Social choice on complex objects: A geometric approach

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We assume that choices are made over a set of n elements or features $F = \{f_1, \ldots, f_n\}$ taking a value out of a finite set of $m+1$ possibilities, i.e. $f_i \in \{0, 1, 2, \ldots, m\}$.
Then the space of possibilities is given by $(m+1)^n$ possible configurations $X = \{x_1, \ldots, x_{(m+1)^n}\}$.

Let us choose in $\mathbb{R}^n$ an hyperplane arrangement

$$\mathcal{A}_{n,m} = \{H_{i,j}\}_{\substack{1 \leq i \leq n \\ 0 \leq j \leq m-1}},$$

where $H_{i,j}$ is the hyperplane of equation $y_i = j$; i.e. an hyperplane parallel to a coordinate hyperplane of an orthogonal Cartesian system in $\mathbb{R}^n$.
Then each configuration $x_i = i_1 \cdots i_n$ corresponds to the chamber C_i which contains the open set

$$\{(y_1, \ldots, y_n) \in \mathbb{R}^n \mid i_j - 1 < y_j < i_j,\ j = 1, \ldots, n\}.$$

If P is a set of transitive preferences, a social decision rule $\mathcal{R}$ is a function:

$$\begin{array}{rcl} \mathcal{R}: \quad P^n & \longrightarrow & \overline{P} \\ (\succeq_1, \ldots, \succeq_k) & \longmapsto & \succeq_{\mathcal{R}(\succeq_1,\ldots,\succeq_k)} \end{array}$$

which associates a societal rule $\succeq_{\mathcal{R}(\succeq_1,\ldots,\succeq_k)}$ to the preferences of k agents.

Let us assume that for any two configurations x_i and x_j it is always possible to say if $x_i \succeq_{\mathcal{R}(\succeq_1,\ldots,\succeq_n)} x_j$, $x_j \succeq_{\mathcal{R}(\succeq_1,\ldots,\succeq_n)} x_i$ or both.

In a very natural way if Δ is the diagonal of the cartesian product $X \times X$, then an element $\succeq_{\mathcal{R}} \in \overline{P}$ defines a subset

$$Y_{1,\succeq_{\mathcal{R}}} \subset X \times X \setminus \Delta$$

as follows: a couple (x_i, x_j) is in $Y_{1,\succeq_{\mathcal{R}}}$ if and only if $x_i \succ_{\mathcal{R}} x_j$; both (x_i, x_j) and (x_j, x_i) are in $Y_{1,\succeq_{\mathcal{R}}}$ iff $x_i \succeq_{\mathcal{R}} x_j$ and $x_j \succeq_{\mathcal{R}} x_i$.

Moreover we can represent the sets X and $Y_{1,\succeq_{\mathcal{R}}}$ respectively as the set of vertices and edges of an oriented graph $\mathcal{Y}_{\succeq_{\mathcal{R}}}$.

Two vertices x_i and x_j in X are connected by an edge if and only if $(x_i, x_j) \in Y_{1,\succeq_{\mathcal{R}}}$ or $(x_j, x_i) \in Y_{1,\succeq_{\mathcal{R}}}$, while the orientation is from x_i to x_j in the first case and from x_j to x_i in the latter.

Then a rule $\succ_{\mathcal{R}}$ of the form:

$(0,0,0)$ preferred to all except $(1,1,0) \succ_{\mathcal{R}} (0,0,0), (0,0,1) \succ_{\mathcal{R}} (0,0,0)$;

$(0,1,0) \prec_{\mathcal{R}} (0,1,1), (0,1,0) \prec_{\mathcal{R}} (1,1,1), (0,1,0) \prec_{\mathcal{R}} (1,0,0)$,

$(0,1,0) \succ_{\mathcal{R}} (1,0,1), (0,1,0) \succ_{\mathcal{R}} (1,1,0), (0,1,0) \prec_{\mathcal{R}} (0,0,1)$;

$(0,1,1) \succ_{\mathcal{R}} (1,1,1), (0,1,1) \succ_{\mathcal{R}} (1,0,0), (0,1,1) \succ_{\mathcal{R}} (1,0,1)$,

$(0,1,1) \succ_{\mathcal{R}} (1,1,0), (0,1,1) \prec_{\mathcal{R}} (0,0,1)$;

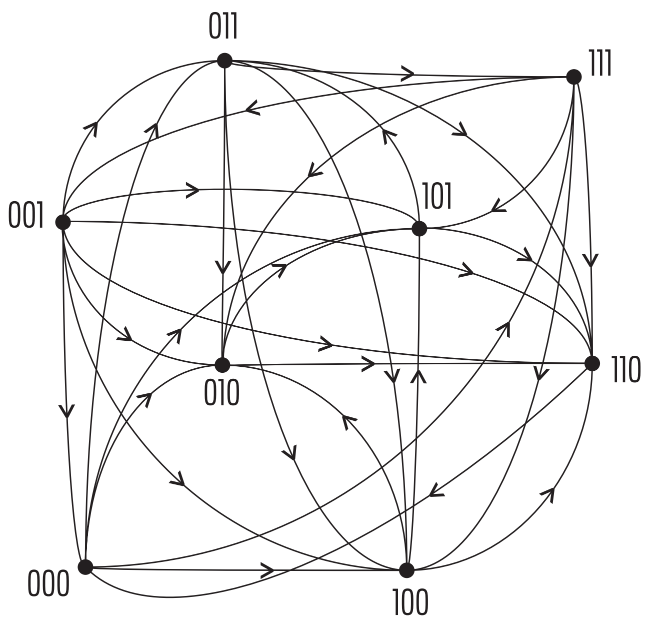
$(1,1,1) \succ_{\mathcal{R}} (1,0,0), (1,1,1) \succ_{\mathcal{R}} (1,0,1), (1,1,1) \succ_{\mathcal{R}} (1,1,0), (1,1,1) \succ_{\mathcal{R}} (0,0$

$(1,0,0) \succ_{\mathcal{R}} (1,0,1), (1,0,0) \succ_{\mathcal{R}} (1,1,0), (1,0,0) \prec_{\mathcal{R}} (0,0,1)$;

$(1,0,1) \succ_{\mathcal{R}} (1,1,0), (1,0,1) \prec_{\mathcal{R}} (0,0,1)$;

$(1,1,0) \prec_{\mathcal{R}} (0,0,1)$.

is described by the following graph:



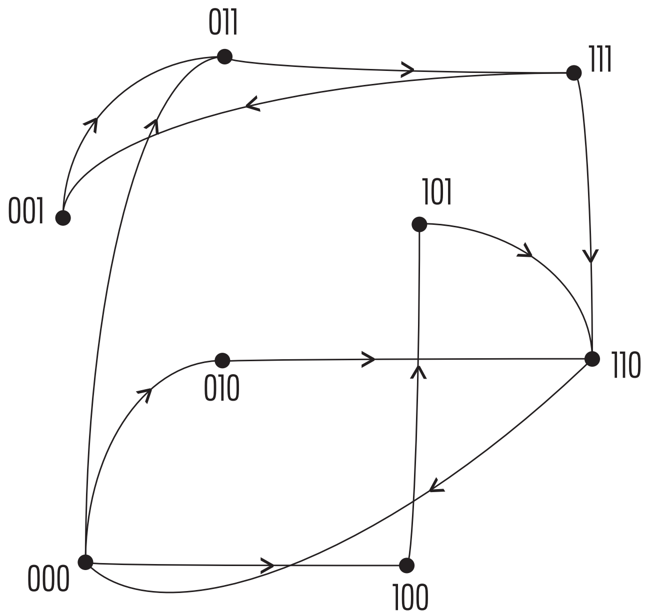
Let us remark that cycles in the oriented graph $\mathcal{Y}_{\succeq_{\mathcal{R}}}$ correspond exactly to cycles *á la* Condorcet-Arrow.

Salvetti's Complex in social choice

The set of generators $\mathcal{S}_0(\mathcal{A}_{n,m})$ of the 0-skeleton of the Salvetti's complex $\mathcal{S}(\mathcal{A}_{n,m})$ is in one to one correspondence with the set of chambers in $\mathcal{A}_{n,m}$, i.e. with the set of configurations X.

While, given a rule $\succeq_{\mathcal{R}}$, any edge $(x_i, x_j) \in Y_{1,\succeq_{\mathcal{R}}}$ can be written as a formal sum of a minimal number of edges in the 1-skeleton $\mathcal{S}_1(\mathcal{A}_{n,m})$. The number of elements is exactly the number of hyperplanes which separate the two configurations $x_i, x_j \in X$.

Then the above graph can be reduced as follows: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The voting process

Definition 1 *Given a subset $I \subset \{1, \ldots, n\}$, a* *decision module* *$\mathcal{A}_I$ is a non empty subset of the arrangement $\mathcal{A}_{n,m}$ of the form*

$$\mathcal{A}_I = \{H_{i,j}\}_{\substack{i \in I \\ 0 \le j \le m-1}}.$$

Definition 2 *A* *modules scheme* *is a set of decision modules $A = \{\mathcal{A}_{I_1}, \ldots, \mathcal{A}_{I_k}\}$ such that $\cup_{j=1}^{k} I_j = \{1, \ldots, n\}$.*

Let A be a scheme, we call *agenda* *α over a modules scheme $A = \{\mathcal{A}_{I_1}, \ldots, \mathcal{A}_{I_k}\}$ an ordered uple of indeces $(h_0, \ldots, h_t)$ in $\{1, \ldots, k\}$ such that the set $\{h_0, \ldots, h_t\} = \{1, \ldots, k\}$. Then an agenda α sets the order in which our society should vote.*

A configuration z is a local optimum for A if and only if it exists a starting configuration $x \in X$ such that the voting process ends up in z.

Given a local optimum z, a modules scheme A and an agenda α, the basin of attraction of z is the set $\Psi(z, A, \alpha)$ of all $x \in X$ such that exists a voting process starting in x and ending in z.

The dipendence of the optimum from the modules scheme is very strong.

Indeed there are many examples in which two different configurations $z_1, z_2 \in X$ are global optima for two different choice of modules schemes.

Indeed we have the following:

Theorem 1 *Let $\mathcal{R}$ be a societal decision rule over $X = \mathcal{S}_0(\mathcal{A}_{n,m})$ and $z \in X$ be a given a configuration. Then z is a local optimum for a modules sheme A_z if and only if for any configuration x such that $x \succ_{\mathcal{R}} z$ then $d_p(x, z) > 1$.*

Definition 3 *Two configurations* $z, x \in X$ *are prominently separate if there exists two hyperplanes* $H_{i_1,j_1}, H_{i_2,j_2} \in \mathcal{A}_{n,m}$ *with* $i_1 \neq i_2$ *and* $z \mid H_{i_1,j_1} \mid x$*,* $z \mid H_{i_2,j_2} \mid x$*.*

The prominent distance $d_p(z, x)$, will be the minimum number of hyperplanes which prominently separate *z* and *x*.

Matters which deserve farther studies:

1. Is it possible to generalize this description? Let us remark that many people started from social choice model obtaining general results in mathematics: for example H. Terao, G. Chichilnisky, S.Weinberger and others.

2. Are there sufficient conditions to characterize a global optimum? (problem in graph theory)

3. How does this model change when we apply it to customers instead of voters?