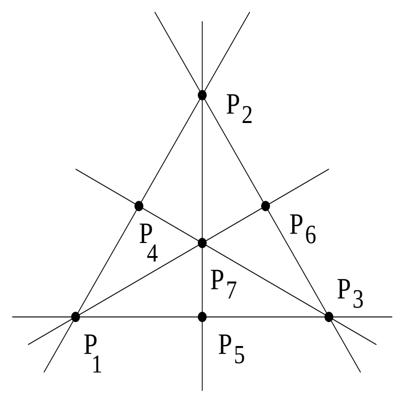
Arrangements and Computations II: Koszul and Lie Algebras



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Let G be a finitely-generated group, with normal subgroups,

$$G = G_1 \geq G_2 \geq G_3 \geq \cdots ,$$

defined inductively by $G_k = [G_{k-1}, G]$.

We obtain an associated Lie algebra

$$gr(G) \otimes \mathbb{Q} := \bigoplus_{k=1}^{\infty} G_k/G_{k+1} \otimes \mathbb{Q},$$

with Lie bracket induced by the commutator map. Let $\phi_k = \phi_k(G)$ denote the rank of the k-th quotient.

Let $X_{\mathcal{A}} = \mathbb{C}^\ell \setminus \mathcal{A}$.

Mission: study the fundamental group $G = \pi_1(X_{\mathcal{A}})$ of the complement $X_{\mathcal{A}}$ of a complex hyperplane arrangement $\mathcal{A}$. Write

$$\mathfrak{g} = gr(\pi_1(X_{\mathcal{A}})) \otimes \mathbb{Q}$$

Lefschetz-type theorem of Hamm-Le implies taking generic two dimensional slice gives isomorphism on π_1, so to study $\pi_1(X_{\mathcal{A}})$, may assume $\mathcal{A} \subseteq \mathbb{C}^2$ or that (coning) $\mathcal{A} \subseteq \mathbb{P}^2$.

WARNING1! Hirzebruch "The topology of the complement of a configuration of lines in the projective plane is very interesting, the investigation of the fundamental group of the complement very difficult."

Presentations for $\pi_1(X_{\mathcal{A}})$ given by

- **Randell**
- **Salvetti**
- **Arvola**
- **Cohen-Suciu**

Braid-Monodromy presentation is simplest, see Suciu's survey

WARNING2! $\pi_1(X_{\mathcal{A}})$ is not combinatorial (Rybnikov).

Recall that the cohomology ring of $X_{\mathcal{A}}$

$$H^*(X_{\mathcal{A}}, \mathbb{C}) = A = E/I$$

is the Orlik–Solomon algebra. E is an exterior algebra with a generator

$$e_i \leftrightarrow H_i \in \mathcal{A}$$

and I is generated by all elements of the form $\partial e_{i_1 \dots i_r} := \sum_q (-1)^{q-1} e_{i_1} \cdots \widehat{e_{i_q}} \cdots e_{i_r}$, for which $\mathrm{codim}(H_{i_1} \cap \cdots \cap H_{i_r}) < r$.

Compute Orlik-Solomon algebra for the arrangement A_3, and compute the Hilbert Series. For A_3, the LCS ranks are

$$6 \quad 4 \quad 10 \quad 21 \quad 54 \quad \cdots$$

General formula for A_3 is $\phi_k = w_k(2) + w_k(3)$.

Consider the series

$$\prod_{k=1}^{\infty} \frac{1}{(1-t^k)^{\phi_k}}$$

For A_3, this is

$$\frac{1}{(1-t)^6}\frac{1}{(1-t^2)^4}\frac{1}{(1-t^3)^{10}}\frac{1}{(1-t^4)^{21}}\frac{1}{(1-t^5)^{54}}\cdots$$

Compute the first few terms of the expansion:

$$1+6t+25t^2+90t^3+301t^4+966t^5+3025t^6+\cdots$$

MAGIC TRICK 1: multiply this with

$$\pi(A_3,-t) = 1 - 6t + 11t^2 - 6t^3$$

Theorem 1 (Kohno's LCS formula) *For the braid arrangement A_{n-1} (graphic arrangement for the complete graph K_n)*

$$\prod_{k=1}^{\infty}(1-t^k)^{\phi_k} = \prod_{i=1}^{n-1}(1-it).$$

Previous example: braid arrangement A_3, so Kohno's result explains the computation that

$$\prod_{k=1}^{\infty}\frac{1}{(1-t^k)^{\phi_k}}\cdot(1-6t+11t^2-6t^3)=1$$

MAGIC TRICK 2: compute

$$\dim_{\mathbb{C}} Tor_i^{A_3}(\mathbb{C},\mathbb{C})_i$$

To do this, look at the top row of the betti diagram for the resolution of $\mathbb{C}$ over A.

LCS formulas for arrangements

- Braid arrangements **[Kohno]**
- Fiber type arrangements **[Falk–Randell]**
- = supersolvable **[Terao]**
- Lower bound for ϕ_k **[Falk]**
- Koszul duality **[Shelton–Yuzvinsky]**
- Hypersolvable **[Jambu–Papadima]**
- Rational $K(\pi, 1)$ **[Papadima–Yuzvinsky]**
- MLS arrangements **[Papadima-Suciu]**
- Graphic arrangements **[Lima-Filho, —]**
- No such formula in general **[Peeva]**

Let $\mathbb{L}(H_1(X_{\mathcal{A}}, \mathbb{C}))$ be the free Lie algebra on $H_1(X_{\mathcal{A}}, \mathbb{C})$. Dual of cup product gives a map

$$H_2(X_{\mathcal{A}}, \mathbb{Q}) \xrightarrow{c} H_1(X_{\mathcal{A}}, \mathbb{Q}) \wedge H_1(X_{\mathcal{A}}, \mathbb{Q}) \longrightarrow \mathbb{L}(H_1(X_{\mathcal{A}}, \mathbb{Q})),$$

Following Chen, define the holonomy Lie algebra

$$\mathfrak{h}_{\mathcal{A}} = \mathbb{L}(H_1(X_{\mathcal{A}}, \mathbb{C}))/I_{\mathcal{A}},$$

where $I_{\mathcal{A}}$ is generated by Im(c).

Theorem 2 (Kohno) *The image of c is generated by*

$$[x_j, \sum_{i=1}^{k} x_i],$$

where x_i is a generator of $\mathbb{L}(H_1(X, \mathbb{C}))$ corresponding to H_i, and $\{H_1, \ldots, H_k\}$ is a maximal dependent set of codimension two, so corresponds to an element of $L_2(\mathcal{A})$.

Note similarity to the Orlik-Solomon algebra!

$$\prod_{k=1}^{\infty} \frac{1}{(1-t^k)^{\phi_k}} = \sum_{i=0}^{\infty} Tor_i^A(\mathbb{C},\mathbb{C})_i t^i.$$

Theorem 3 (Kohno) $\phi_k(\mathfrak{g}) = \phi_k(\mathfrak{h}_{\mathcal{A}})$.

$X_{\mathcal{A}}$ is formal (Brieskorn). Use Sullivan's work and analysis of bigrading on Hirsch extensions.

- **Kohno**: $\prod_{k=1}^{\infty} \frac{1}{(1-t^k)^{\phi_k}} = HS(U(\mathfrak{h}_{\mathcal{A}}, t))$ follows from previous theorem and PBW.

- **Shelton–Yuzvinsky**: $U(\mathfrak{h}_{\mathcal{A}}) = \overline{A}^!$ quadratic dual of quadratic OS-algebra.

- **Priddy, Löfwall**: quadratic dual is related to diagonal Yoneda Ext-algebra via
$$\overline{A}^! \cong \bigoplus_i Ext^i_{\overline{A}}(\mathbb{C},\mathbb{C})_i.$$

- **Peeva**: Nonfano shows DNE standard graded algebra satisfying LCS formula.

Koszul algebras

Definition 4 *Quadratic algebra: quotient of* $T(V)$ *by* $I \subseteq V \otimes V$.

Quadratic algebra has a quadratic dual $T(V^*)/I^{\perp}$:

$$\langle \alpha \otimes \beta \mid \alpha(a) \cdot \beta(b) = 0 \rangle = I^{\perp} \subseteq V^* \otimes V^*$$

Definition 5 *Quadratic algebra* A *is Koszul if*

$$Tor_i^A(\mathbb{C}, \mathbb{C})_j = 0, \; j \neq i$$

A Koszul $\leftrightarrow$ minimal free resolution of $\mathbb{C}$ over A has matrices with only linear entries. This happens exactly when the betti diagram has nonzero entries only in the top row.

Example 6 For

$$S = T(V)/\langle x_i \otimes x_j - x_j \otimes x_i \rangle,$$

Compute resolution and Hilbert Series of $\mathbb{C} = S/\langle x_1, \ldots, x_m \rangle$. Clear that

$$I^{\perp} = \langle x_i \otimes x_j + x_j \otimes x_i \rangle, \text{ so } E = S^!.$$

Compute resolution and Hilbert Series.

Theorem 7 *If A is Koszul, so is $A^!$, and*

$$HS(A,t) \cdot HS(A^!, -t) = 1$$

Example 8 Compute resolution of $\mathbb{C}$ over Orlik-Solomon algebra of A_3 and Nonfano.

Example 9 Via upper semicontinuity, can show quadratic GB $\rightarrow$ Koszul. Pinched Veronese (Caviglia): Koszul but no QGB. (also Eisenbud, Reeves, Totaro)

Problem Formula for LCS ranks for classes of arrangements.
Problem Formula for $Tor_i^A(\mathbb{C},\mathbb{C})_j$ for $A =$ OS-algebra.

We have seen that the numbers above grow very fast. Is there a simpler set of numbers? Yes!

$$Tor_i^E(A,\mathbb{C})_j$$

Problem Formula for $Tor_i^E(A,\mathbb{C})_j$.

Example 10 Compute $Tor_i^E(A,\mathbb{C})_j$ for A_3, D_3.

The spaces

$$Tor_i^E(A,\mathbb{C}) \text{ and } Tor_i^A(\mathbb{C},\mathbb{C})$$

are related via the change of rings spectral sequence

$$\mathrm{Tor}_i^A\left(\mathrm{Tor}_j^E(A,\mathbb{C}),\mathbb{C}\right) \Longrightarrow \mathrm{Tor}_{i+j}^E(\mathbb{C},\mathbb{C}).$$

Change of rings spectral sequence

Take (minimal) free resolutions for $\mathbb{C}$:

$$P_\bullet : 0 \longleftarrow \mathbb{C} \longleftarrow A \longleftarrow A^n(-1) \longleftarrow A^{\binom{n+1}{2}}(-2) \oplus A^{a_2}(-2) \cdots$$

$$Q_\bullet : 0 \longleftarrow \mathbb{C} \longleftarrow E \longleftarrow E^n(-1) \longleftarrow E^{\binom{n+1}{2}}(-2) \longleftarrow \cdots$$

An easy analysis shows that

$$a_2 = \dim_{\mathbb{C}} Tor_1^E(A, \mathbb{C})_2,$$

the number of minimal quadratic generators of the Orlik-Solomon ideal. Pictorially, we have

$$\begin{array}{ccccccc} Q_\bullet & \longrightarrow & \mathbb{C} & & \mathbb{C} & \longleftarrow & P_\bullet \\ & & \uparrow & & \uparrow & & \\ & & E & \longrightarrow & A & & \end{array}$$

which gives a double complex

$$\begin{array}{ccccc} P_0 \otimes (A \otimes Q_2) & \xleftarrow{\delta} & P_1 \otimes (A \otimes Q_2) & & P_2 \otimes (A \otimes Q_2) \\ & & \downarrow d & & \\ P_0 \otimes (A \otimes Q_1) & & P_1 \otimes (A \otimes Q_1) & \xleftarrow{\delta} & P_2 \otimes (A \otimes Q_1) \\ & & & & \downarrow d \\ P_0 \otimes (A \otimes Q_0) & & P_1 \otimes (A \otimes Q_0) & & P_2 \otimes (A \otimes Q_0) \end{array}$$

$A \otimes_E Q_i$ are free A-modules, so the rows are exact, except in leftmost column. This means that ${}^1_{hor}E^{i,j} = 0$ unless $i = 0$, thus

$$ {}^2_{hor}E^{i,j} = {}^\infty_{hor} E^{i,j} = \begin{cases} Tor_j^E(\mathbb{C}, \mathbb{C}) & i = 0 \\ 0 & i \neq 0. \end{cases} $$

Compute: $Tor_j^E(\mathbb{C}, \mathbb{C}) \neq 0$ only in degree j. Conclude that

$$ \dim_{\mathbb{C}} gr(H_j(Tot))_k = \begin{cases} \binom{n+k-1}{k} & k = j \\ 0 & k \neq j. \end{cases} $$

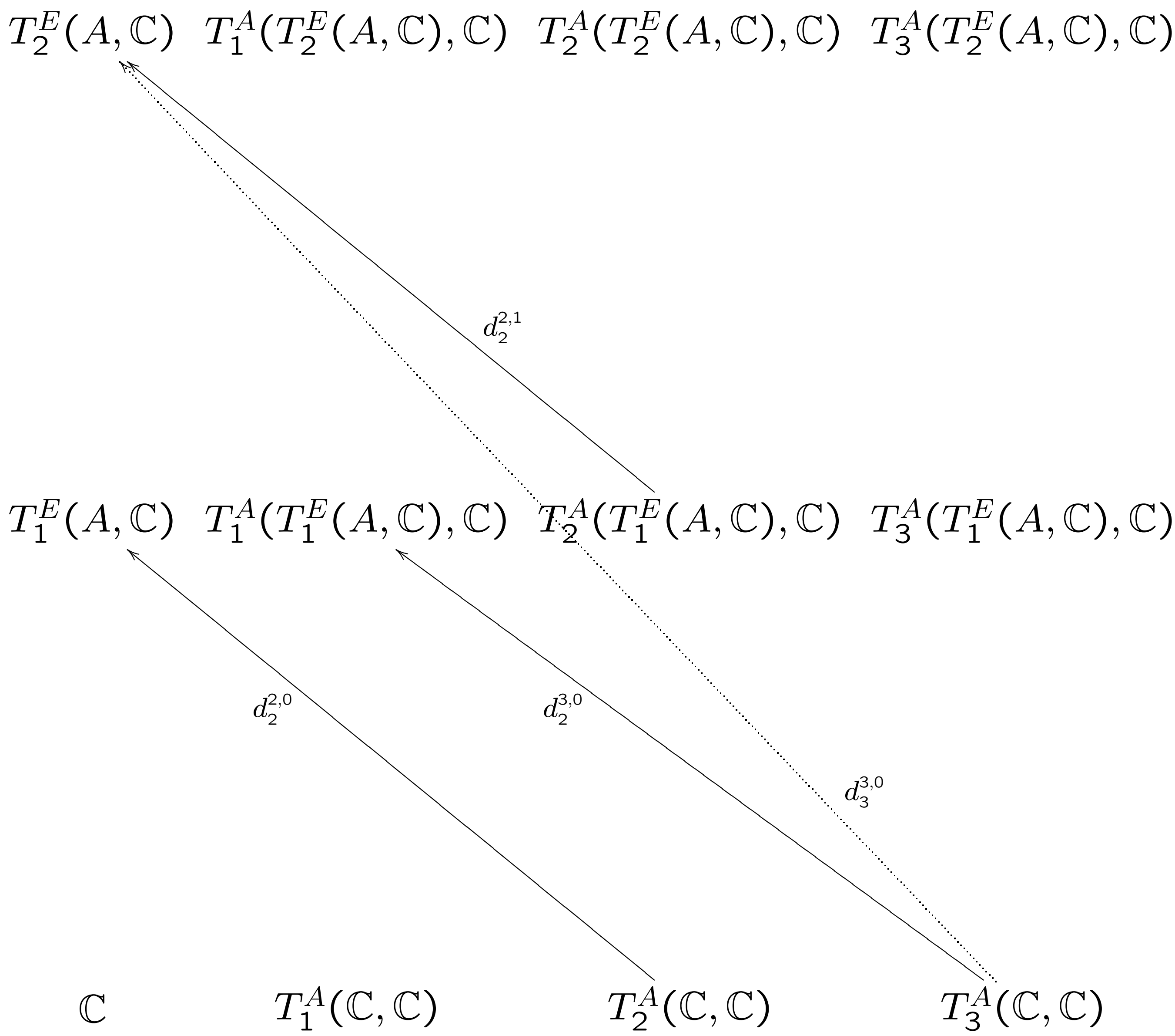
On the other hand,

$$ \begin{aligned} {}^1_{vert}E^{i,j} &= H_j(P_i \otimes_A (A \otimes_E Q_\bullet)) \\ &= P_i \otimes_A H_j(A \otimes_E Q_\bullet)) \\ &= P_i \otimes_A Tor_j^E(A, \mathbb{C}) \end{aligned} $$

Thus,

$$ {}^2_{vert}E^{i,j} = Tor_i^A(Tor_j^E(A, \mathbb{C}), \mathbb{C}). $$

Writing T for Tor, the ${}^{2}_{vert}E$ page is:



Can we compute this?

For ϕ_k, only need $Tor_i^A(\mathbb{C},\mathbb{C})_i$. Differentials are graded. Consider a simple case

Example 11 Consider the sequence

$$0 \to K(d_2) \to Tor_2^A(\mathbb{C},\mathbb{C}) \xrightarrow{d_2} Tor_1^E(A,\mathbb{C}) \to C(d_2) \to 0$$

$C(d_2) = {}^\infty E^{0,1}$, and $gr(H_1(Tot))_k$ nonzero only for $k = 1$, so $C(d_2)$ must vanish (it is generated in degree ≥ 2).

$K(d_2) = {}^\infty E^{2,0}$, nonzero only in degree 2. Both $Tor_1^A(Tor_1^E(A,\mathbb{C}),\mathbb{C})$ and $Tor_1^E(A,\mathbb{C})$ are generated in degree ≥ 3, so

$$K(d_2) \simeq gr(H_2(Tot))_2,$$

which has dimension $\binom{n+1}{2}$. Thus

$$\dim_{\mathbb{C}} Tor_2^A(\mathbb{C},\mathbb{C})_2 = \binom{n+1}{2} + a_2,$$

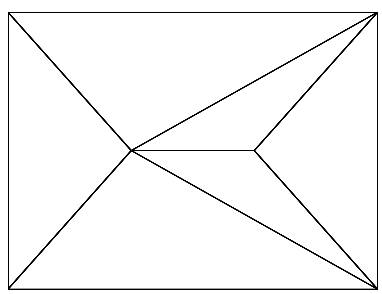
where $a_2 = \dim Tor_1^E(A,\mathbb{C})_2$. Compute!

G a simple graph on ℓ vertices, with edges E. Define $\mathcal{A}_G = \{z_i - z_j = 0 \mid (i,j) \in \mathrm{E} \subseteq \mathbb{C}^\ell\}$, and let $\kappa_s =$ clique $\sharp$'s of G.

Theorem 12 (Lima-Filho, –)

$$U_G(t) = \prod_{j=1}^{\ell-1} (1 - jt)^{\sum_{s=j}^{\ell-1} (-1)^{s-j} \binom{s}{j} \kappa_s}$$

Example 13 $G' =$ Egypt pyramid, $G'' = K_4$.



$U_{G'}(t) = ((1-t)(1-2t))^4/(1-t)^4 = (1-2t)^4$,

$U_{G''}(t) = (1-t)(1-2t)(1-3t)$.

Gluing along a Δ, we have

$$\begin{aligned} U_G(t) &= \frac{(1-2t)^4 \cdot (1-t)(1-2t)(1-3t)}{(1-t)(1-2t)} \\ &= (1-2t)^4(1-3t) \text{ (compute!)} \end{aligned}$$

For each element $a = \sum a_i e_i \in A_1$, we can consider the **Aomoto** complex (A, a).

The i^{th} term is A_i, and differential is $\wedge a$:

$$(A,a): \qquad 0 \longrightarrow A_0 \xrightarrow{a} A_1 \xrightarrow{a} A_2 \xrightarrow{a} \cdots \xrightarrow{a} A_\ell \longrightarrow 0\,.$$

Yuzvinsky: for generic a, (A, a) is exact. The *resonance varieties* of $\mathcal{A}$ are the loci of points $a = \sum_{i=1}^{n} a_i e_i \leftrightarrow (a_1 : \cdots : a_n)$ in $\mathbb{P}(A_1) \cong \mathbb{P}^{n-1}$ for which (A, a) fails to be exact.

Definition 14 *For each* $k \geq 1$*,*

$$R^k(\mathcal{A}) = \{a \in \mathbb{P}^{n-1} \mid H^k(A, a) \neq 0\}.$$

Falk: necessary conditions for $R^1(\mathcal{A})$, conjectured $R^1(\mathcal{A})$ is a union of linear components. Proved by **Cohen–Suciu** & **Libgober–Yuzvinsky**, and for $R^{\geq 2}(\mathcal{A})$ by **Cohen–Orlik**.

Conjecture 15 (Suciu) *If* $\phi_4 = Tor_3^E(A, \mathbb{C})_4$*,*

then

$$\prod_{k \geq 1} (1 - t^k)^{\phi_k} = \prod_{L_i \in R^1(\mathcal{A})} (1 - (dim(L_i)t)$$

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