Arrangements of hyperplanes and solutions of the Fuchsian diferential equations free from accessary parameters

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Euler integral of the Gauss Hypergeometric function

$$\int_1^\infty t^a (t-1)^b (t-z)^c \, dt$$

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Differential equation (scalar valued or Vector valued)

Cycles

Series expansion

Evaluation of the integral at the special value

Monodromy representation

Connection problem (connection matrices)

Asymptotic behaviour

How can we generalize these?

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$\Longrightarrow$ Differential equations free from accessory parameters

(rigid local systems)

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$$

where $(a)_k = a(a+1)\cdots(a+k-1)$.

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where $(a)_k = a(a+1)\cdots(a+k-1)$.

Differential equation ${}_{n+1}E_n$

$$ \left\{\theta_z\Big\{\prod_{1\le i\le n}(\theta_z+\beta_i-1)\Big\}-z\Big\{\prod_{1\le i\le n+1}(\theta_z+\alpha_i)\Big\}\right\}F=0 $$

where $\theta = zd/dz$.

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Differential equation ${}_{n+1}E_n$

$$ \left\{ \theta_z \Big\{ \prod_{1\le i\le n} (\theta_z+\beta_i-1) \Big\} - z \Big\{ \prod_{1\le i\le n+1} (\theta_z+\alpha_i) \Big\} \right\} F = 0 $$

where $\theta = zd/dz$. (Rank=n+1)

Differential equations free from accessory parameters
(rigid local systems, rigid differential equations)

are studied by

Okubo, Takano, S.Yoshida, Sasai, Yokoyama, Haraoka, N.Katzs, Kostov, Simpson, Dettweiler-Reiter, Gleiser, Crawley-Boevey, Oshima etc.

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$\Rightarrow$ Kimura's talk on Tuesday for the confluence

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Characteristic exponents of ${}_{n+1}F_n$

$$
\begin{array}{lll}
0,\ 1-\beta_1,\ 1-\beta_2,\ \ldots,\ 1-\beta_n & \text{at} & z=0, \\
0,\ 1,\ \ldots,\ n-1,\ \sum_{i=1}^{n}\beta_i-\sum_{i=1}^{n+1}\alpha_i & \text{at} & z=1, \\
\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_{n+1} & \text{at} & z=\infty
\end{array}
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$\Longrightarrow$ Spectral type is $(1^{n+1}\ ;\ 1, n\ ;\ 1^{n+1})$.

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$\Longrightarrow$ Spectral type is $(1^{n+1}\ ;\ 1, n\ ;\ 1^{n+1})$, where $1^{n+1} = \underbrace{1, 1, \ldots, 1}_{n+1}$.

Rigid local system

of irreducible rigid Fuchsian differential systems with 3 singularities on $\mathbb{P}^1$

order	2	3	4	5	6	7	8	9	10	11	12	13	14
#	1	1	3	5	13	20	45	74	142	212	421	588	1004

# of irreducible rigid Fuchsian differential systems

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These lists are by Oshima (2008).

Yokoyama's list (1995)

	rank	# of singularities on $\mathbb{P}^1$
I (HGF)	n	3
I* (Pochhammer)	n	$n-1$
II	$2n$	3
II*	$2n$	4
III	$2n+1$	3
III*	$2n+1$	4
IV	6	3
IV*	6	4

Yokoyama's list (1995)

	rank	# of singularities on $\mathbb{P}^1$	spectrale type
I (HGF)	n	3	$1^n\ ; 1, n-1\ ; 1^n$
I* (Pochhammer)	n	$n-1$	$1, n-1; 1, n-1; \ldots; 1, n-1$
II	$2n$	3	$1^n, n; 1^n, n; 1, n-1, n$
II*	$2n$	4	$1^n, n; 1^{n-1}, n+1; 1, 2n-1; n, n$
III	$2n+1$	3	$1^{n+1}, n; 1^n, n+1; 1, n, n$
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IV*	6	4	$1^2, 4; 1^2, 4; 2, 4$

(I) Euler integral of the generalized HGF

$$
{}_{n+1}F_n\left(\begin{array}{c}\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_{n+1}\\ \beta_1,\ \beta_2,\ \ldots,\ \beta_n\end{array};z\right)=\prod_{s=1}^{n}\frac{\Gamma(\beta_s)}{\Gamma(\alpha_s)\Gamma(\beta_s-\alpha_s)}
$$

$$
\times\int_{1<t_1<t_2<\cdots<t_n<\infty}\prod_{i=1}^{n}t_i^{\alpha_{i+1}-\beta_i}\prod_{i=1}^{n+1}(t_i-t_{i-1})^{\beta_i-\alpha_i-1}\,dt_1\,\cdots\,dt_n
$$

where $t_0=1,\ t_{n+1}=z$.

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$$ {}_{n+1}F_n \left(\begin{matrix} \alpha_1, \ \alpha_2, \ \ldots, \ \alpha_{n+1} \\ \beta_1, \ \beta_2, \ \ldots, \ \beta_n \end{matrix} ; z \right) = \prod_{s=1}^{n} \frac{\Gamma(\beta_s)}{\Gamma(\alpha_s)\Gamma(\beta_s - \alpha_s)} $$

$$ \times \int_{1<t_1<t_2<\cdots<t_n<\infty} \prod_{i=1}^{n} t_i^{\alpha_{i+1}-\beta_i} \prod_{i=1}^{n+1} (t_i - t_{i-1})^{\beta_i - \alpha_i - 1} \, dt_1 \cdots dt_n $$

where $t_0 = 1, \ t_{n+1} = z$.

(I) Generalized HGF

$$\prod_{i=1}^{n} t_i^{\alpha_{i+1}-\beta_i} \prod_{i=1}^{n+1} (t_i - t_{i-1})^{\beta_i-\alpha_i-1}$$

where $t_0 = 1,\ t_{n+1} = z$.

(I) Generalized HGF

$$\prod_{i=1}^{n} t_i^{\lambda_i} \prod_{i=1}^{n+1} (t_i - t_{i-1})^{\lambda_{i-1,i}}$$

where $t_0 = 1,\ t_{n+1} = z$.

(I*) Pochhammer function

$$\prod_{j=0}^{n+1} (t - c_j)^{\lambda_j}$$

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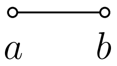
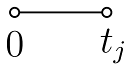
IV	6	3	$1^2, 4; 2^3; 1^4, 2$
IV*	6	4	$1^2, 4; 1^2, 4; 2, 4$

Diagramatic expression

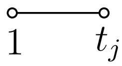


$\Leftrightarrow \quad (a-b)^{\lambda_{ab}}$ (or $(b-a)^{\lambda_{ab}}$)

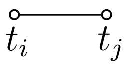
Examples:



$\Leftrightarrow \quad t_j^{\lambda_j}$



$\Leftrightarrow \quad (1-t_j)^{\lambda'_j}$



$\Leftrightarrow \quad (t_i-t_j)^{\lambda_{ij}}$

(I) Generalized HGF

$\prod_{i=1}^{n} t_i^{\lambda_i} \prod_{i=1}^{n+1} (t_i - t_{i-1})^{\lambda_{i-1,i}}$

where $t_0 = 1,\ t_{n+1} = z$.

(I*) Pochhammer function

$\prod_{j=0}^{n+1} (t - c_j)^{\lambda_j}$

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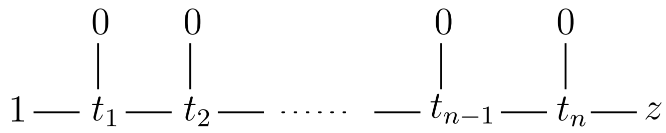

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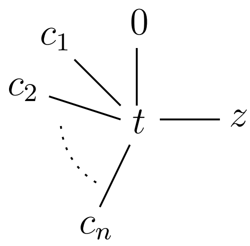

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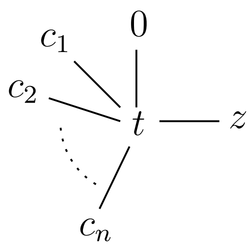
(I) Generalized Hypergeometric function

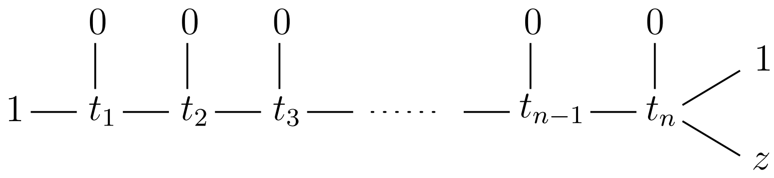
1^{n+1}	at 0
$1, n$	at 1
1^{n+1}	at ∞

(I*) Pochhammmer function



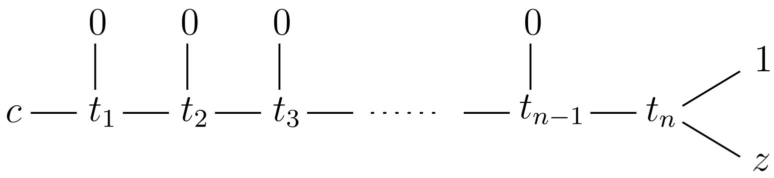
$1, n-1$	at 0
$1, n-1$	at c_1
$\cdots$	$\cdots$
$1, n-1$	at c_n
$1, n-1$	at ∞

(II)



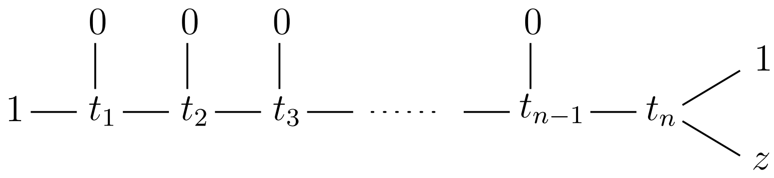
$1^n, n$	at 0
$1, n-1, n$	at 1
$1^n, n$	at ∞

(II*)



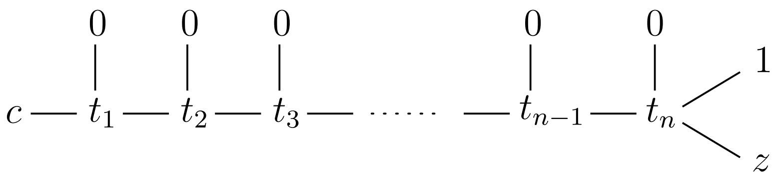
$1^{n-1}, n+1$	at 0
n, n	at 1
$1, 2n-1$	at c
$1^n, n$	at ∞

(III)



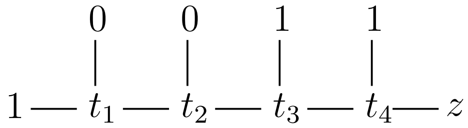
$1^{n-1}, n$	at 0
$1, n-1, n-1$	at 1
$1^n, n-1$	at ∞

(III*)



$1^n, n+1$	at 0
$n, n+1$	at 1
$1, 2n$	at c
$1^n, n+1$	at ∞

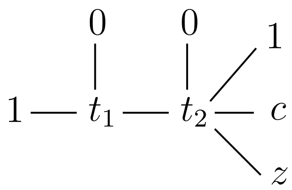
(IV)



(with $\lambda_1' + \lambda_{12} + \lambda_{23} + \lambda_3' + 2 = 0$)

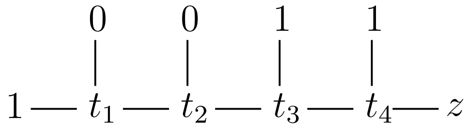
$1^2, 4$	at 0
2^3	at 1
$1^4, 2$	at ∞

(IV*)



$1, 1, 4$	at 0
$1, 1, 4$	at 1
$1, 1, 4$	at c
$2, 4$	at ∞

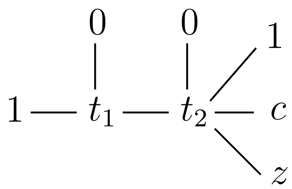
(IV)



(with $\lambda_1' + \lambda_{12} + \lambda_{23} + \lambda_3' + 2 = 0$)
resonance condition

$1^2, 4$	at 0
2^3	at 1
$1^4, 2$	at ∞

(IV*)



$1, 1, 4$	at 0
$1, 1, 4$	at 1
$1, 1, 4$	at c
$2, 4$	at ∞

Resonances and subsytems

$$\begin{array}{ccccccccccc} & & 0 & & 0 & & 1 & & 1 & & \\ & & | & & | & & | & & | & & \\ 1 & — & t_1 & — & t_2 & — & t_3 & — & t_4 & — & z \end{array} \quad .$$

The resonance $\lambda_{01} + \lambda_{12} + \lambda_{23} + \lambda_{03} + 2 = 0$ induces the subsystem.

$$\begin{array}{llcclll} 1,1,5 & \text{at } 0 & & & 1,1,4 & \text{at } 0 \\ 1,2,2,2 & \text{at } 1 & \longrightarrow & (\text{IV}) & 2,2,2 & \text{at } 1 \\ 1,1,1,2,2 & \text{at } \infty & & & 1,1,1,1,2 & \text{at } \infty \end{array}$$

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$$\begin{array}{ccccccccccc} & & 0 & & 0 & & 1 & & 1 & & \\ & & | & & | & & | & & | & & \\ 1 & — & t_1 & — & t_2 & — & t_3 & — & t_4 & — & z \end{array} \quad .$$

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$$\begin{array}{llcll} 1, 1, 5 & \text{at } 0 & & 1, 1, 4 & \text{at } 0 \\ 1, 2, 2, 2 & \text{at } 1 & \longrightarrow \quad \text{(IV)} & 2, 2, 2 & \text{at } 1 \\ 1, 1, 1, 2, 2 & \text{at } \infty & & 1, 1, 1, 1, 2 & \text{at } \infty \end{array}$$

subsytem

$$\iota \ : \ H_n(T, \mathcal{L}) \longrightarrow H_n^{\mathrm{lf}}(T, \mathcal{L})$$

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$\operatorname{Im} \iota \ni$ regularizable cycle

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$\mathrm{Im}\, \iota$: space of regularizable cycles

$\mathrm{reg}(\sigma) := \iota^{-1}(\sigma)$: a regularization of $\sigma \in \mathrm{Im}\, \iota$

If the exponent of the irreducible component of the divisor $\widetilde{D} = \pi^{-1}(D)$, where $\pi : \widetilde{(\mathbb{P}^1(\mathbb{C}))^m} \to (\mathbb{P}^1(\mathbb{C}))^m$ is the minimal blow-up along the non-normally crossing loci of D, is an ineger, the irreducible component or the exponent itself is said to be resonant.

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$$T = (\mathbb{P}^1(\mathbb{C}))^n \backslash D, \quad D = \cup_i \{f_i(t) = 0\}, \quad N^\circ : \text{tubular nbd of } D$$

$$\cdots \to H_{n+1}(T, N^\circ, \mathcal{L}) \to H_n(N^\circ, \mathcal{L}) \to H_n(T, \mathcal{L}) \to H_n(T, N^\circ, \mathcal{L}) \to$$

$$H_k(T, N^\circ, \mathcal{L}) \sim H_k^{\mathrm{lf}}(T, \mathcal{L}),$$

$$H_{n+1}^{\mathrm{lf}}(T, \mathcal{L}) \longrightarrow H_n(N^\circ, \mathcal{L}) \longrightarrow H_n(T, \mathcal{L}) \longrightarrow H_n^{\mathrm{lf}}(T, \mathcal{L})$$

Simpson's list

	rank	spectral type
HGF	n	1^n ; 1^n ; $n-1, 1$
Even family	$2n$	1^{2n} ; $n, n-1, 1$; n, n
Odd family	$2n+1$	1^{2n+1} ; $n, n, 1$; $n+1, n$
Extra case	6	1^6 ; 2^3 ; $4, 2$

Simpson's list

	rank	spectral type
HGF	n	$1^n\ ;\quad 1^n\ ;\quad n-1,1$
Even family	$2n$	$1^{2n}\ ;\quad n,n-1,1\ ;\quad n,n$
Odd family	$2n+1$	$1^{2n+1}\ ;\quad n,n,1\ ;\quad n+1,n$
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The even family of rank $2n$ corresponds to the restriction of the Heckman-Opdam HGF of BC_n-type.

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(by Oshima and Shimeno)

The rank of H_n in case of

$$\begin{array}{ccccccccccccc}
 & & 1 & & 0 & & 1 & & & & 1 & & 0 & & \\
 & & | & & | & & | & & & & | & & | & & \\
0 & — & t_1 & — & t_2 & — & t_3 & — & \cdots\cdots & — & t_{n-1} & — & t_n & — & z
\end{array}$$

or

n : even

$$\begin{array}{ccccccccccccc}
 & & 1 & & 0 & & 1 & & & & 0 & & 1 & & \\
 & & | & & | & & | & & & & | & & | & & \\
0 & — & t_1 & — & t_2 & — & t_3 & — & \cdots\cdots & — & t_{n-1} & — & t_n & — & z
\end{array}$$

n : odd

is a_{n+2}. Here a_n is the Fibonacci number: $a_1 = a_2 = 1, a_3 = 2, a_4 = 3, a_5 = 5, a_6 = 8, a_7 = 13, a_8 = 21, a_9 = 34, a_{10} = 55, a_{11} = 89, \ldots$.

The rank of H_n in case of

$$\begin{array}{ccccccccccccc}
 & & 1 & & 0 & & 1 & & & & 1 & & 0 & & \\
 & & | & & | & & | & & & & | & & | & & \\
0 & — & t_1 & — & t_2 & — & t_3 & — & \cdots\cdots & — & t_{n-1} & — & t_n & — & 1
\end{array}$$

n : even

or

$$\begin{array}{ccccccccccccc}
 & & 1 & & 0 & & 1 & & & & 0 & & 1 & & \\
 & & | & & | & & | & & & & | & & | & & \\
0 & — & t_1 & — & t_2 & — & t_3 & — & \cdots\cdots & — & t_{n-1} & — & t_n & — & 0
\end{array}$$

n : odd

is a_{n+1}. $(a_1 = a_2 = 1, a_3 = 2, a_4 = 3, a_5 = 5, a_6 = 8, a_7 = 13, a_8 = 21, a_9 = 34, a_{10} = 55, a_{11} = 89, \ldots.)$

(Odd family)

$1, n, n$	at 0
$n, n+1$	at 1
1^{2n+1}	at ∞

Resonance condition

$$\begin{aligned}
&\lambda'_1 + \lambda_{12} + \lambda_{23} + \lambda'_3 + 2 = 0, \quad && \lambda_2 + \lambda_{23} + \lambda_{34} + \lambda_4 + 1 = 0,\\
&\lambda'_3 + \lambda_{34} + \lambda_{45} + \lambda'_5 = 0, && \lambda_4 + \lambda_{45} + \lambda_{56} + \lambda_6 = 0,\\
&\qquad\cdots && \qquad\cdots\\
&\lambda'_{2n-3} + \cdots + \lambda'_{2n-1} = 0, && \lambda_{2n-2} + \cdots + \lambda_{2n} = 0.
\end{aligned}$$

(Even family)

$1, n, n+1$	at 0
$n+1, n+1$	at 1
1^{2n+2}	at ∞

Resonance condition

$$
\begin{array}{ll}
\lambda'_1 + \lambda_{12} + \lambda_{23} + \lambda'_3 + 2 = 0, & \lambda_2 + \lambda_{23} + \lambda_{34} + \lambda_4 + 1 = 0, \\
\lambda'_3 + \lambda_{34} + \lambda_{45} + \lambda'_5 = 0, & \lambda_4 + \lambda_{45} + \lambda_{56} + \lambda_6 = 0, \\
\cdots & \cdots \\
\lambda'_{2n-3} + \cdots + \lambda'_{2n-1} = 0, & \lambda_{2n-2} + \cdots + \lambda_{2n} = 0 \\
\lambda'_{2n-1} + \cdots + \lambda'_{2n+1} = 0. &
\end{array}
$$

(Exitra X_6)

$$\begin{array}{ccccccccccc} & & 1 & & 0 & & 1 & & 0 & & 0 & & \\ & & | & & | & & | & & | & & | & & \\ 0 & — & t_1 & — & t_2 & — & t_3 & — & t_4 & — & t_5 & — & z \end{array}$$

$1, 2, 3$	at 0
$3, 3$	at 1
1^6	at ∞

Resonance condition

$$\lambda_1 + \lambda_{12} + \lambda_{45} + \lambda_5 + 4 = \lambda_2 + \lambda_{23} + \lambda_{45} + \lambda_4 + 2 = 0$$

Connection formulas

Examples.

(1) ${}_{n+1}F_n$

$$f_i^{(\infty)}(z) = \sum_{j=1}^{n+1} \prod_{s\neq i} \frac{\Gamma(\alpha_i - \alpha_s + 1)}{\Gamma(\beta_j - \alpha_s)} \prod_{s\neq j} \frac{\Gamma(\beta_j - \beta_s)}{\Gamma(\alpha_i - \beta_s + 1)} \times f_j^{(0)}(z)\,,$$

where $f_i^{(0)}(z) = (-z)^{1-\beta_i}(1+O(z)), f_i^{(\infty)}(z) = (-z)^{-\alpha_i}(1+O(z^{-1}))$.

$$f_1^{(1)}(z) = \sum_{j=1}^{n+1} \prod_{s\neq i} \frac{\Gamma(1+\sum_{s=1}^{n}\beta_s - \sum_{s=1}^{n+1}\alpha_s)\prod_{\substack{1\leq s\leq n+1\\ s\neq i}}\Gamma(\beta_j-\beta_s)}{\prod_{1\leq s\leq n+1}\Gamma(\beta_j-\alpha_s)} \times f_j^{(0)}(z)\,,$$

where $f_i^{(0)}(z) = (-z)^{1-\beta_i}(1+O(z)), f_1^{(1)}(z) = (1-z)^{\sum_{i=1}^{n}\beta_i - \sum_{i=1}^{n+1}\alpha_i}(1+O(1-z))$.

(2) Even family of rank=4 (joint work with Haraoka):

$$t_1^{\lambda_1}(t_1-1)^{\lambda_2}(t_1-t_2)^{\lambda_3}t_2^{\lambda_4}(t_2-t_3)^{\lambda_5}(t_3-1)^{\lambda_6}(t_3-z)^{\lambda_7}$$

$(\lambda_{2356}+2=0,\ \lambda_{ij\cdots k}=\lambda_i+\lambda_j+\cdots+\lambda_k)$

$$F_1^{(0)}(z)=(-z)^{\lambda_{13457}+3}(1+O(z)),$$

$$\begin{aligned}
F_1^{(\infty)}(z)&=(-z)^{\lambda_{1234567}+3}\left(1+O(z^{-1})\right),\\
F_2^{(\infty)}(z)&=(-z)^{\lambda_{34567}+2}\left(1+O(z^{-1})\right),\\
F_3^{(\infty)}(z)&=(-z)^{\lambda_{567}+1}\left(1+O(z^{-1})\right),\\
F_4^{(\infty)}(z)&=\left(1+O(z^{-1})\right).
\end{aligned}$$

$$F_1^{(0)}(z)=\sum_{j=1}^{4}p_{1j}F_j^{(\infty)}(z).$$

(2) Even family of rank=4 (joint work with Haraoka):

$t_1^{\lambda_1}(t_1-1)^{\lambda_2}(t_1-t_2)^{\lambda_3}t_2^{\lambda_4}(t_2-t_3)^{\lambda_5}(t_3-1)^{\lambda_6}(t_3-z)^{\lambda_7} \quad (\lambda_{2356}+2=0)$

$$F_1^{(0)}(z) = \sum_{j=1}^{4} p_{1j} F_j^{(\infty)}(z),$$

where

$$p_{11} = \frac{\Gamma(1+\lambda_{12}, 1+\lambda_{14}, 2+\lambda_{13}, 1+\lambda_{1234}, 4+\lambda_{13457})}{\Gamma(1+\lambda_1, 2+\lambda_{123}, 2+\lambda_{134}, 2+\lambda_{147}, 3+\lambda_{12345})},$$
$$p_{12} = \frac{\Gamma(1+\lambda_{34}, 2+\lambda_{13}, 2+\lambda_{3456}, 4+\lambda_{1357}, -1-\lambda_{12})}{\Gamma(1+\lambda_3, 2+\lambda_{134}, 2+\lambda_{345}, 3+\lambda_{34567}, -\lambda_2)},$$
$$p_{13} = \frac{\Gamma(1+\lambda_{56}, 2+\lambda_{13}, 4+\lambda_{13457}, -1-\lambda_{34}, -2-\lambda_{1234})}{\Gamma(1+\lambda_1, 1+\lambda_5, 2+\lambda_{567}, -\lambda_2, -\lambda_4)},$$
$$p_{14} = \frac{\Gamma(2+\lambda_{13}, 4+\lambda_{13457}, -2-\lambda_{3456}, -1-\lambda_{56}, -1-\lambda_{14})}{\Gamma(1+\lambda_3, 1+\lambda_7, 2+\lambda_{123}, -\lambda_4, -\lambda_6)},$$

with $\quad \Gamma(a_1, a_2, \cdots, a_m) = \Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_m).$

${}_{n+1}F_n$ case

Derivation of $f_i^{(\infty)}(z) = \sum_{j=1}^{n+1} \prod_{s\neq i} \frac{\Gamma(\alpha_i-\alpha_s+1)}{\Gamma(\beta_j-\alpha_s)} \prod_{s\neq j} \frac{\Gamma(\beta_j-\beta_s)}{\Gamma(\alpha_i-\beta_s+1)} \times f_j^{(0)}(z)$ by use of the intersection number.

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$H_n^{\mathrm{lf}}(T,\mathcal{L})$ or $H_n(T,\mathcal{L})$, where $\mathcal{L}$ is determined by

$$u(t) = \prod_{i=1}^{n} t_i^{\alpha_{i+1}-\beta_i} \prod_{i=1}^{n+1} (t_i - t_{i-1})^{\beta_i-\alpha_i-1}, \quad (\beta_{n+1}=1, t_0=1, t_{n+1}=z),$$

$$T = \mathbb{C}^n \setminus \cup_{i=1}^{n} \{\, t_i = 0 \,\} \cup \cup_{i=1}^{n+1} \{\, t_i - t_{i-1} = 0 \,\}.$$

reg : $H_n^{\mathrm{lf}}(T,\mathcal{L}) \longrightarrow H_n(T,\mathcal{L})$ (identified)

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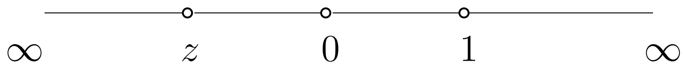
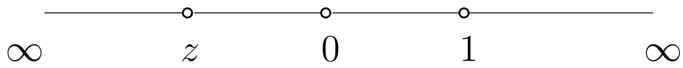
In what follows, z is fixed to be $\infty < z < 0$.

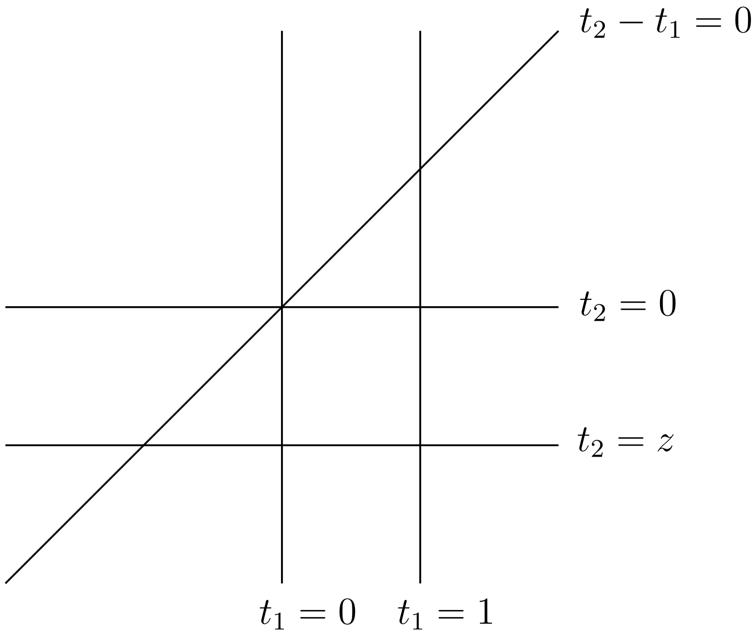
$n = 1$



$n = 1$



$n = 2$



Bases of $H_n^{\mathrm{lf}}(T, \mathcal{L})$:

$$\left\{ D_1^{(0)}, D_2^{(0)}, \ldots, D_{n+1}^{(0)} \;\middle|\; D_i^{(0)} = \begin{pmatrix} \infty < z < t_n < \cdots < t_i < 0 \\ 1 < t_1 < \cdots < t_{i-1} < \infty \end{pmatrix} \right\},$$

$$\left\{ D_1^{(\infty)}, D_2^{(\infty)}, \ldots, D_{n+1}^{(\infty)} \;\middle|\; D_i^{(\infty)} = \begin{pmatrix} \infty < t_i < \cdots < t_n < z \\ 0 < t_{i-1} < \cdots < t_1 < 1 \end{pmatrix} \right\}.$$

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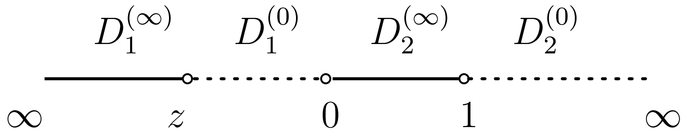
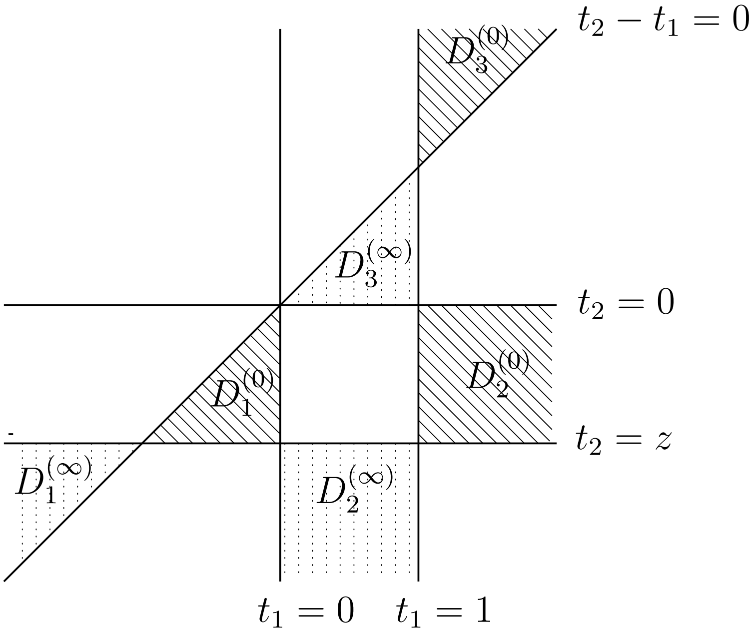
$n = 1$



$n = 2$



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$$\left\{ D_1^{(\infty)}, D_2^{(\infty)}, \ldots, D_{n+1}^{(\infty)} \;\middle|\; D_i^{(\infty)} = \begin{pmatrix} \infty < t_i < \cdots < t_n < z \\ 0 < t_{i-1} < \cdots < t_1 < 1 \end{pmatrix} \right\}.$$

$\Longrightarrow \exists\, c_{ij}$ such that

$$D_i^{(\infty)} = \sum_{1 \le j \le n+1} c_{ij}\, D_j^{(0)}$$

$$I_i^{(0)}(z) = \int_{D_i^{(0)}} u_{D_i^{(0)}}(t)\, dt_1 \cdots dt_n = \prod_{\substack{1 \le s \le n+1 \\ s \neq i}} B(\alpha_s - \beta_i + 1, \beta_s - \alpha_s) \times f_i^{(0)}(z),$$

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For $u(t) = \prod_i f_i(t)^{\alpha_i}$, $u_D(t) = \prod_i (\epsilon_i\, f_i(t))^{\alpha_i}$, where $\epsilon_i = \pm$ is determined so that $\epsilon_i\, f_i(t) > 0$ on D.

$$D_i^{(\infty)} = \sum_{1 \leq j \leq n+1} c_{ij} \, D_j^{(0)}$$

Intersection form (Intersection numbers)

$$\langle\quad,\quad\rangle : H_n^{\mathrm{lf}}(T,\mathcal{L}) \times H_n^{\mathrm{lf}}(T,\mathcal{L}) \longrightarrow \mathbb{C} \quad \text{(intersection form)}$$

$$(C, C') \longmapsto \langle C, C'\rangle = \textstyle\sum_{\rho,\sigma} a_\rho\, \overline{a'_\sigma} \sum_{t\in\rho\cap\sigma} I_t(\rho,\sigma) v_\rho(t)\overline{v'_\sigma(t)}/|u|^2,$$

$$\mathrm{reg}\ C = \textstyle\sum_\rho a_\rho\, \rho \otimes v_\rho, \quad C' = \sum_\sigma a'_\sigma\, \sigma \otimes v'_\sigma,$$

where $a_\rho, a'_\sigma \in \mathbb{C}$, ρ, σ:n-simplex, v_ρ, v'_σ: a section of $\mathcal{L}$ on ρ, σ, $^-$: the complex conjugation, $I_t(\rho,\sigma)$: the topological intersection number of ρ and σ at t.

The value $\langle C, C'\rangle$ is called the intersection number of C and C' and written also by $C \bullet C'$

Example. $T = \mathbb{C}\backslash\{0,1\}$, $u(t) = t^{\alpha}(1-t)^{\beta}$

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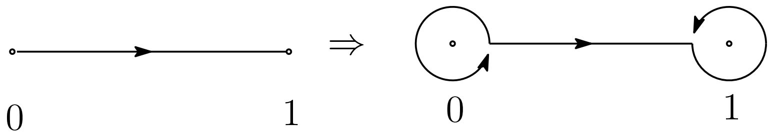
$\overrightarrow{(0,1)} \bullet \overrightarrow{(0,1)} \quad ?$

Example. $T = \mathbb{C} \backslash \{0, 1\}$, $u(t) = t^{\alpha}(1-t)^{\beta}$

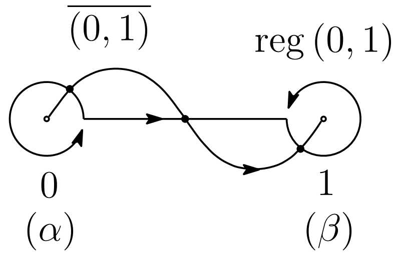
$$\overrightarrow{(0,1)} \quad \Rightarrow \quad \mathrm{reg}\,\overrightarrow{(0,1)} = \left\{ \frac{1}{d_\alpha} S(\epsilon\,; 0) + \overrightarrow{[\epsilon, 1-\epsilon]} - \frac{1}{d_\beta} S(1-\epsilon\,; 1) \right\}$$



$d_a = e(a) - 1, \quad e(a) = \exp(2\pi\sqrt{-1}a).$

$$\overrightarrow{(0,1)} \bullet \overrightarrow{(0,1)} = -\tfrac{1}{d_\alpha} - 1 + \tfrac{-1}{d_\beta}$$
$$= -\tfrac{d_{\alpha+\beta}}{d_\alpha d_\beta} = -\tfrac{s(\alpha+\beta)}{s(\alpha)s(\beta)},$$

where $s(a) = \sin(\pi a)$



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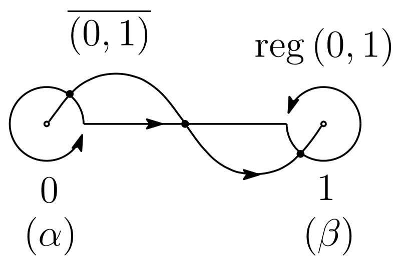
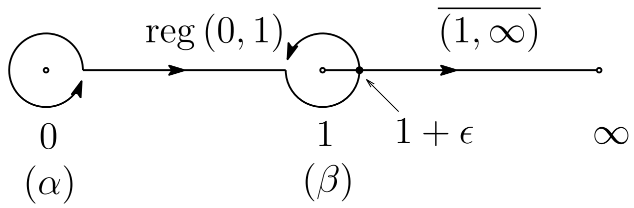


$$\overrightarrow{(0,1)} \bullet \overrightarrow{(1,\infty)} = \tfrac{e(\beta/2)}{e(\beta)-1}$$



$$D_i^{(\infty)} = \sum_{1 \leq j \leq n+1} c_{ij}\, D_j^{(0)}$$

$$D_i^{(\infty)} = \sum_{1 \leq j \leq n+1} c_{ij} \, D_j^{(0)}, \qquad C = (c_{ij}),$$

$$\begin{pmatrix} D_1^{(\infty)} \\ \vdots \\ \vdots \\ D_{n+1}^{(\infty)} \end{pmatrix} \bullet (D_1^{(0)}, \ldots, D_{n+1}^{(0)}) = C \begin{pmatrix} D_1^{(0)} \\ \vdots \\ \vdots \\ D_{n+1}^{(0)} \end{pmatrix} \bullet (D_1^{(0)}, \ldots, D_{n+1}^{(0)})$$

$$\begin{pmatrix} D_1^{(\infty)} \bullet D_1^{(0)} & \cdots & D_1^{(\infty)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(\infty)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(\infty)} \bullet D_{n+1}^{(0)} \end{pmatrix} = C \begin{pmatrix} D_1^{(0)} \bullet D_1^{(0)} & \cdots & D_1^{(0)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(0)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(0)} \bullet D_{n+1}^{(0)} \end{pmatrix}$$

$$\begin{pmatrix} D_1^{(\infty)} \bullet D_1^{(0)} & \cdots & D_1^{(\infty)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(\infty)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(\infty)} \bullet D_{n+1}^{(0)} \end{pmatrix} = C \begin{pmatrix} D_1^{(0)} \bullet D_1^{(0)} & \cdots & D_1^{(0)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(0)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(0)} \bullet D_{n+1}^{(0)} \end{pmatrix}$$

$$C = \begin{pmatrix} D_1^{(\infty)} \bullet D_1^{(0)} & \cdots & D_1^{(\infty)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(\infty)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(\infty)} \bullet D_{n+1}^{(0)} \end{pmatrix} \begin{pmatrix} D_1^{(0)} \bullet D_1^{(0)} & \cdots & D_1^{(0)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(0)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(0)} \bullet D_{n+1}^{(0)} \end{pmatrix}^{-1}$$

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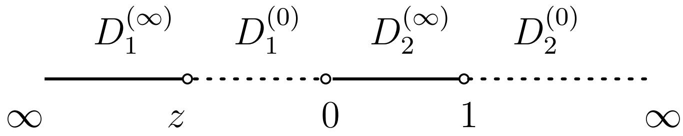
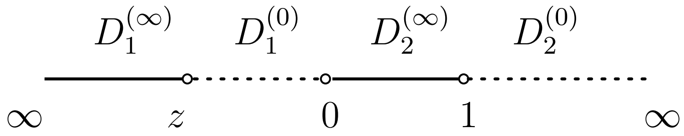
$$C = \begin{pmatrix} D_1^{(\infty)} \bullet D_1^{(0)} & \cdots & D_1^{(\infty)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(\infty)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(\infty)} \bullet D_{n+1}^{(0)} \end{pmatrix} \begin{pmatrix} D_1^{(0)} \bullet D_1^{(0)} & \cdots & D_1^{(0)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(0)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(0)} \bullet D_{n+1}^{(0)} \end{pmatrix}^{-1}$$

$n = 1$

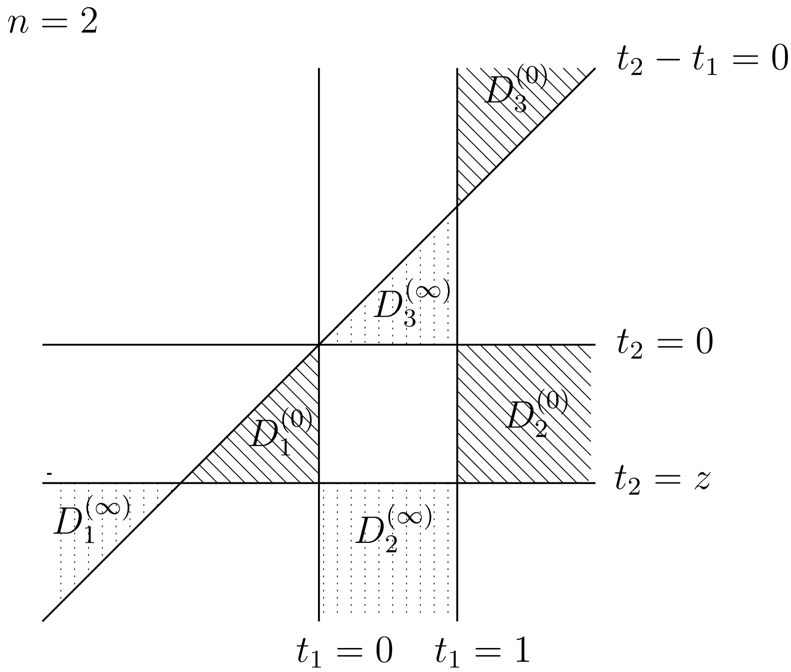


$$C = \begin{pmatrix} D_1^{(\infty)} \bullet D_1^{(0)} & \cdots & D_1^{(\infty)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(\infty)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(\infty)} \bullet D_{n+1}^{(0)} \end{pmatrix} \begin{pmatrix} D_1^{(0)} \bullet D_1^{(0)} & \cdots & D_1^{(0)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(0)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(0)} \bullet D_{n+1}^{(0)} \end{pmatrix}^{-1}$$

$n = 1$

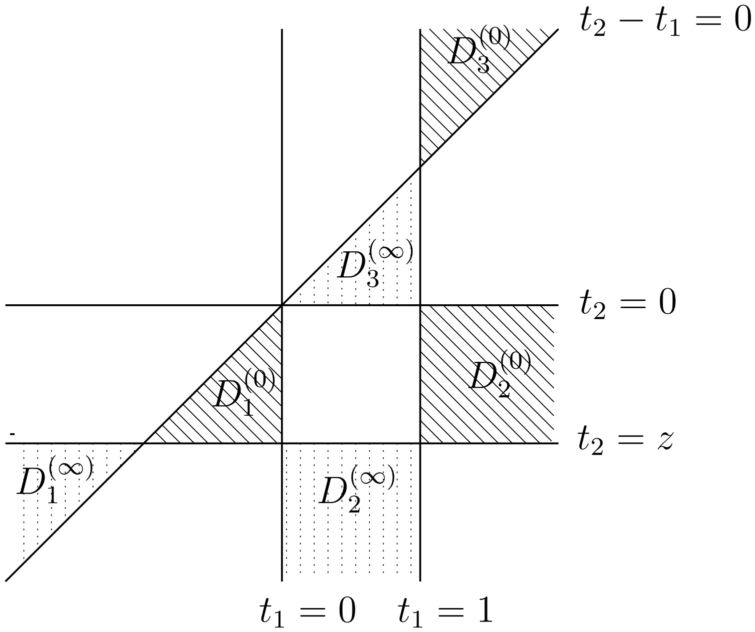


$$\Longrightarrow \quad \begin{aligned} D_1^{(0)} \bullet D_2^{(0)} &= 0 \\ D_2^{(0)} \bullet D_1^{(0)} &= 0 \end{aligned}$$



$n = 2$
$t_2 - t_1 = 0$
$D_3^{(0)}$
$D_3^{(\infty)}$
$t_2 = 0$
$D_1^{(0)}$
$D_2^{(0)}$
$t_2 = z$
$D_1^{(\infty)}$
$D_2^{(\infty)}$
$t_1 = 0$
$t_1 = 1$

$n = 2$



$$\Longrightarrow \quad D_i^{(0)} \bullet D_j^{(0)} = 0 \quad (i \neq j)$$

$$C = \begin{pmatrix} D_1^{(\infty)} \bullet D_1^{(0)} & \cdots & D_1^{(\infty)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(\infty)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(\infty)} \bullet D_{n+1}^{(0)} \end{pmatrix} \begin{pmatrix} D_1^{(0)} \bullet D_1^{(0)} & \cdots & D_1^{(0)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(0)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(0)} \bullet D_{n+1}^{(0)} \end{pmatrix}^{-1}$$

$$C = \begin{pmatrix} D_1^{(\infty)} \bullet D_1^{(0)} & \cdots & D_1^{(\infty)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(\infty)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(\infty)} \bullet D_{n+1}^{(0)} \end{pmatrix} \begin{pmatrix} D_1^{(0)} \bullet D_1^{(0)} & \cdots & D_1^{(0)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(0)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(0)} \bullet D_{n+1}^{(0)} \end{pmatrix}^{-1}$$

$$\Uparrow \quad D_i^{(0)} \bullet D_j^{(0)} = 0 \quad (i \neq j)$$

$$C = \begin{pmatrix} D_1^{(\infty)} \bullet D_1^{(0)} & \cdots & D_1^{(\infty)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(\infty)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(\infty)} \bullet D_{n+1}^{(0)} \end{pmatrix} \begin{pmatrix} D_1^{(0)} \bullet D_1^{(0)} & \cdots & D_1^{(0)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(0)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(0)} \bullet D_{n+1}^{(0)} \end{pmatrix}^{-1}$$

$$D_i^{(0)} \bullet D_j^{(0)} = \delta_{ij} \left(\frac{\sqrt{-1}}{2}\right)^n \prod_{\substack{1 \leq s \leq n+1 \\ s \neq j}} \frac{\sin(\beta_s - \beta_j)}{\sin(\beta_s - \alpha_s)\sin(\alpha_s - \beta_j)} \qquad \Uparrow$$

$$C = \begin{pmatrix} D_1^{(\infty)} \bullet D_1^{(0)} & \cdots & D_1^{(\infty)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(\infty)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(\infty)} \bullet D_{n+1}^{(0)} \end{pmatrix} \begin{pmatrix} D_1^{(0)} \bullet D_1^{(0)} & \cdots & D_1^{(0)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(0)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(0)} \bullet D_{n+1}^{(0)} \end{pmatrix}^{-1}$$

$$D_i^{(0)} \bullet D_j^{(0)} = \delta_{ij} \left(\frac{\sqrt{-1}}{2}\right)^n \prod_{\substack{1 \le s \le n+1 \\ s \neq j}} \frac{\sin(\beta_s - \beta_j)}{\sin(\beta_s - \alpha_s)\sin(\alpha_s - \beta_j)},$$

$$D_i^{(\infty)} \bullet D_j^{(0)} = \left(\frac{\sqrt{-1}}{2}\right)^n \frac{1}{\sin(\beta_j - \alpha_i)} \prod_{\substack{1 \le s \le n+1 \\ s \neq i,j}} \frac{1}{\sin(\beta_s - \alpha_s)}.$$

$$C = \begin{pmatrix} D_1^{(\infty)} \bullet D_1^{(0)} & \cdots & D_1^{(\infty)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(\infty)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(\infty)} \bullet D_{n+1}^{(0)} \end{pmatrix} \begin{pmatrix} D_1^{(0)} \bullet D_1^{(0)} & \cdots & D_1^{(0)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(0)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(0)} \bullet D_{n+1}^{(0)} \end{pmatrix}^{-1}$$

$$D_i^{(0)} \bullet D_j^{(0)} = \delta_{ij} \left(\tfrac{\sqrt{-1}}{2}\right)^n \prod_{\substack{1 \le s \le n+1 \\ s \ne j}} \tfrac{\sin(\beta_s - \beta_j)}{\sin(\beta_s - \alpha_s)\sin(\alpha_s - \beta_j)},$$

$$D_i^{(\infty)} \bullet D_j^{(0)} = \left(\tfrac{\sqrt{-1}}{2}\right)^n \tfrac{1}{\sin(\beta_j - \alpha_i)} \prod_{\substack{1 \le s \le n+1 \\ s \ne i,j}} \tfrac{1}{\sin(\beta_s - \alpha_s)}.$$

$$\Longrightarrow \quad c_{ij} = \frac{D_i^{(\infty)} \bullet D_j^{(0)}}{D_j^{(0)} \bullet D_j^{(0)}}$$

$$C = \begin{pmatrix} D_1^{(\infty)} \bullet D_1^{(0)} & \cdots & D_1^{(\infty)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(\infty)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(\infty)} \bullet D_{n+1}^{(0)} \end{pmatrix} \begin{pmatrix} D_1^{(0)} \bullet D_1^{(0)} & \cdots & D_1^{(0)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(0)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(0)} \bullet D_{n+1}^{(0)} \end{pmatrix}^{-1}$$

$$D_i^{(0)} \bullet D_j^{(0)} = \delta_{ij} \left(\tfrac{\sqrt{-1}}{2}\right)^n \prod_{\substack{1 \le s \le n+1 \\ s \ne j}} \tfrac{\sin(\beta_s - \beta_j)}{\sin(\beta_s - \alpha_s)\sin(\alpha_s - \beta_j)},$$

$$D_i^{(\infty)} \bullet D_j^{(0)} = \left(\tfrac{\sqrt{-1}}{2}\right)^n \tfrac{1}{\sin(\beta_j - \alpha_i)} \prod_{\substack{1 \le s \le n+1 \\ s \ne i,j}} \tfrac{1}{\sin(\beta_s - \alpha_s)}.$$

$$\Longrightarrow \quad c_{ij} = \frac{D_i^{(\infty)} \bullet D_j^{(0)}}{D_j^{(0)} \bullet D_j^{(0)}} = \frac{\sin(\beta_i - \alpha_i)}{\sin(\beta_j - \alpha_i)} \prod_{\substack{1 \le s \le n+1 \\ s \ne j}} \frac{\sin(\alpha_s - \beta_j)}{\sin(\beta_s - \beta_j)}$$

$$C = \begin{pmatrix} D_1^{(\infty)} \bullet D_1^{(0)} & \cdots & D_1^{(\infty)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(\infty)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(\infty)} \bullet D_{n+1}^{(0)} \end{pmatrix} \begin{pmatrix} D_1^{(0)} \bullet D_1^{(0)} & \cdots & D_1^{(0)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(0)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(0)} \bullet D_{n+1}^{(0)} \end{pmatrix}^{-1}$$

$$D_i^{(0)} \bullet D_j^{(0)} = \delta_{ij} \left(\tfrac{\sqrt{-1}}{2}\right)^n \prod_{\substack{1 \le s \le n+1 \\ s \ne j}} \tfrac{\sin(\beta_s - \beta_j)}{\sin(\beta_s - \alpha_s)\sin(\alpha_s - \beta_j)},$$

$$D_i^{(\infty)} \bullet D_j^{(0)} = \left(\tfrac{\sqrt{-1}}{2}\right)^n \tfrac{1}{\sin(\beta_j - \alpha_i)} \prod_{\substack{1 \le s \le n+1 \\ s \ne i,j}} \tfrac{1}{\sin(\beta_s - \alpha_s)}.$$

$$\Longrightarrow \quad c_{ij} = \tfrac{D_i^{(\infty)} \bullet D_j^{(0)}}{D_j^{(0)} \bullet D_j^{(0)}} = \tfrac{\sin(\beta_i - \alpha_i)}{\sin(\beta_j - \alpha_i)} \prod_{\substack{1 \le s \le n+1 \\ s \ne j}} \tfrac{\sin(\alpha_s - \beta_j)}{\sin(\beta_s - \beta_j)}$$

$$\Longrightarrow \quad f_i^{(\infty)}(z) = \sum_{j=1}^{n+1} \prod_{s \ne i} \tfrac{\Gamma(\alpha_i - \alpha_s + 1)}{\Gamma(\beta_j - \alpha_s)} \prod_{s \ne j} \tfrac{\Gamma(\beta_j - \beta_s)}{\Gamma(\alpha_i - \beta_s + 1)} \times f_j^{(0)}(z)$$

$$C = \begin{pmatrix} D_1^{(\infty)} \bullet D_1^{(0)} & \cdots & D_1^{(\infty)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(\infty)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(\infty)} \bullet D_{n+1}^{(0)} \end{pmatrix} \begin{pmatrix} D_1^{(0)} \bullet D_1^{(0)} & \cdots & D_1^{(0)} \bullet D_{n+1}^{(0)} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ D_{n+1}^{(0)} \bullet D_1^{(0)} & \cdots & D_{n+1}^{(0)} \bullet D_{n+1}^{(0)} \end{pmatrix}^{-1}$$

$$\Longrightarrow \quad c_{ij} = \tfrac{D_i^{(\infty)} \bullet D_j^{(0)}}{D_j^{(0)} \bullet D_j^{(0)}} = \tfrac{\sin(\beta_i - \alpha_i)}{\sin(\beta_j - \alpha_i)} \prod_{\substack{1 \le s \le n+1 \\ s \neq j}} \tfrac{\sin(\alpha_s - \beta_j)}{\sin(\beta_s - \beta_j)}$$

$$\Longrightarrow \quad f_i^{(\infty)}(z) = \sum_{j=1}^{n+1} \prod_{s \neq i} \frac{\Gamma(\alpha_i - \alpha_s + 1)}{\Gamma(\beta_j - \alpha_s)} \prod_{s \neq j} \frac{\Gamma(\beta_j - \beta_s)}{\Gamma(\alpha_i - \beta_s + 1)} \times f_j^{(0)}(z)$$

Differential equation of (II) of rank=4:

$$\frac{d}{dz}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \left\{\frac{A_0}{z} + \frac{A_1}{z-1}\right\}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix},$$

where

$$\begin{aligned} y_1 &= ((0 < t_1 < t_2 < z)), \\ y_2 &= ((0 < t_2 < z, t_2 < t_1 < 1)), \\ y_3 &= ((0 < t_1 < t_2 < z, z < t_2 < 1)), \\ y_4 &= ((z < t_2 < t_1 < 1)) \end{aligned}$$

and

$$A_0 = \begin{pmatrix} \lambda_1 + \lambda_2 + \lambda_{12} + \lambda_{23} & 0 & 0 & 0 \\ \lambda_1 & \lambda_2 + \lambda_{23} & 0 & 0 \\ \lambda_1 + \lambda_2 + \lambda_{12} & 0 & 0 & 0 \\ \lambda_1 & \lambda_2 & 0 & 0 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0 & 0 & \lambda_{02} & \lambda_{01} \\ 0 & 0 & 0 & \lambda_{01} + \lambda_{02} + \lambda_{12} \\ 0 & 0 & \lambda_{02} + \lambda_{23} & \lambda_{01} \\ 0 & 0 & 0 & \lambda_{01} + \lambda_{02} + \lambda_{12} + \lambda_{23} \end{pmatrix}$$