Bruhat order and Hyperplane arrangements

SuHo Oh, Alexander Postnikov, Hwanchul Yoo

Massachusetts Insitute of Technology

August 10, 2009

Introduction

- Link Schubert varieties with Hyperplane arrangements

Introduction

- Link Schubert varieties with Hyperplane arrangements

- Schubert varieties
- Hyperplane arrangements

Introduction

- Link Schubert varieties with Hyperplane arrangements

- Schubert varieties
- X_w, $w \in W$

- Hyperplane arrangements
- $\mathcal{A}_w$, $w \in W$

Introduction

- Link Schubert varieties with Hyperplane arrangements

- Schubert varieties
- X_w, $w \in W$
- Poincare Polynomial

- Hyperplane arrangements
- $\mathcal{A}_w$, $w \in W$
- Distance Enumerator Polynomial

Introduction

- Link Schubert varieties with Hyperplane arrangements

- Schubert varieties
- X_w, $w \in W$
- Poincare Polynomial
- Setting $q = 1$, # of elements in $[id, w]$.

- Hyperplane arrangements
- $\mathcal{A}_w$, $w \in W$
- Distance Enumerator Polynomial
- Setting $q = 1$, # of regions in the arrangement.

Introduction

- Link Schubert varieties with Hyperplane arrangements

- Schubert varieties
- X_w, $w \in W$
- Poincare Polynomial
- Setting $q = 1$, # of elements in $[id, w]$.

- Hyperplane arrangements
- $\mathcal{A}_w$, $w \in W$
- Distance Enumerator Polynomial
- Setting $q = 1$, # of regions in the arrangement.

- Type A :

Introduction

- Link Schubert varieties with Hyperplane arrangements

- Schubert varieties
- X_w, $w \in W$
- Poincare Polynomial
- Setting $q = 1$, # of elements in $[id, w]$.

- Hyperplane arrangements
- $\mathcal{A}_w$, $w \in W$
- Distance Enumerator Polynomial
- Setting $q = 1$, # of regions in the arrangement.

- Type A :
  - Two polynomials equal iff w avoids $3412, 4231$.

Introduction

- Link Schubert varieties with Hyperplane arrangements

- Schubert varieties
- X_w, $w \in W$
- Poincare Polynomial
- Setting $q = 1$, # of elements in $[id, w]$.

- Hyperplane arrangements
- $\mathcal{A}_w$, $w \in W$
- Distance Enumerator Polynomial
- Setting $q = 1$, # of regions in the arrangement.

- Type A :
  - Two polynomials equal iff w avoids $3412, 4231$.
  - Nice way to compute Poincare polynomial

Introduction

- Link Schubert varieties with Hyperplane arrangements

- Schubert varieties
- X_w, $w \in W$
- Poincare Polynomial
- Setting $q = 1$, # of elements in $[id, w]$.

- Hyperplane arrangements
- $\mathcal{A}_w$, $w \in W$
- Distance Enumerator Polynomial
- Setting $q = 1$, # of regions in the arrangement.

- Type A :
  - Two polynomials equal iff w avoids $3412, 4231$.
  - Nice way to compute Poincare polynomial
- A conjecture(?) for general W

Bruhat order and Poincaré polynomials

- $X_w = B\bar{w}B/B$

Bruhat order and Poincaré polynomials

- $X_w = B\bar{w}B/B$
- $[\mathrm{id}, w] := \{u \in S_n \mid u \leq w\}$

Bruhat order and Poincaré polynomials

- $X_w = B\bar{w}B/B$
- $[\mathrm{id}, w] := \{u \in S_n \mid u \leq w\}$
- *Poincaré polynomial* of X_w : Rank generating function for $[\mathrm{id}, w]$

$$P_w(q) = \sum_{u \leq w} q^{\ell(u)}.$$

Bruhat order and Poincaré polynomials

- $X_w = B\bar{w}B/B$
- $[\mathrm{id}, w] := \{u \in S_n \mid u \leq w\}$
- *Poincaré polynomial* of X_w : Rank generating function for $[\mathrm{id}, w]$

$$P_w(q) = \sum_{u \leq w} q^{\ell(u)}.$$

- (Lakshmibai-Sandhya) X_w smooth iff w avoids $3412, 4231$.

Bruhat order and Poincaré polynomials

- $X_w = B\bar{w}B/B$
- $[\mathrm{id}, w] := \{u \in S_n \mid u \leq w\}$
- *Poincaré polynomial* of X_w : Rank generating function for $[\mathrm{id}, w]$

$$P_w(q) = \sum_{u \leq w} q^{\ell(u)}.$$

- (Lakshmibai-Sandhya) X_w smooth iff w avoids $3412, 4231$.
- (Carrell-Peterson) X_w smooth iff P_w palindromic.

Bruhat order and Poincaré polynomials

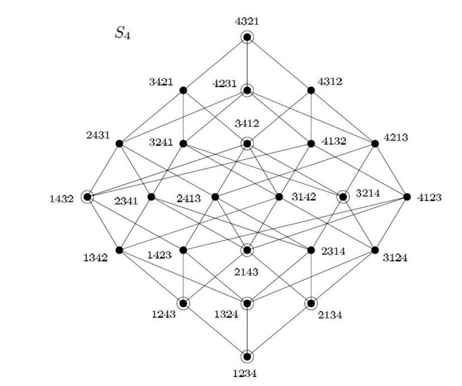
- $X_w = B\bar{w}B/B$
- $[\mathrm{id}, w] := \{u \in S_n \mid u \leq w\}$
- *Poincaré polynomial* of X_w : Rank generating function for $[\mathrm{id}, w]$

$$P_w(q) = \sum_{u \leq w} q^{\ell(u)}.$$

- (Lakshmibai-Sandhya) X_w smooth iff w avoids $3412, 4231$.
- (Carrell-Peterson) X_w smooth iff P_w palindromic.
- (Gasharov) $P_w(q)$ factors as products of form $[a]_q$ where $[a]_q := (1-q^a)/(1-q) = 1 + q + q^2 + \cdots + q^{a-1}$

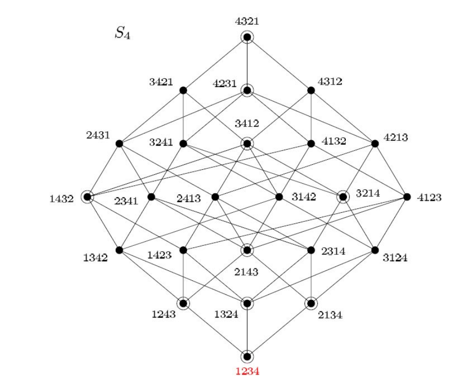
Example : S_4

X_w smooth $\leftrightarrow$ w avoids $3412, 4231$ $\leftrightarrow$ P_w palindromic.



Example : S_4

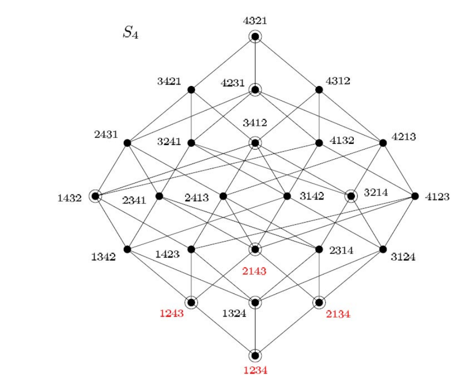
X_w smooth $\leftrightarrow$ w avoids $3412, 4231$ $\leftrightarrow$ P_w palindromic.



- $P_{1234} = 1$

Example : S_4

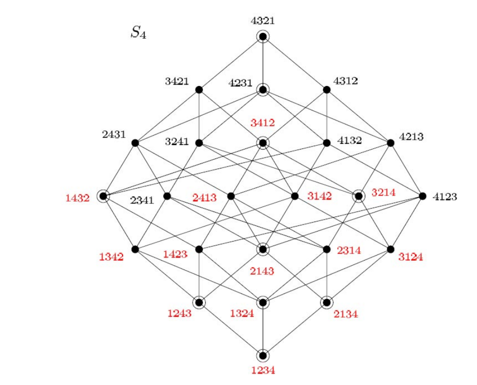
X_w smooth $\leftrightarrow$ w avoids $3412, 4231$ $\leftrightarrow$ P_w palindromic.



- $P_{1234} = 1$
- $P_{2143} = 1 + 2q + q^2$

Example : S_4

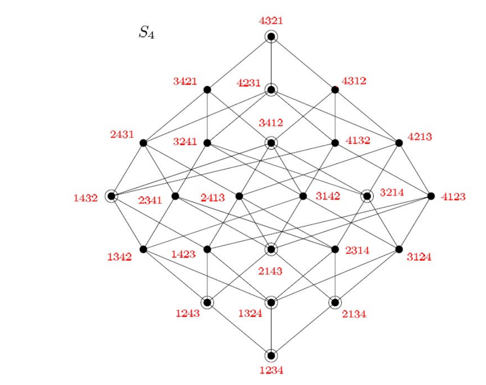
X_w smooth $\leftrightarrow$ w avoids $3412, 4231$ $\leftrightarrow$ P_w palindromic.



- $P_{1234} = 1$
- $P_{2143} = 1 + 2q + q^2$
- $P_{3412} =$
  $1 + 3q + 5q^2 + 4q^3 + q^4$

Example : S_4

X_w smooth $\leftrightarrow$ w avoids $3412, 4231$ $\leftrightarrow$ P_w palindromic.



- $P_{1234} = 1$
- $P_{2143} = 1 + 2q + q^2$
- $P_{3412} = 1 + 3q + 5q^2 + 4q^3 + q^4$
- $P_{4321} = 1 + 3q + 5q^2 + 6q^3 + 5q^4 + 3q^5 + q^6$

Inversion Hyperplane arrangement

- $\mathcal{A}_w : x_i - x_j = 0$ for each $1 \leq i < j \leq n$, $w(i) > w(j)$.

Inversion Hyperplane arrangement

- $\mathcal{A}_w : x_i - x_j = 0$ for each $1 \leq i < j \leq n$, $w(i) > w(j)$.
- (Hultman-Linusson-Shareshian-Sjöstrand) #Regions of $\mathcal{A}_w$ = $P_w(1)$ iff w avoids 4231, 35142, 42513, 351624.

Inversion Hyperplane arrangement

- $\mathcal{A}_w$: $x_i - x_j = 0$ for each $1 \leq i < j \leq n$, $w(i) > w(j)$.
- (Hultman-Linusson-Shareshian-Sjöstrand) #Regions of $\mathcal{A}_w$ = $P_w(1)$ iff w avoids 4231, 35142, 42513, 351624.
- Lets define q-analog of R_w :

Inversion Hyperplane arrangement

- $\mathcal{A}_w : x_i - x_j = 0$ for each $1 \le i < j \le n$, $w(i) > w(j)$.
- (Hultman-Linusson-Shareshian-Sjöstrand) #Regions of $\mathcal{A}_w$ = $P_w(1)$ iff w avoids 4231, 35142, 42513, 351624.
- Lets define q-analog of R_w :
  - Regions r, r', $d(r, r') :=$ Minimal # hyperplanes crossed to go from r to r'.

Inversion Hyperplane arrangement

- $\mathcal{A}_w : x_i - x_j = 0$ for each $1 \leq i < j \leq n$, $w(i) > w(j)$.
- (Hultman-Linusson-Shareshian-Sjöstrand) #Regions of $\mathcal{A}_w$ = $P_w(1)$ iff w avoids 4231, 35142, 42513, 351624.
- Lets define q-analog of R_w :
  - Regions r, r', $d(r, r') :=$ Minimal # hyperplanes crossed to go from r to r'.
  - r_0 : Region containing $(1, \ldots, n)$.

Inversion Hyperplane arrangement

- $\mathcal{A}_w : x_i - x_j = 0$ for each $1 \leq i < j \leq n$, $w(i) > w(j)$.
- (Hultman-Linusson-Shareshian-Sjöstrand) #Regions of $\mathcal{A}_w$ = $P_w(1)$ iff w avoids 4231, 35142, 42513, 351624.
- Lets define q-analog of R_w :
  - Regions r, r', $d(r, r') :=$ Minimal # hyperplanes crossed to go from r to r'.
  - r_0 : Region containing $(1, \ldots, n)$.
  - Distance Enumerator Polynomial of r_0 :

$$R_{\mathcal{A}_w}(q) := \sum_r q^{d(r, r_0)}.$$

Inversion Hyperplane arrangement

- $\mathcal{A}_w : x_i - x_j = 0$ for each $1 \leq i < j \leq n$, $w(i) > w(j)$.
- (Hultman-Linusson-Shareshian-Sjöstrand) #Regions of $\mathcal{A}_w$ = $P_w(1)$ iff w avoids 4231, 35142, 42513, 351624.
- Lets define q-analog of R_w :
  - Regions r, r', $d(r, r') :=$ Minimal # hyperplanes crossed to go from r to r'.
  - r_0 : Region containing $(1, \ldots, n)$.
  - Distance Enumerator Polynomial of r_0 :

$$R_{\mathcal{A}_w}(q) := \sum_r q^{d(r, r_0)}.$$

  - Let G be inversion graph of w. $\mathcal{O}$ be an acyclic orientation. $des(\mathcal{O}) := \#$ edges oriented rightwards.

Inversion Hyperplane arrangement

- $\mathcal{A}_w : x_i - x_j = 0$ for each $1 \leq i < j \leq n$, $w(i) > w(j)$.
- (Hultman-Linusson-Shareshian-Sjöstrand) #Regions of $\mathcal{A}_w$ = $P_w(1)$ iff w avoids 4231, 35142, 42513, 351624.
- Lets define q-analog of R_w :
  - Regions r, r', $d(r, r') :=$ Minimal # hyperplanes crossed to go from r to r'.
  - r_0 : Region containing $(1, \ldots, n)$.
  - Distance Enumerator Polynomial of r_0 :

$$R_{\mathcal{A}_w}(q) := \sum_r q^{d(r, r_0)}.$$

  - Let G be inversion graph of w. $\mathcal{O}$ be an acyclic orientation. $des(\mathcal{O}) := \#$ edges oriented rightwards.
  - Then $R_{\mathcal{A}_w}$ equals :

$$R_G(q) := \sum_{\mathcal{O}} q^{des(\mathcal{O})}$$

Examples

Examples

- 

- $R_{\mathcal{A}_{1234}} = 1$

Examples

- 

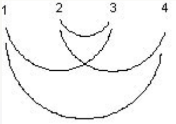
- 

- $R_{\mathcal{A}_{1234}} = 1$
- $R_{\mathcal{A}_{2143}} = 1 + 2q + q^2$

Examples

- 

- 

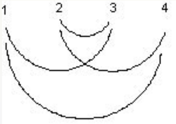
- 

- $R_{\mathcal{A}_{1234}} = 1$
- $R_{\mathcal{A}_{2143}} = 1 + 2q + q^2$
- $R_{\mathcal{A}_{3412}} =$
$1 + 4q + 4q^2 + 4q^3 + q^4$

Examples

- 

- 

- 

- $R_{\mathcal{A}_{1234}} = 1$
- $R_{\mathcal{A}_{2143}} = 1 + 2q + q^2$
- $R_{\mathcal{A}_{3412}} = 1 + 4q + 4q^2 + 4q^3 + q^4$
- $R_{\mathcal{A}_{4321}} = 1 + 3q + 5q^2 + 6q^3 + 5q^4 + 3q^5 + q^6$

Very brief outline of the proof

Lemma (Björner-Edelman-Ziegler)

G on vertex set $[n]$ has vertex v adjacent to m vertices such that

1. *Set of all neighbors of v form a clique in G*
2. 1. *All neighbors of v less than v, or*
   2. *All neighbors of v greater than v.*

Then $R_G(q) = [m+1]_q R_{G \setminus v}(q)$.

Very brief outline of the proof

Lemma (Björner-Edelman-Ziegler)

G on vertex set $[n]$ has vertex v adjacent to m vertices such that

1. *Set of all neighbors of v form a clique in G*
2. 1. *All neighbors of v less than v, or*
   2. *All neighbors of v greater than v.*

Then $R_G(q) = [m+1]_q R_{G \setminus v}(q)$.

- Find a nice ordering on inversion graph G such that :

Very brief outline of the proof

Lemma (Björner-Edelman-Ziegler)

G on vertex set $[n]$ has vertex v adjacent to m vertices such that

1. *Set of all neighbors of v form a clique in G*
2. 1. *All neighbors of v less than v, or*
   2. *All neighbors of v greater than v.*

Then $R_G(q) = [m+1]_q R_{G \setminus v}(q)$.

- Find a nice ordering on inversion graph G such that :
  - Perfect elimination ordering : $\forall i$, $Nbr(v_i) \cap \{v_1, \cdots, v_{i-1}\}$ form clique.

Very brief outline of the proof

Lemma (Björner-Edelman-Ziegler)

G on vertex set $[n]$ has vertex v adjacent to m vertices such that

1. *Set of all neighbors of v form a clique in G*
2. 1. *All neighbors of v less than v, or*
   2. *All neighbors of v greater than v.*

Then $R_G(q) = [m+1]_q R_{G\setminus v}(q)$.

- Find a nice ordering on inversion graph G such that :
  - Perfect elimination ordering : $\forall i$, $Nbr(v_i) \cap \{v_1, \cdots, v_{i-1}\}$ form clique.
  - $\forall i$, $Nbr(v_i) \cap \{v_1, \cdots, v_{i-1}\}$ are all greater(or less) than v_i.

Very brief outline of the proof

Lemma (Björner-Edelman-Ziegler)

G on vertex set $[n]$ has vertex v adjacent to m vertices such that

1. *Set of all neighbors of v form a clique in G*
2. 1. *All neighbors of v less than v, or*
   2. *All neighbors of v greater than v.*

Then $R_G(q) = [m+1]_q R_{G\setminus v}(q)$.

- Find a nice ordering on inversion graph G such that :
  - Perfect elimination ordering : $\forall i$, $Nbr(v_i) \cap \{v_1, \cdots, v_{i-1}\}$ form clique.
  - $\forall i$, $Nbr(v_i) \cap \{v_1, \cdots, v_{i-1}\}$ are all greater(or less) than v_i.
- So we have a recursive factorization of $R_w(q)$.

Very brief outline of the proof

Lemma (Björner-Edelman-Ziegler)

G on vertex set $[n]$ has vertex v adjacent to m vertices such that

1. *Set of all neighbors of v form a clique in G*
2.
   1. *All neighbors of v less than v, or*
   2. *All neighbors of v greater than v.*

Then $R_G(q) = [m+1]_q R_{G\setminus v}(q)$.

- Find a nice ordering on inversion graph G such that :
  - Perfect elimination ordering : $\forall i$, $Nbr(v_i) \cap \{v_1, \cdots, v_{i-1}\}$ form clique.
  - $\forall i$, $Nbr(v_i) \cap \{v_1, \cdots, v_{i-1}\}$ are all greater(or less) than v_i.
- So we have a recursive factorization of $R_w(q)$.
- The recurrence is same as Gasharov's for $P_w(q)$.

General Conjecture

General Conjecture

- X_w, $w \in S_n$
- X_w, $w \in W$, for other types

General Conjecture

- X_w, $w \in S_n$
- $\mathcal{A}_w : x_i - x_j = 0$ for each $1 \leq i < j \leq n$, $w(i) > w(j)$.

- X_w, $w \in W$, for other types
- $\mathcal{A}_w : \alpha(x) = 0$ s.t. $\alpha > 0, w(\alpha) < 0$

General Conjecture

- X_w, $w \in S_n$
- $\mathcal{A}_w : x_i - x_j = 0$ for each $1 \leq i < j \leq n$, $w(i) > w(j)$.
- Theorem : $P_w(q) = R_w(q)$ iff X_w smooth

- X_w, $w \in W$, for other types
- $\mathcal{A}_w : \alpha(x) = 0$ s.t. $\alpha > 0, w(\alpha) < 0$
- Conjecture : $P_w(q) = R_w(q)$ iff X_w rationally smooth

General Conjecture

- X_w, $w \in S_n$
- $\mathcal{A}_w : x_i - x_j = 0$ for each $1 \leq i < j \leq n$, $w(i) > w(j)$.
- Theorem : $P_w(q) = R_w(q)$ iff X_w smooth
- w avoiding $3412, 4231$

- X_w, $w \in W$, for other types
- $\mathcal{A}_w : \alpha(x) = 0$ s.t. $\alpha > 0, w(\alpha) < 0$
- Conjecture : $P_w(q) = R_w(q)$ iff X_w rationally smooth
- w avoiding some patterns

Type B,D ...

From (Billey) Pattern avoidance and rational smoothness of Schubert varieties:

- X_w rat.smooth iff $P_w(q)$ factors as $[e_i]_q$'s.

Type B,D ...

From (Billey) Pattern avoidance and rational smoothness of Schubert varieties:

- X_w rat.smooth iff $P_w(q)$ factors as $[e_i]_q$'s.
- Type B : X_w rat.smooth iff w avoids

Type B,D ...

From (Billey) Pattern avoidance and rational smoothness of Schubert varieties:

- X_w rat.smooth iff $P_w(q)$ factors as $[e_i]_q$'s.
- Type B : X_w rat.smooth iff w avoids

$\bar{1}2\bar{3}$	$1\bar{2}\bar{3}$	$12\bar{3}$	$1\bar{3}\bar{2}$	$\bar{2}\bar{1}\bar{3}$	$\bar{2}1\bar{3}$	$2\bar{1}\bar{3}$
$2\bar{3}\bar{1}$	$\bar{3}1\bar{2}$	$\bar{3}\bar{2}\bar{1}$	$\bar{3}\bar{2}1$	$\bar{3}2\bar{1}$	$3\bar{2}\bar{1}$	$3\bar{2}1$
$\bar{2}\bar{4}31$	$2\bar{4}31$	$\bar{3}\bar{4}\bar{1}\bar{2}$	$\bar{3}4\bar{1}2$	$\bar{3}412$	$34\bar{1}2$	3412
$4\bar{1}3\bar{2}$	$413\bar{2}$	$\bar{4}231$	$423\bar{1}$	4231		

Type B,D ...

From (Billey) Pattern avoidance and rational smoothness of Schubert varieties:

- X_w rat.smooth iff $P_w(q)$ factors as $[e_i]_q$'s.
- Type B : X_w rat.smooth iff w avoids

$\bar{1}2\bar{3}$	$1\bar{2}\bar{3}$	$12\bar{3}$	$1\bar{3}\bar{2}$	$\bar{2}\bar{1}\bar{3}$	$\bar{2}1\bar{3}$	$2\bar{1}\bar{3}$
$2\bar{3}\bar{1}$	$\bar{3}1\bar{2}$	$\bar{3}\bar{2}\bar{1}$	$\bar{3}\bar{2}1$	$\bar{3}2\bar{1}$	$3\bar{2}\bar{1}$	$3\bar{2}1$
$\bar{2}\bar{4}31$	$2\bar{4}31$	$\bar{3}\bar{4}\bar{1}\bar{2}$	$\bar{3}4\bar{1}2$	$\bar{3}412$	$34\bar{1}2$	3412
$4\bar{1}3\bar{2}$	$413\bar{2}$	$\bar{4}231$	$423\bar{1}$	4231		

- Type D : X_w rat.smooth iff w avoids

Type B,D ...

From (Billey) Pattern avoidance and rational smoothness of Schubert varieties:

- X_w rat.smooth iff $P_w(q)$ factors as $[e_i]_q$'s.
- Type B : X_w rat.smooth iff w avoids

$\bar{1}2\bar{3}$	$1\bar{2}\bar{3}$	$12\bar{3}$	$1\bar{3}\bar{2}$	$\bar{2}\bar{1}\bar{3}$	$\bar{2}1\bar{3}$	$2\bar{1}\bar{3}$
$2\bar{3}\bar{1}$	$\bar{3}1\bar{2}$	$\bar{3}\bar{2}\bar{1}$	$\bar{3}\bar{2}1$	$\bar{3}2\bar{1}$	$3\bar{2}\bar{1}$	$3\bar{2}1$
$\bar{2}\bar{4}31$	$2\bar{4}31$	$\bar{3}\bar{4}\bar{1}\bar{2}$	$\bar{3}4\bar{1}2$	$\bar{3}412$	$34\bar{1}2$	3412
$4\bar{1}3\bar{2}$	$413\bar{2}$	$\bar{4}231$	$423\bar{1}$	4231		

- Type D : X_w rat.smooth iff w avoids

$12\bar{3}$	$\bar{1}2\bar{3}$	$\bar{1}\bar{3}\bar{2}$	$1\bar{3}\bar{2}$	$\bar{2}\bar{1}\bar{3}$	$\bar{3}\bar{2}\bar{1}$	
$\bar{1}4\bar{3}2$	$\bar{2}1\bar{3}\bar{4}$	$2\bar{1}\bar{3}\bar{4}$	$21\bar{3}\bar{4}$	$\bar{2}\bar{3}1\bar{4}$	$2\bar{3}1\bar{4}$	
$2\bar{4}31$	$\bar{2}\bar{4}3\bar{1}$	$\bar{2}4\bar{3}\bar{1}$	$24\bar{3}\bar{1}$	$2\bar{4}3\bar{1}$	$\bar{2}4\bar{3}1$	
$\bar{2}\bar{4}31$	$3\bar{1}\bar{2}\bar{4}$	$31\bar{2}\bar{4}$	$3\bar{2}1\bar{4}$	$3\bar{2}\bar{4}1$	$\bar{3}\bar{4}1\bar{2}$	
$3\bar{4}\bar{1}\bar{2}$	$\bar{3}412$	$34\bar{1}2$	$\bar{3}4\bar{1}2$	3412	$\bar{3}\bar{4}\bar{1}\bar{2}$	
$3\bar{4}1\bar{2}$	$\bar{3}\bar{4}\bar{2}1$	$34\bar{2}\bar{1}$	$\bar{3}4\bar{2}1$	$3\bar{4}\bar{2}1$	$\bar{4}\bar{1}\bar{3}2$	
$413\bar{2}$	$\bar{4}\bar{1}3\bar{2}$	$4\bar{1}3\bar{2}$	$\bar{4}13\bar{2}$	$4\bar{1}\bar{3}2$	$\bar{4}1\bar{3}2$	$4\bar{2}1\bar{3}$
$4\bar{2}\bar{3}\bar{1}$	$\bar{4}\bar{2}\bar{3}1$	$\bar{4}231$	$423\bar{1}$	$\bar{4}23\bar{1}$	4231	$4\bar{2}\bar{3}1$
$4\bar{3}\bar{1}\bar{2}$	$4\bar{3}\bar{1}2$	$\bar{4}\bar{3}12$	$4\bar{3}1\bar{2}$	$4\bar{3}\bar{2}1$		

Type B,D ...

From (Billey) Pattern avoidance and rational smoothness of Schubert varieties:

- X_w rat.smooth iff $P_w(q)$ factors as $[e_i]_q$'s.
- Type B : X_w rat.smooth iff w avoids

$\bar{1}2\bar{3}$	$1\bar{2}\bar{3}$	$12\bar{3}$	$1\bar{3}\bar{2}$	$\bar{2}\bar{1}\bar{3}$	$\bar{2}1\bar{3}$	$2\bar{1}\bar{3}$
$2\bar{3}\bar{1}$	$\bar{3}1\bar{2}$	$\bar{3}\bar{2}\bar{1}$	$\bar{3}\bar{2}1$	$\bar{3}2\bar{1}$	$3\bar{2}\bar{1}$	$3\bar{2}1$
$\bar{2}\bar{4}31$	$2\bar{4}31$	$\bar{3}\bar{4}\bar{1}\bar{2}$	$\bar{3}4\bar{1}2$	$\bar{3}412$	$34\bar{1}2$	3412
$4\bar{1}3\bar{2}$	$413\bar{2}$	$\bar{4}231$	$423\bar{1}$	4231		

- Type D : X_w rat.smooth iff w avoids

$12\bar{3}$	$\bar{1}2\bar{3}$	$\bar{1}\bar{3}\bar{2}$	$1\bar{3}\bar{2}$	$\bar{2}\bar{1}\bar{3}$	$\bar{3}\bar{2}\bar{1}$	
$\bar{1}4\bar{3}2$	$\bar{2}1\bar{3}\bar{4}$	$2\bar{1}\bar{3}\bar{4}$	$21\bar{3}\bar{4}$	$\bar{2}\bar{3}1\bar{4}$	$2\bar{3}1\bar{4}$	
$2\bar{4}31$	$\bar{2}\bar{4}3\bar{1}$	$\bar{2}4\bar{3}\bar{1}$	$24\bar{3}\bar{1}$	$\bar{2}\bar{4}3\bar{1}$	$\bar{2}4\bar{3}1$	
$\bar{2}\bar{4}31$	$3\bar{1}\bar{2}\bar{4}$	$31\bar{2}\bar{4}$	$3\bar{2}1\bar{4}$	$3\bar{2}\bar{4}1$	$\bar{3}\bar{4}1\bar{2}$	
$3\bar{4}\bar{1}\bar{2}$	$\bar{3}412$	$34\bar{1}2$	$\bar{3}4\bar{1}2$	3412	$\bar{3}\bar{4}\bar{1}\bar{2}$	
$3\bar{4}1\bar{2}$	$\bar{3}\bar{4}\bar{2}1$	$34\bar{2}\bar{1}$	$\bar{3}4\bar{2}1$	$3\bar{4}\bar{2}1$	$\bar{4}\bar{1}\bar{3}2$	
$413\bar{2}$	$\bar{4}\bar{1}3\bar{2}$	$4\bar{1}3\bar{2}$	$\bar{4}13\bar{2}$	$4\bar{1}\bar{3}2$	$\bar{4}1\bar{3}2$	$4\bar{2}1\bar{3}$
$4\bar{2}\bar{3}\bar{1}$	$\bar{4}\bar{2}\bar{3}1$	$\bar{4}231$	$423\bar{1}$	$\bar{4}23\bar{1}$	4231	$4\bar{2}\bar{3}1$
$4\bar{3}\bar{1}\bar{2}$	$4\bar{3}\bar{1}2$	$\bar{4}\bar{3}12$	$4\bar{3}1\bar{2}$	$4\bar{3}\bar{2}1$		

- Almost done for B,D.

Further remarks

- Conjecture : For $w \in S_n$, $R_w(q)$ factorizes into $[a]_q$'s iff w is smooth.

Further remarks

- Conjecture : For $w \in S_n$, $R_w(q)$ factorizes into $[a]_q$'s iff w is smooth.
- When w smooth, a's equal the roots of χ_G.

Further remarks

- Conjecture : For $w \in S_n$, $R_w(q)$ factorizes into $[a]_q$'s iff w is smooth.
- When w smooth, a's equal the roots of χ_G.
- G chordal implies $\mathcal{A}_G$ is free.

Further remarks

- Conjecture : For $w \in S_n$, $R_w(q)$ factorizes into $[a]_q$'s iff w is smooth.
- When w smooth, a's equal the roots of χ_G.
- G chordal implies $\mathcal{A}_G$ is free.
- G_{4231} is chordal, but $R_{4231}(q) = 1 + 4q + 4q^2 + 4q^3 + 4q^4 + q^5$.

Further remarks

- Conjecture : For $w \in S_n$, $R_w(q)$ factorizes into $[a]_q$'s iff w is smooth.
- When w smooth, a's equal the roots of χ_G.
- G chordal implies $\mathcal{A}_G$ is free.
- G_{4231} is chordal, but $R_{4231}(q) = 1 + 4q + 4q^2 + 4q^3 + 4q^4 + q^5$.
- For type D, also allow $(1 + q + \cdots + q^{k-1} + 2q^k + q^{k+1} + \cdots + q^{2k})$.

Further remarks

- Conjecture : For $w \in S_n$, $R_w(q)$ factorizes into $[a]_q$'s iff w is smooth.
- When w smooth, a's equal the roots of χ_G.
- G chordal implies $\mathcal{A}_G$ is free.
- G_{4231} is chordal, but $R_{4231}(q) = 1 + 4q + 4q^2 + 4q^3 + 4q^4 + q^5$.
- For type D, also allow $(1 + q + \cdots + q^{k-1} + 2q^k + q^{k+1} + \cdots + q^{2k})$.
- There may be more linking Schubert variety or Kazhdan-Lustzig Polynomials to inversion Hyperplane arrangements!

Appendix : Nice way to compute $P_w(q)$

- Records : Left-to-right maxima

Appendix : Nice way to compute $P_w(q)$

- Records : Left-to-right maxima
- $i = 1, \ldots, n$, $r \leq i < r'$ closest records.

$$e_i := \#\{j \mid r \leq j < i,\ w(j) > w(i)\} + \#\{k \mid r' \leq k \leq n,\ w(k) < w(i)\}.$$

Appendix : Nice way to compute $P_w(q)$

- Records : Left-to-right maxima
- $i = 1, \ldots, n$, $r \leq i < r'$ closest records.

$$e_i := \#\{j \mid r \leq j < i,\ w(j) > w(i)\} + \#\{k \mid r' \leq k \leq n,\ w(k) < w(i)\}.$$

Theorem

$w \in S_n$ smooth. $e_1, \ldots, e_n$ as above. Then

$$P_w(q) = R_w(q) = [e_1 + 1]_q\,[e_2 + 1]_q \cdots [e_n + 1]_q\,.$$

Appendix : Nice way to compute $P_w(q)$

- Records : Left-to-right maxima
- $i = 1, \ldots, n$, $r \leq i < r'$ closest records.

$$e_i := \#\{j \mid r \leq j < i,\ w(j) > w(i)\} + \#\{k \mid r' \leq k \leq n,\ w(k) < w(i)\}.$$

Theorem

$w \in S_n$ smooth. $e_1, \ldots, e_n$ as above. Then

$$P_w(q) = R_w(q) = [e_1 + 1]_q\, [e_2 + 1]_q \cdots [e_n + 1]_q\, .$$

- $w = 5\,1\,6\,4\,7\,3\,2$

Appendix : Nice way to compute $P_w(q)$

- Records : Left-to-right maxima
- $i = 1, \ldots, n$, $r \leq i < r'$ closest records.

$$e_i := \#\{j \mid r \leq j < i,\ w(j) > w(i)\} + \#\{k \mid r' \leq k \leq n,\ w(k) < w(i)\}.$$

Theorem

$w \in S_n$ smooth. $e_1, \ldots, e_n$ as above. Then

$$P_w(q) = R_w(q) = [e_1 + 1]_q \, [e_2 + 1]_q \cdots [e_n + 1]_q \, .$$

- $w = \hat{5}\,1\,\hat{6}\,4\,\hat{7}\,3\,2$

Appendix : Nice way to compute $P_w(q)$

- Records : Left-to-right maxima
- $i = 1, \ldots, n$, $r \leq i < r'$ closest records.

$$e_i := \#\{j \mid r \leq j < i,\ w(j) > w(i)\} + \#\{k \mid r' \leq k \leq n,\ w(k) < w(i)\}.$$

Theorem

$w \in S_n$ smooth. $e_1, \ldots, e_n$ as above. Then

$$P_w(q) = R_w(q) = [e_1 + 1]_q \, [e_2 + 1]_q \cdots [e_n + 1]_q \,.$$

- $w = \hat{5}\,1\,\hat{6}\,4\,\hat{7}\,3\,2$
- $(e_1, \ldots, e_7) = (0+3,\ 1+0,\ 0+2,\ 1+2,\ 0+0,\ 1+0,\ 2+0)$.

Appendix : Nice way to compute $P_w(q)$

- Records : Left-to-right maxima
- $i = 1, \dots, n$, $r \le i < r'$ closest records.

$$e_i := \#\{j \mid r \le j < i,\ w(j) > w(i)\} + \#\{k \mid r' \le k \le n,\ w(k) < w(i)\}.$$

Theorem

$w \in S_n$ smooth. $e_1, \dots, e_n$ as above. Then

$$P_w(q) = R_w(q) = [e_1 + 1]_q\,[e_2 + 1]_q \cdots [e_n + 1]_q\,.$$

- $w = \hat{5}\,1\,\hat{6}\,4\,\hat{7}\,3\,2$
- $(e_1, \dots, e_7) = (0+3,\ 1+0,\ 0+2,\ 1+2,\ 0+0,\ 1+0,\ 2+0)$.
- $P_w(q) = R_w(q) = [4]_q\,[2]_q\,[3]_q\,[4]_q\,[1]_q\,[2]_q\,[3]_q$.