell)$ and recall that $L(\mathcal{A}_\ell)$ is the partition lattice.

For each subspace X_i define an equivalence relation $\sim_i$ on $[\ell]$ by $r \sim_i s$ if and only if $X_i \subseteq \{x_r - x_s = 0\}$.
Associate X_i with the partition given by the equivalence classes of $\sim_i$ and denote this partition by $\pi_i = \{B^i_1, \dots, B^i_{p_i}\}$.
The associated hypergraph is $\mathcal{H}_\mathcal{A}$ has vertex set $[\ell]$ and edges

$$E = \{B^i_j | i \in \{1, \dots, n\}, j \in \{1, \dots, p_i\} \text{ and } |B^i_j| > 1\}$$

let $\Lambda = [n]$ be a set of colors. To distinguish between the edges we define an edge coloring of $\mathcal{H}_\mathcal{A}$ by $C_\mathcal{A} : E \to \Lambda$ where $B^i_j \mapsto i$

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Examples

- Graphic hyperplane arrangements: if $\mathcal{A} \subseteq \mathcal{A}_\ell$ then $(\mathcal{H}_\mathcal{A}, C_\mathcal{A})$ is a graph where each edge is colored differently. (many authors)
- Hypergraph arrangements or diagonal arrangements: $(\mathcal{H}_\mathcal{A}, C_\mathcal{A})$ is a hypergraph where $C_\mathcal{A}$ gives each edge a different color. (Brjörner, Lovász, Yao, Kozlov, Hultman, Peeva, Reiner, Welker ...)
- Orbit arrangements: all partitions of a certain type. (Li, Peeva, Sidman, Björner,...)
- k-equal arrangements: all partitions with exactly one non-trivial block of size k or $\mathcal{H}$ has all edges of size k (Björner, Yuzvinsky,...)

These arrangements have been studied from many perspectives including combinatorics, algebra, topology, and even computational complexity theory.

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Let $\ell = 4$ and $(\mathcal{H}, \mathcal{C})$ be the hypergraph defined by $E = \{\{1,2\},\{2,3\},\{3,4\}\}$ where the colors set is $\Lambda = \{R, B\}$ and the color function is given by $\mathcal{C}(\{1,2\}) = R$, $\mathcal{C}(\{2,3\}) = B$, and $\mathcal{C}(\{3,4\}) = R$. The corresponding arrangement $\mathcal{A} = \{X_1, X_2\}$ is the collection of the codimension 2 space $X_1 = \{v \in V | v_1 = v_2 \text{ and } v_3 = v_4\}$ and the codimension 1 space $X_2 = \{v \in V | v_2 = v_3\}$

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Let $\mathcal{H} = ([4], \{a, b, c\})$ where $a = \{1, 2, 3\}$, $b = \{3, 4\}$, and $c = \{2, 4\}$, and let each edge have its own color.

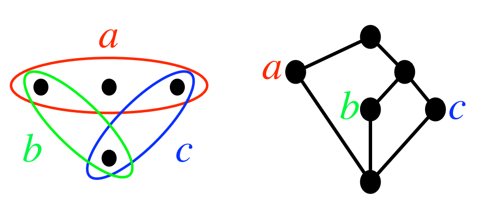

Figure: On the left is the smallest hypergraph that is not geometric with the corresponding intersection lattice on the right.

Characteristic Polynomials

Theorem (Blass-Sagan)

If $\mathcal{A} \subseteq L(B_\ell)$ then

$$\chi(\mathcal{A}, t) = \#([-s, s]^\ell \setminus \bigcup \mathcal{A})$$

where $t = 2s + 1$.

Theorem (Zaslavsky)

Let $\mathcal{A} \subseteq B_\ell$. Then

$$\chi(G_\mathcal{A}, t) = \chi(\mathcal{A}, t).$$

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$\chi(\mathcal{A}, t)$ for edge colored hypergraph arrangements

Definition

A proper vertex coloring of an edge colored hypergraph $(\mathcal{H}, C)$ has for every color there exists a connected component that has two different colors.

$\chi(\mathcal{H}_\mathcal{A}, C_\mathcal{A}, t) = \#$(proper vertex colorings with t colors)

Theorem (Miller-W)

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Special color sets

Definition

For $\Gamma, \Gamma' \subseteq \Lambda$, we say that Γ and Γ' are *multiplicative* if

$$\operatorname{codim} \bigcap_{\gamma \in \Gamma} X_\gamma + \operatorname{codim} \bigcap_{\gamma' \in \Gamma'} X_{\gamma'} = \operatorname{codim} \bigcap_{\gamma \in \Gamma \cup \Gamma'} X_\gamma .$$

For two sets of edges e and e' we write $e \Subset e'$ if e is a refinement of e'.

For two color sets Γ and Γ' we write $\Gamma \Subset \Gamma'$ if $C^{-1}(\Gamma) \Subset C^{-1}(\Gamma')$.

Special color sets

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Massey Color System

Definition

Let $(\mathcal{H}, C)$ be an edge colored hypergraph with edge colors Λ. Let $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$. We call $(\lambda_1, \lambda_2, \lambda_3)$ a Massey color system if the pairs λ_1, λ_2 and $\{\lambda_1, \lambda_2\}$, λ_3 are multiplicative and there exists $\lambda_4, \lambda_5 \in \Lambda$ such that

$$\{\lambda_1, \lambda_2\} \ni \lambda_4 \qquad \{\lambda_2, \lambda_4\} \not\ni \lambda_1 \qquad \{\lambda_1, \lambda_4\} \not\ni \lambda_2$$
$$\{\lambda_2, \lambda_3\} \ni \lambda_5 \qquad \{\lambda_3, \lambda_5\} \not\ni \lambda_2 \qquad \{\lambda_2, \lambda_5\} \not\ni \lambda_3.$$

We call λ_4 and λ_5 *embedded colors* for the triple $\lambda_1, \lambda_2, \lambda_3$.

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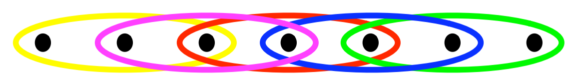
We call λ_4 and λ_5 *embedded colors* for the triple $\lambda_1, \lambda_2, \lambda_3$.

Examples

Example

Let $(\mathcal{H}, C)$ be the edge colored hypergraph below, the edge color sets are given by $\lambda_1 = \text{green}$, $\lambda_2 = \text{red}$, $\lambda_3 = \text{yellow}$, $\lambda_4 = \text{blue}$, and $\lambda_5 = \text{magenta}$.



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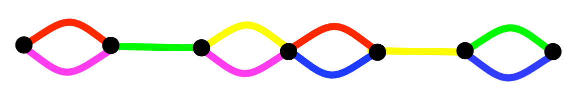

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Massey Products

Theorem (Miller-W)

Let $\mathcal{A}$ be an edge colored hypergraphic arrangement and $(\lambda_1, \lambda_2, \lambda_3)$ a Massey color system with embedded colors λ_4 and λ_5 . Let $\Gamma := \Lambda \backslash \{\lambda_1, \dots, \lambda_5\}$. If the set

$$\{\Psi \subseteq \Gamma \mid \Psi \Subset \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \text{ or } \Psi \Subset \{\lambda_1, \lambda_2, \lambda_3, \lambda_5\}\}$$

is empty then $M(\mathcal{A})$ admits a non-trivial Massey product.

Idea of Proof:

- View $D_{\mathcal{A}}$ in terms of the edge colored hypergraph.
- Apply a functor engineered by Sinha-Walter to $D_{\mathcal{A}}$ that gives a differential graded Lie coalgebra $E(D_{\mathcal{A}})$ (which actually has 2 differentials)
- Find a non-zero differential in the spectral sequence of $E(D_{\mathcal{A}})$
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Pascal Arrangements

Definition

Let n be a positive integer and let $\ell = 2n - 1$. For $1 \leq k \leq n$ let X_k be the subspace defined by

$$X_k = \{(v_1, \dots, v_\ell) \in V \mid v_k = \cdots = v_{k+n-1}\}.$$

Define the subspace arrangement $\mathcal{P}_n$ to be the collection $\{X_1, \dots, X_n\}$.

Properties:

- $L(\mathcal{P}_n)$ is the top n rows of Pascal's triangle. Hence not geometric.
- $\chi(\mathcal{P}_n, t) = (n-1)t^{n-1} - nt^n + t^{2n-1}$

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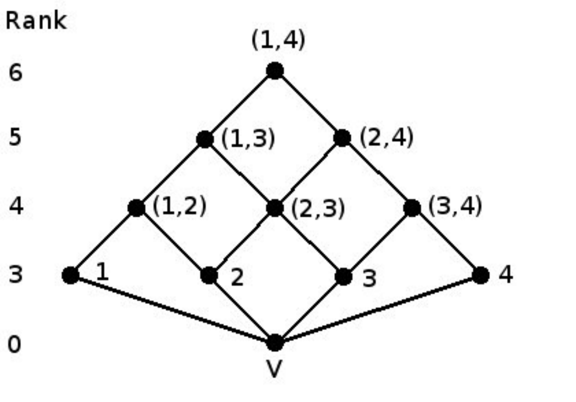
Example

Let $n = 4$ then $\ell = 7$ and there are 4 subspaces $X_1, X_2, X_3,$ and X_4. The Möbius values of the atoms are all -1, the Möbius values of the codimension 4 level elements are 1, and the Möbius values of the higher codimension levels elements are all 0. Hence the characteristic polynomial is $\chi(\mathcal{P}_4, t) = 3t^3 - 4t^4 + t^7$.



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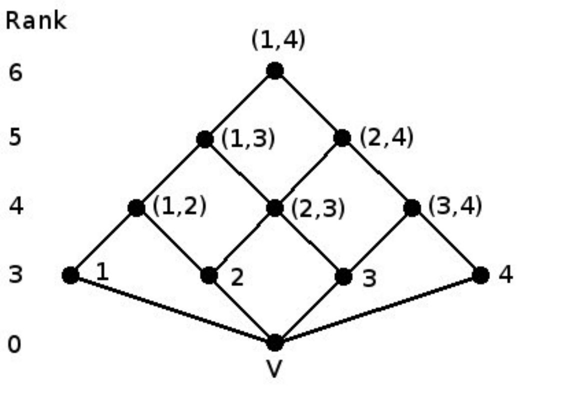

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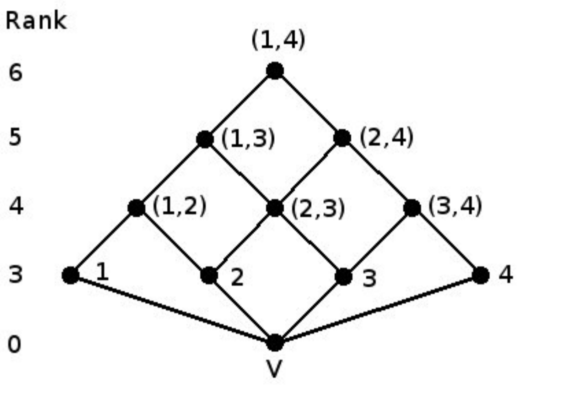

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$M(\mathcal{P}_n)$ is formal for all n.

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THANK YOU!!