# On the geometry and topology of cohomology jumping loci

Alex Suciu

Northeastern University

MSJ Seasonal Institute on Arrangements of Hyperplanes
Hokkaido University
Sapporo, Japan
August 9, 2009

1. Cohomology jumping loci
   - Characteristic varieties
   - Tangent cones
   - Resonance varieties
   - Tangent cone theorem

2. Bieri–Neumann–Strebel–Renz invariants
   - $\Sigma$-invariants
   - Exponential tangent cone upper bound
   - Resonance upper bound

3. Applications
   - Kähler and quasi-Kähler manifolds
   - Toric complexes and right-angled Artin groups
   - Three-dimensional manifolds
   - Hyperplane arrangements

# Characteristic varieties

- $X$ connected CW-complex with finite $k$-skeleton ($k \geq 1$)
- $G = \pi_1(X, x_0)$: a finitely generated group
- $\mathbb{k}$ field; Hom$(G, \mathbb{k}^\times)$ character variety

## Definition

The *characteristic varieties* of $X$ (over $\mathbb{k}$):

$$\mathcal{V}^i_d(X, \mathbb{k}) = \{\rho \in \operatorname{Hom}(G, \mathbb{k}^\times) \mid \dim_{\mathbb{k}} H_i(X, \mathbb{k}_\rho) \geq d\},$$

for $0 \leq i \leq k$ and $d > 0$.

- For each $i$, get stratification $\operatorname{Hom}(G, \mathbb{k}^\times) \supseteq \mathcal{V}^i_1 \supseteq \mathcal{V}^i_2 \supseteq \cdots$
- If $\mathbb{k} \subseteq \mathbb{K}$ extension: $\mathcal{V}^i_d(X, \mathbb{k}) = \mathcal{V}^i_d(X, \mathbb{K}) \cap \operatorname{Hom}(G, \mathbb{k}^\times)$
- For $G$ of type $\mathrm{F}_k$, set: $\mathcal{V}^i_d(G, \mathbb{k}) := \mathcal{V}^i_d(K(G, 1), \mathbb{k})$
- Note: $\mathcal{V}_d(X, \mathbb{k}) := \mathcal{V}^1_d(X, \mathbb{k}) = \mathcal{V}^1_d(\pi_1(X), \mathbb{k})$

Let $X^{\mathrm{ab}} \to X$ be the maximal abelian cover.

## Definition

The *Alexander varieties* of $X$ (over $\mathbb{k}$):

$$\mathcal{W}^i_d(X,\mathbb{k}) = V(E_{d-1}(H_i(X^{\mathrm{ab}},\mathbb{k}))),$$

the subvariety of $\operatorname{Spec}\Lambda = \operatorname{Hom}(G,\mathbb{k}^\times)$ defined by the ideal of codim $d-1$ minors of a presentation matrix for $H_i(X^{\mathrm{ab}},\mathbb{k})$, viewed as module over $\Lambda = \mathbb{k}H_1(X,\mathbb{Z})$.

## Proposition (Papadima–S. 2008)

$$\bigcup_{i=0}^q \mathcal{V}^i_1(X,\mathbb{k}) = \bigcup_{i=0}^q \mathcal{W}^i_1(X,\mathbb{k}), \quad \forall\, 0 \le q \le k$$

$\implies \quad \mathcal{V}^1_1(X,\mathbb{C}) \setminus \{1\} = \mathcal{W}^1_1(X,\mathbb{C}) \setminus \{1\}$ [E. Hironaka 1997]

# Tangent cones and exponential tangent cones

The homomorphism $\mathbb{C} \to \mathbb{C}^\times$, $z \mapsto e^z$ induces

$$\exp\colon \operatorname{Hom}(G, \mathbb{C}) \to \operatorname{Hom}(G, \mathbb{C}^\times), \quad \exp(0) = 1$$

Let $W = V(I)$ be a Zariski closed subset in $\operatorname{Hom}(G, \mathbb{C}^\times)$.

## Definition

- The *tangent cone* at 1 to $W$:

$$TC_1(W) = V(\operatorname{in}(I))$$

- The *exponential tangent cone* at 1 to $W$:

$$\tau_1(W) = \{z \in \operatorname{Hom}(G, \mathbb{C}) \mid \exp(tz) \in W,\ \forall t \in \mathbb{C}\}$$

Both types of tangent cones

- are homogeneous subvarieties of $\mathrm{Hom}(G, \mathbb{C})$
- are non-empty iff $1 \in W$
- depend only on the analytic germ of $W$ at 1
- commute with finite unions and arbitrary intersections

Moreover,

- $\tau_1(W) \subseteq TC_1(W)$
  - $=$ if all irred components of $W$ are subtori
  - $\neq$ in general
- $\tau_1(W)$ is a finite union of rationally defined subspaces

# Computing $\tau_1(W)$

Let $W$ be a Zariski closed subset of $(\mathbb{C}^\times)^n$.
Since $\tau_1$ commutes with intersections, may assume $W = V(f)$, where

$$f = \sum_{u \in S} c_u t_1^{u_1} \cdots t_n^{u_n}$$

is a non-zero Laurent polynomial, with $f(1) = 0$, and support $S \subseteq \mathbb{Z}^n$.
Let $\mathcal{P}$ be the set of partitions $p = S_1 \coprod \cdots \coprod S_r$ of $S$, satisfying

$$\sum_{u \in S_i} c_u = 0, \quad \text{for } i = 1, \dots, r.$$

For each such partition $p$, set

$$L(p) := \{z \in \mathbb{C}^n \mid \langle u - v, z \rangle = 0,\ \forall u, v \in S_i,\ \forall 1 \le i \le r\}.$$

Clearly, $L(p)$ is a rational linear subspace in $\mathbb{C}^n$. Then:

$$\tau_1(W) = \bigcup_{p \in \mathcal{P}} L(p).$$

# Resonance varieties

Let $A = H^*(X, \Bbbk)$. If char $\Bbbk = 2$, assume $H_1(X, \mathbb{Z})$ has no 2-torsion. Then: $a \in A^1 \Rightarrow a^2 = 0$. Get cochain complex ("Aomoto complex")

$$(A, \cdot a)\colon \; A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \longrightarrow \cdots$$

## Definition

The *resonance varieties* of $X$ (over $\Bbbk$):

$$\mathcal{R}^i_d(X, \Bbbk) = \{a \in A^1 \mid \dim_{\Bbbk} H^i(A, \cdot a) \geq d\}$$

Homogeneous subvarieties of $A^1 = H^1(X, \Bbbk)$: $\mathcal{R}^i_1 \supseteq \mathcal{R}^i_2 \supseteq \cdots$

## Theorem (Libgober 2002)

$$TC_1(\mathcal{V}^i_d(X, \mathbb{C})) \subseteq \mathcal{R}^i_d(X, \mathbb{C})$$

Equality does not hold in general (Matei–S. 2002)

# Formality

## Definition

1. A group $G$ is 1-*formal* if its Malcev Lie algebra, $\mathfrak{m}_G = \text{Prim}(\widehat{\mathbb{Q}G})$, is quadratic.
2. A space $X$ is *formal* if its minimal model is quasi-isomorphic to $(H^*(X, \mathbb{Q}), 0)$.

- $X$ formal $\implies$ $\pi_1(X)$ is 1-formal.
- $X_1, X_2$ formal $\implies$ $X_1 \times X_2$ and $X_1 \vee X_2$ are formal
- $G_1, G_2$ 1-formal $\implies$ $G_1 \times G_2$ and $G_1 * G_2$ are 1-formal
- $M_1, M_2$ formal, closed $n$-manifolds $\implies$ $M_1 \# M_2$ formal

# Tangent cone theorem

## Theorem (Dimca–Papadima–S., Duke 2009)

*If $G$ is 1-formal, then* $\exp\colon (\mathcal{R}^1_d(G,\mathbb{C}),0) \xrightarrow{\simeq} (\mathcal{V}^1_d(G,\mathbb{C}),1)$. *Hence*

$$\tau_1(\mathcal{V}^1_d(G,\mathbb{C})) = TC_1(\mathcal{V}^1_d(G,\mathbb{C})) = \mathcal{R}^1_d(G,\mathbb{C})$$

In particular, $\mathcal{R}^1_d(G,\mathbb{C})$ is a union of rationally defined subspaces in $H^1(G,\mathbb{C}) = \mathrm{Hom}(G,\mathbb{C})$.

## Example

Let $G = \langle x_1, x_2, x_3, x_4 \mid [x_1,x_2], [x_1,x_4][x_2^{-2},x_3], [x_1^{-1},x_3][x_2,x_4]\rangle$. Then

$$\mathcal{R}^1_1(G,\mathbb{C}) = \{x \in \mathbb{C}^4 \mid x_1^2 - 2x_2^2 = 0\}$$

splits into subspaces over $\mathbb{R}$ but not over $\mathbb{Q}$. Thus, $G$ is *not* 1-formal.

## Example

- $X = F(\Sigma_g, n)$: the configuration space of $n$ labeled points of a Riemann surface of genus $g$ (a smooth, quasi-projective variety).
- $\pi_1(X) = P_{g,n}$: the pure braid group on $n$ strings on $\Sigma_g$.

Using computation of $H^*(F(\Sigma_g, n), \mathbb{C})$ by Totaro (1996), get

$$\mathcal{R}_1^1(P_{1,n}, \mathbb{C}) = \left\{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n \;\middle|\; \begin{array}{l} \sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 0, \\ x_i y_j - x_j y_i = 0, \text{ for } 1 \leq i < j < n \end{array} \right\}$$

For $n \geq 3$, this is an irreducible, non-linear variety (a rational normal scroll). Hence, $P_{1,n}$ is not 1-formal.

# Bieri–Neumann–Strebel–Renz invariants

$G$ finitely generated group $\rightsquigarrow \mathcal{C}(G)$ Cayley graph.
$\chi\colon G \to \mathbb{R}$ homomorphism $\rightsquigarrow \mathcal{C}_\chi(G)$ induced subgraph on vertex set $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$.

### Definition

$\Sigma^1(G) = \{\chi \in \mathrm{Hom}(G,\mathbb{R}) \setminus \{0\} \mid \mathcal{C}_\chi(G) \text{ is connected}\}$

An open, conical subset of $\mathrm{Hom}(G,\mathbb{R}) = H^1(G,\mathbb{R})$, independent of choice of generating set for $G$.

### Definition

$\Sigma^k(G,\mathbb{Z}) = \{\chi \in \mathrm{Hom}(G,\mathbb{R}) \setminus \{0\} \mid \text{the monoid } G_\chi \text{ is of type } \mathrm{FP}_k\}$

Here, $G$ is of type $\mathrm{FP}_k$ if there is a projective $\mathbb{Z}G$-resolution $P_\bullet \to \mathbb{Z}$, with $P_i$ finitely generated for all $i \leq k$.

- The BNSR invariants $\Sigma^q(G,\mathbb{Z})$ form a descending chain of *open* subsets of $\operatorname{Hom}(G,\mathbb{R})\setminus\{0\}$.
- $\Sigma^k(G,\mathbb{Z})\neq\emptyset \implies G$ is of type $\mathrm{FP}_k$.
- $\Sigma^1(G,\mathbb{Z})=\Sigma^1(G)$.
- The Σ-invariants control the finiteness properties of normal subgroups $N\triangleleft G$ with $G/N$ is abelian:

$$N \text{ is of type } \mathrm{FP}_k \iff S(G,N)\subseteq\Sigma^k(G,\mathbb{Z})$$

  where $S(G,N)=\{\chi\in\operatorname{Hom}(G,\mathbb{R})\setminus\{0\}\mid \chi(N)=0\}$.
- In particular:

$$\ker(\chi\colon G\twoheadrightarrow\mathbb{Z}) \text{ is f.g.} \iff \{\pm\chi\}\subseteq\Sigma^1(G)$$

Let $X$ be a connected CW-complex with finite 1-skeleton, $G = \pi_1(X)$.

## Definition

The *Novikov-Sikorav completion* of $\mathbb{Z}G$:

$$\widehat{\mathbb{Z}G}_\chi = \left\{\lambda \in \mathbb{Z}^G \mid \{g \in \operatorname{supp}\lambda \mid \chi(g) < c\} \text{ is finite, } \forall c \in \mathbb{R}\right\}$$

$\widehat{\mathbb{Z}G}_\chi$ is a ring, contains $\mathbb{Z}G$ as a subring $\implies$ $\widehat{\mathbb{Z}G}_\chi$ is a $\mathbb{Z}G$-module.

## Definition

$$\Sigma^q(X, \mathbb{Z}) = \{\chi \in \operatorname{Hom}(G, \mathbb{R}) \setminus \{0\} \mid H_i(X, \widehat{\mathbb{Z}G}_{-\chi}) = 0, \ \forall\, i \leq q\}$$

Bieri: $G$ of type $\mathrm{FP}_k \implies \Sigma^q(G, \mathbb{Z}) = \Sigma^q(K(G,1), \mathbb{Z}), \forall q \leq k$.

# Exponential tangent cone upper bound

## Theorem (Papadima–S. 2008)

*If $X$ has finite $k$-skeleton, then, for every $q \leq k$,*

$$\Sigma^q(X,\mathbb{Z}) \subseteq \Big(\tau_1^{\mathbb{R}}\big(\bigcup_{i\leq q} \mathcal{V}_1^i(X,\mathbb{C})\big)\Big)^{\complement}. \qquad (*)$$

Thus: Each $\Sigma$-invariant is contained in the complement of a union of rationally defined subspaces. Bound is sharp:

## Example

Let $G$ be a finitely generated nilpotent group. Then

$$\Sigma^q(G,\mathbb{Z}) = \operatorname{Hom}(G,\mathbb{R}) \setminus \{0\}, \quad V_1^q(G,\mathbb{C}) = \{1\}, \quad \forall q$$

and so (*) holds as an equality.

# Resonance upper bound

## Corollary

*Suppose* $\exp\colon (\mathcal{R}^i_1(X,\mathbb{C}),0) \xrightarrow{\simeq} (\mathcal{V}^i_1(X,\mathbb{C}),1)$, *for* $i \le q$. *Then:*

$$\Sigma^q(X,\mathbb{Z}) \subseteq \Big(\bigcup_{i\le q} \mathcal{R}^i_1(X,\mathbb{R})\Big)^{\complement}.$$

## Corollary

*Suppose $G$ is a 1-formal group. Then* $\Sigma^1(G) \subseteq \mathcal{R}^1_1(G,\mathbb{R})^{\complement}$.
*In particular, if* $\mathcal{R}^1_1(G,\mathbb{R}) = H^1(G,\mathbb{R})$, *then* $\Sigma^1(G) = \emptyset$.

## Example

The above inclusion may be strict: Let $G = \langle a, b \mid aba^{-1} = b^2\rangle$.
Then $G$ is 1-formal, $\Sigma^1(G) = (-\infty, 0)$, yet $\mathcal{R}^1_1(G,\mathbb{R}) = \{0\}$.

# Kähler and quasi-Kähler manifolds

- A compact, connected, complex manifold *M* is *Kähler* if there is a Hermitian metric *h* such that $\omega = \mathfrak{Im}(h)$ is a closed 2-form.
- A manifold *X* is called *quasi-Kähler* if $X = \overline{X} \setminus D$, where $\overline{X}$ is Kähler and *D* is a divisor with normal crossings.

Formality properties:

- *M* Kähler $\Rightarrow$ *M* is formal
  (Deligne, Griffiths, Morgan, Sullivan 1975)
- $X = \mathbb{CP}^n \setminus \{\text{hyperplane arrangement}\} \Rightarrow X$ is formal
  (Brieskorn 1973)
- *X* quasi-projective, $W_1(H^1(X, \mathbb{C})) = 0 \Rightarrow \pi_1(X)$ is 1-formal
  (Morgan 1978)
- $X = \mathbb{CP}^n \setminus \{\text{hypersurface}\} \Rightarrow \pi_1(X)$ is 1-formal
  (Kohno 1983)

## Theorem (Arapura 1997)

*Let $X = \overline{X} \setminus D$ be a quasi-Kähler manifold. Then:*

1. *Each component of $\mathcal{V}^1_1(X)$ is either an isolated unitary character, or of the form $\rho \cdot f^*(H^1(C, \mathbb{C}^\times))$, for some torsion character $\rho$ and some admissible map $f \colon X \to C$.*
2. *If either $X = \overline{X}$ or $b_1(\overline{X}) = 0$, then each component of $\mathcal{V}^i_d(X)$ is of the form $\rho \cdot f^*(H^1(T, \mathbb{C}^\times))$, for some unitary character $\rho$ and some holomorphic map $f \colon X \to T$ to a complex torus.*

Here, $f \colon X \to C$ is *admissible* (or, a *pencil*) if $f$ is a holomorphic, surjective map to a connected, smooth complex curve $C$, and there is a holomorphic, surjective extension $\overline{f} \colon \overline{X} \to \overline{C}$ with connected fibers.

## Corollary

*All components of $\mathcal{V}^i_d(X, \mathbb{C})$ passing through* 1 *are subtori of* $\operatorname{Hom}(G, \mathbb{C}^\times)$*, provided $i = d = 1$, or $X$ is Kähler, or $W_1(H^1(X, \mathbb{C})) = 0$.*

Let $X$ be a quasi-Kähler manifold, $G = \pi_1(X)$.

## Theorem (Dimca–Papadima–S., Duke 2009)

*Let $\{\mathcal{V}^\alpha\}_\alpha$ be the irred components of $\mathcal{V}^1_1(G)$ containing $1$. Set $\mathcal{T}^\alpha = TC_1(\mathcal{V}^\alpha)$. Then:*

1. *Each $\mathcal{T}^\alpha$ is a p-isotropic subspace of $H^1(G,\mathbb{C})$, of $\dim \geq 2p+2$, for some $p = p(\alpha) \in \{0,1\}$.*
2. *If $\alpha \neq \beta$, then $\mathcal{T}^\alpha \bigcap \mathcal{T}^\beta = \{0\}$.*

*Assume further that G is $1$-formal. Let $\{\mathcal{R}^\alpha\}_\alpha$ be the irred components of $\mathcal{R}^1_1(G)$. Then:*

3. $\{\mathcal{T}^\alpha\}_\alpha = \{\mathcal{R}^\alpha\}_\alpha$.
4. $\mathcal{R}^1_d(G) = \{0\} \cup \bigcup_{\alpha:\ \dim \mathcal{R}^\alpha > d+p(\alpha)} \mathcal{R}^\alpha$.

Let $X$ be a quasi-Kähler manifold, $G = \pi_1(X)$.

## Theorem (Papadima–S. 2008)

1. $\Sigma^1(G) \subseteq TC_1^{\mathbb{R}}(\mathcal{V}_1^1(G, \mathbb{C}))^{\complement}$.
2. *If $X$ is Kähler, or $W_1(H^1(X, \mathbb{C})) = 0$, then $\mathcal{R}_1^1(G, \mathbb{R})$ is a finite union of rationally defined linear subspaces, and $\Sigma^1(G) \subseteq \mathcal{R}_1^1(G, \mathbb{R})^{\complement}$.*

## Example

Assumption from (2) is necessary. E.g., let $X$ be the complex Heisenberg manifold: bundle $\mathbb{C}^{\times} \to X \to (\mathbb{C}^{\times})^2$ with $e = 1$. Then:

1. $X$ is a smooth quasi-projective variety;
2. $G = \pi_1(X)$ is nilpotent (and not 1-formal);
3. $\Sigma^1(G) = \mathbb{R}^2 \setminus \{0\}$ and $\mathcal{R}_1^1(G, \mathbb{R}) = \mathbb{R}^2$.

Thus, $\Sigma^1(G) \not\subseteq \mathcal{R}_1^1(G, \mathbb{R})^{\complement}$.

For Kähler manifolds, we can say precisely when the resonance upper bound for $\Sigma^1$ is attained.

**Theorem (Papadima–S. 2008)**

*Let $M$ be a compact Kähler manifold with $b_1(M) > 0$, and $G = \pi_1(M)$. The following are equivalent:*

1. $\Sigma^1(G) = \mathcal{R}^1_1(G, \mathbb{R})^{\complement}$.
2. *If $f\colon M \to C$ is an elliptic pencil, then $f$ has no multiple fibers.*

Proof uses results of Arapura, DPS, and Delzant.

# Toric complexes and right-angled Artin groups

## Definition

Let $L$ be simplicial complex on $n$ vertices. The associated *toric complex*, $T_L$, is the subcomplex of the $n$-torus obtained by deleting the cells corresponding to the missing simplices of $L$.

- Special case of "generalized moment angle complex".
- $\pi_1(T_L)$ is the *right-angled Artin group* associated to graph $\Gamma = L^{(1)}$:

$$G_\Gamma = \langle v \in V(\Gamma) \mid vw = wv \text{ if } \{v, w\} \in E(\Gamma)\rangle.$$

- $K(G_\Gamma, 1) = T_{\Delta_\Gamma}$, where $\Delta_\Gamma$ is the *flag complex* of $\Gamma$.
- $H^*(T_L, \Bbbk)$ is the *exterior Stanley-Reisner ring* of $L$, with generators the duals $v^*$, and relations the monomials corresponding to the missing simplices of $L$.
- $T_L$ is formal, and so $G_\Gamma$ is 1-formal.

## Example

- $\Gamma = \overline{K}_n \Rightarrow G_\Gamma = F_n$
- $\Gamma = K_n \Rightarrow G_\Gamma = \mathbb{Z}^n$
- $\Gamma = \Gamma' \coprod \Gamma'' \Rightarrow G_\Gamma = G_{\Gamma'} * G_{\Gamma''}$
- $\Gamma = \Gamma' * \Gamma'' \Rightarrow G_\Gamma = G_{\Gamma'} \times G_{\Gamma''}$

Using a result of Aramova, Avramov, Herzog (2000), we get:

## Theorem (Papadima–S., Adv. Math. 2009)

$$\mathcal{R}^i_d(T_L, \Bbbk) = \bigcup_{\substack{\mathsf{W} \subset \mathsf{V} \\ \sum_{\sigma \in L_{\mathsf{V}\setminus \mathsf{W}}} \dim_{\Bbbk} \widetilde{H}_{i-1-|\sigma|}(\mathrm{lk}_{L_{\mathsf{W}}}(\sigma), \Bbbk) \geq d}} \Bbbk^{\mathsf{W}},$$

*where $L_{\mathsf{W}}$ is the subcomplex induced by $L$ on $\mathsf{W}$, and $\mathrm{lk}_K(\sigma)$ is the link of a simplex $\sigma$ in a subcomplex $K \subseteq L$.*

Similar formula holds for $\mathcal{V}^i_d(T_L, \Bbbk)$, with $\Bbbk^{\mathsf{W}}$ replaced by $(\Bbbk^{\times})^{\mathsf{W}}$.
In particular: $\exp\colon (\mathcal{R}^i_d(T_L, \mathbb{C}), 0) \xrightarrow{\simeq} (\mathcal{V}^i_d(T_L, \mathbb{C}), 1)$.

# Non-propagation of resonance

## Remark

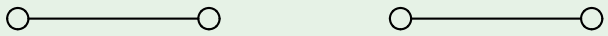
Given a graded algebra $A$, we say resonance “propagates" if

$$\mathcal{R}^i_1(A) \subseteq \mathcal{R}^k_1(A),\ \forall i \le k, \quad \text{provided } A^j \ne 0, \text{ for } i \le j \le k$$

If $A$ is the Orlik-Solomon algebra, then resonance propagates.
But this is not the case for the exterior Stanley-Reisner ring.

## Example

Let $\Gamma = \Gamma_1 \coprod \Gamma_2$, where $\Gamma_j = K_{n_j}$ and $n_j \ge 2$, e.g.:



Then:

$$\mathcal{R}^i_1(G_\Gamma, \Bbbk) = \begin{cases} \Bbbk^{n_1+n_2}, & \text{if } i = 1, \\ \Bbbk^{n_1} \times \{0\} \cup \{0\} \times \Bbbk^{n_2}, & \text{if } 1 < i \le \min(n_1, n_2). \end{cases}$$

Using (1) resonance upper bound, and (2) computation of $\Sigma^k(G_\Gamma, \mathbb{Z})$ by Meier, Meinert, VanWyk (1998), we get:

## Corollary (Papadima-S. 2008)

$$\Sigma^k(T_L, \mathbb{Z}) \subseteq \Big(\bigcup_{i\le k} \mathcal{R}^i_1(T_L, \mathbb{R})\Big)^{\complement}$$
$$\Sigma^k(G_\Gamma, \mathbb{Z}) = \Big(\bigcup_{i\le k} \mathcal{R}^i_1(T_{\Delta_\Gamma}, \mathbb{R})\Big)^{\complement}$$

## Theorem (Dimca–Papadima–S. Duke 2009)

*The following are equivalent:*

1. $G_\Gamma$ *is a quasi-Kähler group*
2. $\Gamma = K_{n_1,\dots,n_r} := \overline{K}_{n_1} * \cdots * \overline{K}_{n_r}$
3. $G_\Gamma = F_{n_1} \times \cdots \times F_{n_r}$

1. $G_\Gamma$ *is a Kähler group*
2. $\Gamma = K_{2r}$
3. $G_\Gamma = \mathbb{Z}^{2r}$

Bestvina–Brady groups: $N_\Gamma = \ker(\nu\colon G_\Gamma \twoheadrightarrow \mathbb{Z})$, where $\nu(v) = 1$

## Theorem (Dimca–Papadima–S., JAG 2008)

*The following are equivalent:*

1. $N_\Gamma$ *is a quasi-Kähler group*
2. $\Gamma$ *is either a tree, or* $\Gamma = K_{n_1,\dots,n_r}$, *with some* $n_i = 1$, *or all* $n_i \geq 2$ *and* $r \geq 3$.

1. $N_\Gamma$ *is a Kähler group*
2. $\Gamma = K_{2r+1}$
3. $N_\Gamma = \mathbb{Z}^{2r}$

## Example

$\Gamma = K_{2,2,2} \rightsquigarrow G_\Gamma = F_2 \times F_2 \times F_2$
$N_\Gamma$ = the Stallings group = group of the $X_3$ arrangement
$N_\Gamma$ is finitely presented, but $H_3(N_\Gamma, \mathbb{Z})$ has infinite rank, so $N_\Gamma$ not $\mathrm{FP}_3$.

# 3-manifolds

## Question (Goldman–Donaldson 1989, Reznikov 1993)

Which 3-manifold groups are Kähler groups?

Reznikov (2002) and Hernández-Lamoneda (2001) gave partial solutions.

## Theorem (Dimca–S., JEMS 2009)

*Let $G$ be the fundamental group of a closed 3-manifold. Then $G$ is a Kähler group $\iff$ $G$ is a finite subgroup of $\mathrm{O}(4)$, acting freely on $S^3$.*

Idea of proof: compare the resonance varieties of (orientable) 3-manifolds to those of Kähler manifolds.

### Proposition

*Let $M$ be a closed, orientable 3-manifold. Then:*

1. *$H^1(M,\mathbb{C})$ is not 1-isotropic.*
2. *If $b_1(M)$ is even, then $\mathcal{R}_1(M,\mathbb{C}) = H^1(M,\mathbb{C})$.*

On the other hand, it follows from [DPS 2009] that:

### Proposition

*Let $M$ be a compact Kähler manifold with $b_1(M) \neq 0$. If $\mathcal{R}_1(M,\mathbb{C}) = H^1(M,\mathbb{C})$, then $H^1(M,\mathbb{C})$ is 1-isotropic.*

But $G = \pi_1(M)$, with $M$ Kähler $\Rightarrow b_1(G)$ even.
Thus, if $G$ is both a 3-mfd group and a Kähler group $\Rightarrow b_1(G) = 0$.
Using work of Fujiwara (1999) and Reznikov (2002) on Kazhdan's property (T), as well as Perelman (2003) $\Rightarrow G$ finite subgroup of O(4).

## Question

Which 3-manifold groups are quasi-Kähler groups?

## Theorem (Dimca–Papadima–S. 2008)

*Let G be the fundamental group of a closed, orientable 3-manifold. Assume G is 1-formal. Then the following are equivalent:*

1. $\mathfrak{m}(G) \cong \mathfrak{m}(\pi_1(X))$, *for some quasi-Kähler manifold X.*
2. $\mathfrak{m}(G) \cong \mathfrak{m}(\pi_1(M))$, *where M is either* $S^3$, $\#^n S^1 \times S^2$, *or* $S^1 \times \Sigma_g$.

# Hyperplane arrangements

Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{C}^\ell$, with complement $X = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$, and group $G = \pi_1(X)$.

- Resonance varieties $\mathcal{R}^1_d(X, \mathbb{C})$ are very much understood.
- Propagation of resonance: $\mathcal{R}^i_1(X, \mathbb{C}) \subseteq \mathcal{R}^j_1(X, \mathbb{C}), \ \forall i < j \leq \ell$.
- Tangent cone formula:

  $$\exp\colon (\mathcal{R}^i_d(X, \mathbb{C}), 0) \xrightarrow{\simeq} (\mathcal{V}^i_d(X, \mathbb{C}), 1), \ \forall i, d > 0$$

  In particular, $TC_1(\mathcal{V}^i_d(X, \mathbb{C})) = \mathcal{R}^i_d(X, \mathbb{C})$.
- Components of $\mathcal{V}^1_d(X, \mathbb{C})$ passing through 1 are combinatorially determined.
- $\mathcal{V}^1_1(X, \mathbb{C})$ may contain translated subtori, e.g., if $\mathcal{A}$ is the deleted $B_3$ arrangement.

Using (1) res upper bound and (2) propagation of resonance, we get:

## Theorem

$$\Sigma^q(X,\mathbb{Z}) \subseteq \mathcal{R}_1^q(X,\mathbb{R})^{\complement}$$

## Problem

*Let $G = G(\mathcal{A})$ be an arrangement group.*

1. *Compute the BNS invariant $\Sigma^1(G)$.*
2. *Does the equality $\Sigma^1(G) = -\Sigma^1(G)$ hold?*
3. *Even stronger, does the equality $\Sigma^1(G) = \mathcal{R}_1(G,\mathbb{R})^{\complement}$ hold?*
4. *If it doesn't, is the BNS invariant combinatorially determined?*

## Remark

For a complexified real arrangement, $\Sigma^1(G) = -\Sigma^1(G)$, which is consistent with the symmetry property of $\mathcal{R}_1(G,\mathbb{R})^{\complement}$.

# Boundary manifold

Let $\mathcal{A} = \{\ell_0, \dots, \ell_n\}$ be an arrangement of lines in $\mathbb{CP}^2$. The *boundary manifold* of $\mathcal{A}$ is the closed, orientable 3-manifold $M = M(\mathcal{A})$ obtained by taking the boundary of a regular neighborhood of $\bigcup_{i=0}^{n} \ell_i$ in $\mathbb{CP}^2$.

## Theorem (Cohen–S., GTM 08, Dimca–Papadima–S., IMRN 08)

*Let $\mathcal{A} = \{\ell_0, \dots, \ell_n\}$ be an arrangement of lines in $\mathbb{CP}^2$, and let $M$ be the corresponding boundary manifold. The following are equivalent:*

1. *The manifold $M$ is formal.*
2. *The group $G = \pi_1(M)$ is 1-formal.*
3. $TC_1(V_1(G, \mathbb{C})) = \mathcal{R}_1(G, \mathbb{C})$.
4. *The group $G$ is quasi-projective.*
5. *$\mathcal{A}$ is either a pencil (and so $M = \sharp^n S^1 \times S^2$), or $\mathcal{A}$ is a near-pencil (and so $M = S^1 \times \Sigma_{n-1}$).*

# Milnor fibration

- $f \in \mathbb{C}[z_0, \dots, z_d]$ weighted homogeneous polynomial of degree $n$, with positive integer weights $(w_0, \dots, w_d)$.
- $V(f)$ the zero-set of $f\colon \mathbb{C}^{d+1} \to \mathbb{C}$.
- $X = \mathbb{C}^{d+1} \setminus V(f)$ its complement.
- Milnor fibration: $f\colon X \to \mathbb{C}^*$.
- Milnor fiber: $F = f^{-1}(1)$. It is a smooth affine variety, with the homotopy type of a $d$-dimensional, finite CW-complex. When $(V(f), 0)$ is reduced, $F$ is connected.
- Geometric monodromy: $h\colon F \to F$, $(z_0, \dots, z_d) \mapsto (\xi^{w_0} z_0, \dots, \xi^{w_d} z_d)$, where $\xi = \exp(2\pi i/n)$.
- If $f$ is homogeneous, $F$ is a regular, $n$-fold cyclic cover of $U = \mathbb{CP}^d \setminus V(f)$.
- Hence, we may compute $b_1(F)$ from $\mathcal{V}^1_d(U, \mathbb{C})$.

## Question (Papadima–S., BMSR 2009)

Is the Milnor fiber of a reduced polynomial always 1-formal?

## Example (Zuber 2009)

Let $\mathcal{A}$ be the monomial arrangement in $\mathbb{C}^3$, defined by the polynomial

$$f(z_0, z_1, z_2) = (z_0^3 - z_1^3)(z_0^3 - z_2^3)(z_1^3 - z_2^3)$$

Then $TC_1(\mathcal{V}_1(F, \mathbb{C})) \neq \mathcal{R}_1(F, \mathbb{C})$.
Hence, by the Tangent Cone Theorem, $F$ is not 1-formal.

## Example (Fernández de Bobadilla 2009)

Let

$$f(z_0,\dots,z_{10}) = z_0z_2z_3z_5z_6 + z_0z_2z_4z_7 + z_1z_2z_4z_8 + z_1z_3z_5z_9 + z_1z_3z_4z_{10}$$

- $f$ is weighted homogeneous of degree 5, with weights $(1,1,1,1,1,1,1,2,2,2,2)$.
- The Milnor fiber $F$ is homotopy equivalent to the complement of the coordinate subspace arrangement $\mathcal{A} = \{H_1,\dots,H_5\}$ in $\mathbb{C}^6$, with $H_i = \{x_i = x_{i+1} = 0\}$.
- $F$ is 2-connected.
- As shown in [Denham–S., PAMQ 2007], there are classes $\alpha,\beta,\gamma \in H^3(F,\mathbb{Z}) = \mathbb{Z}^5$ such that the triple Massey product $\langle\alpha,\beta,\gamma\rangle \in H^8(F,\mathbb{Z}) = \mathbb{Z}$ is non-zero.
- Hence, $F$ is not formal.