Hyperbolic volumes, configuration spaces and iterated integrals

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(3) Nilpotent flat connections for volume functions.

(4) Asymptotic behavior of volumes.

M^n_κ : spherical, Euclidean or hyperbolic space of constant curvature κ and of dimension $n \geq 2$.

$\kappa = 1 \quad S^n$ (unit sphere in $\mathbf{R}^{n+1}$)

$\kappa = -1 \quad \mathcal{H}^n$

(hyperboloid model in the Minkowski space)

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(hyperboloid model in the Minkowski space)

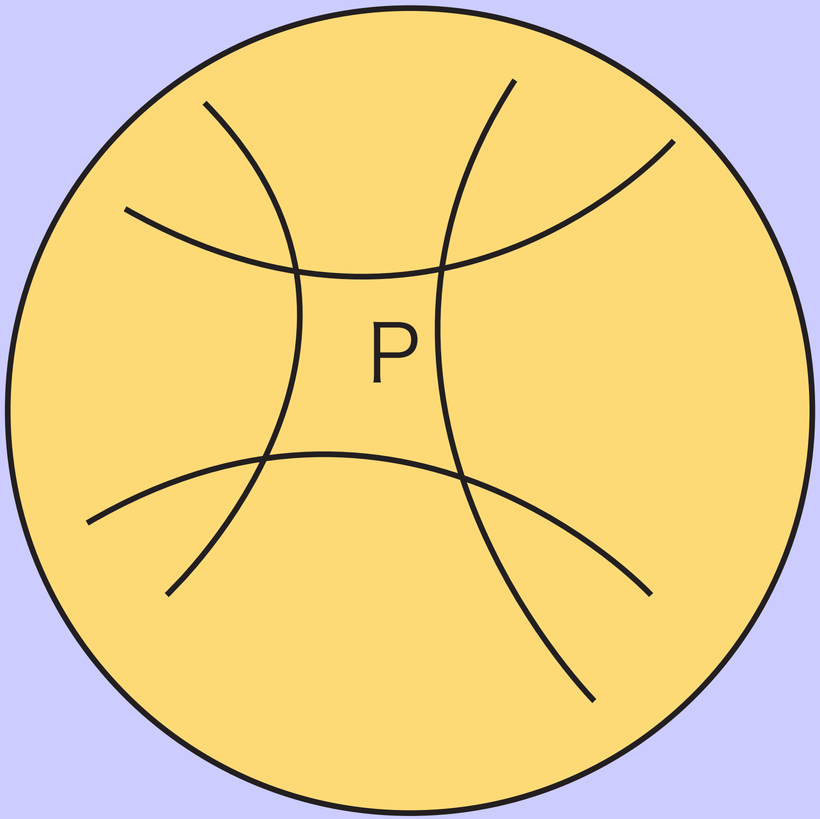
H_j, $1 \leq j \leq m$: hyperplanes in $\mathbf{R}^{n+1}$ defined by linear forms $f_j : \mathbf{R}^{n+1} \to \mathbf{R}$

C : the intersection of the half spaces

$$f_j \geq 0, \quad 1 \leq j \leq m$$

spherical polyhedron $P = S^n \cap C$

hyperbolic polyhedron $P = \mathcal{H}^n \cap C$



Volume of P :

function on the moduli space

of arrangements

1 Schläfli formula

$\{P\}$: a family of smoothly parametrized compact n dimensional polyhedra in M^n_κ

$V_n(P)$ the n-dimensional volume of P

$$\kappa \, dV_n(P) = \frac{1}{n-1} \sum_F V_{n-2}(F) \, d\theta_F$$

F : $(n-2)$-dimensional face of P

θ_F : the dihedral angle of the two $(n-1)$-dimensional faces meeting at F

2 The space of Gram matrices

Δ : n-dimensional simplex in M^n_κ
$E_1, \cdots, E_{n+1}$: $(n-1)$-dimensional faces of Δ
θ_{ij} : the dihedral angle between E_i and E_j
The Gram matrix $A = (a_{ij})$ of the simplex Δ :
$(n+1) \times (n+1)$ matrix defined by

$$a_{ij} = -\cos\theta_{ij}$$

2 The space of Gram matrices

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$X_n(\mathbf{R})$: the set of all symmetric unidiagonal
$(n+1) \times (n+1)$ matirices

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Lemma

The union $C_n = C_n^+ \cup C_n^0 \cup C_n^-$ is a convex open set in $X_n(\mathbf{R})$ and the codimension one Euclidean locus C_n^0 is a topological cell which cuts C_n into two open cells C_n^+ and C_n^-.

3 Iterated integrals

$\omega_1, \cdots, \omega_k$: differential 1-forms on a smooth manifold M

$\gamma : [0,1] \longrightarrow M$ a smooth path

pull-back $\gamma^*\omega_i = f_i(t)dt$, $1 \leq i \leq k$.

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The iterated integral of the 1-forms $\int_\gamma \omega_1\omega_2 \cdots \omega_k$ is defined as

$$\int_{0 \leq t_1 \leq \cdots \leq t_k \leq 1} f_1(t_1) f_2(t_2) \cdots f_k(t_k)\, dt_1 dt_2 \cdots dt_k$$

$\omega_1, \cdots, \omega_k$: differential forms on M
$\mathcal{P}M$: the space of smooth paths on M

$$\Delta_k = \{(t_1, \cdots, t_k) \in \mathbf{R}^k \,;\, 0 \le t_1 \le \cdots \le t_k \le 1\}$$

$$\varphi : \Delta_k \times \mathcal{P}M \to \underbrace{M \times \cdots \times M}_{k}$$

defined by $\varphi(t_1, \cdots, t_k; \gamma) = (\gamma(t_1), \cdots, \gamma(t_k))$
The iterated integral of $\omega_1, \cdots, \omega_k$ is defined as

$$\int \omega_1 \cdots \omega_k = \int_{\Delta_k} \varphi^*(\omega_1 \times \cdots \times \omega_k)$$

Proposition [K. T. Chen]

As a differential form on the space of paths fixing endpoints $d\int \omega_1 \cdots \omega_k$ is

$$\sum_{j=1}^{k}(-1)^{\nu_{j-1}+1}\int \omega_1 \cdots \omega_{j-1} d\omega_j \ \omega_{j+1} \cdots \omega_k$$

$$+\sum_{j=1}^{k-1}(-1)^{\nu_j+1}\int \omega_1 \cdots \omega_{j-1}(\omega_j \wedge \omega_{j+1})\omega_{j+2} \cdots \omega_k$$

where $\nu_j = \deg \omega_1 + \cdots + \deg \omega_j - j$.

4 Volumes of spherical simplices

For a positive integer m with $2m \leq n+1$ let

$$I_0 \subset I_1 \subset \cdots \subset I_k \subset \cdots \subset I_m$$

be an increasing sequence of subsets of $I = \{1, 2, \cdots, n+1\}$ such that $|I_k| = 2k$.

$\mathcal{F}_m[n]$: the set of all such sequences $(I_0 \cdots I_m)$

Δ : n-dimensional spherical simplex with faces

$$E_1, \cdots, E_{n+1}$$

Put

$$\Delta(I_k) = \bigcap_{j \in I_k} E_j, \quad k = 1, 2, \cdots$$

$$I_k = \{a_1, b_1, \cdots, a_k, b_k\}, \quad k = 1, 2, \cdots$$

$\theta(I_{k-1}, I_k)$: the dihedral angle between the faces $\Delta(I_{k-1}) \cap E_{a_k}$ and $\Delta(I_{k-1}) \cap E_{b_k}$

Define a 1-form on C_n^+ by

$$\omega(I_{k-1}, I_k) = d\theta(I_{k-1}, I_k)$$

Take a path connecting $A \in C_n^+$ and $\mathbf{x}_0 \in C_n^0$.

$\Delta(A) \subset S^n$ spherical simplex with a Gram matrix A

Theorem [Aomoto (1977)] Put $m = \left[\frac{n+1}{2}\right]$. The volume of the n-dimensional spherical simplex $\Delta(A)$ with a Gram matrix A is expressed as

$$\sum \frac{1}{(n-1)!!} \int_{\mathbf{x}_0}^{A} \omega(I_{m-1}, I_m) \cdots \omega(I_0, I_1)$$

where the sum is for any

$$(I_0 \cdots I_m) \in \mathcal{F}_m[n]$$

5 Logarithmic forms

$$I_k = \{a_1, b_1, \cdots, a_k, b_k\}, \quad k = 1, 2, \cdots$$

$D(I_k)$: the small determinant of the Gram matrix A with rows and columns indexed by I_k

$D(I_{k-1}, I_k)$: the small determinant of A with rows and and columns indexed by $I_{k-1} \cup \{a_k\}$ and $I_{k-1} \cup \{b_k\}$

Proposition $\omega(I_{k-1}, I_k) = d\theta(I_{k-1}, I_k)$ is expressed by means of small determinants of the Gram matrix as

$$\frac{1}{2i}\ d\log\left(\frac{-D(I_{k-1}, I_k) + i\sqrt{D(I_{k-1})D(I_k)}}{-D(I_{k-1}, I_k) - i\sqrt{D(I_{k-1})D(I_k)}}\right)$$

$$= d\arctan\left(-\frac{\sqrt{D(I_{k-1})D(I_k)}}{D(I_{k-1}, I_k)}\right)$$

6 Example – volume of 3-simplex

$$dV = \frac{1}{2} \sum_{1 \le i < j \le 4} \arctan\left(- \frac{\sin \theta_{ij} \sqrt{\det A}}{D_{ij}} \right) d\theta_{ij}$$

θ_{ij}: dihedral angle between 2 faces
D_{ij}: the small determinant $D(\{i, j\}, \{1, 2, 3, 4\})$

Formula for V in terms of θ_{ij} due to Y.Cho-H.Kim, J. Murakami-M.Yano

7 Analytic continuation to hyperbolic volumes

$S(A)$: Schläfli function on C_n^+ defined by

$$\sum \frac{1}{(n-1)!!} \int_{\mathbf{x}_0}^{A} \omega(I_{m-1}, I_m) \cdots \omega(I_0, I_1)$$

for $(I_0 \cdots I_m) \in \mathcal{F}_m[n]$

$X_{n+1}(\mathbf{C})$: the set of $(n+1) \times (n+1)$ symmetric unidiagonal complex matrices.

For a subset J of $I = \{1, 2, \cdots, n+1\}$ define $Z(J)$ to be the set consisting of $A \in X_{n+1}(\mathbf{C})$ such that the small determinant $D(J)$ of A vanishes. Set

$$\mathcal{Z} = \bigcup_{J \subset I,\ |J| \equiv 1 \bmod 2} Z(J).$$

$\mathcal{Z}$ will be a singular locus of the volume function.

$\gamma : [0,1] \longrightarrow X_{n+1}(\mathbf{C})$ a smooth path such that $\gamma(0) = \mathbf{x}_0$, $\gamma(1) = A$ and $\gamma(t) \in X_{n+1}(\mathbf{C}) \setminus \mathcal{Z}$ for $t > 0$.
Fix $I_p, I_q \subset I$ with $I_p \subset I_q$.

Theorem The iterated integral

$$F = \sum_{I_p \subset I_{p+1} \subset \cdots \subset I_q} \int_\gamma \omega(I_{q-1}, I_q) \cdots \omega(I_p, I_{p+1}),$$

is invariant under the homotopy of a path γ fixing the endpoints $\mathbf{x}_0$ and A.

The integrability condition $dF = 0$ follows from:

Lemma For $(I_0, \cdots, I_m) \in \mathcal{F}_m[n]$ there is a relation

$$\sum_K \omega(I_k, K) \wedge \omega(K, I_{k+2}) = 0$$

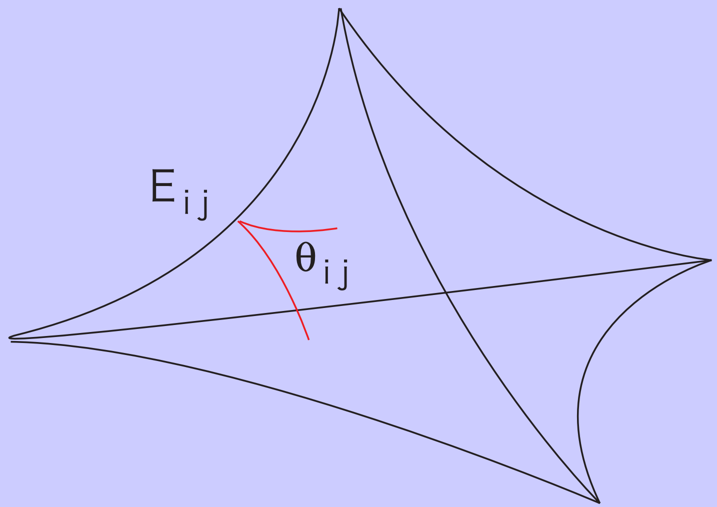
where the sum is taken for any K with $|K| = 2k + 2$ and $I_k \subset K \subset I_{k+2}$, $k = 0, 1, \cdots, m-2$.

Idea of proof of Lemma

Reduce to the case of 3-dimensional simplex.

The Schläfli's formula implies

$$\sum_{i<j} d\ell(E_{ij}) \wedge d\theta_{ij} = 0$$



Theorem The Schläfli function $S(A)$ is analytically continued to a multi-valued holomorphic function on $X_{n+1}(\mathbf{C}) \setminus \mathcal{Z}$.

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Corollary For $A \in C_n^-$ the volume of the hyperbolic simplex $\Delta(A) \subset \mathcal{H}^n$ is related to the analytic continuation of the Schläfli function by

$$i^n V(\Delta(A)) = S(A)$$

The volume corrected curvature $\kappa V(\Delta(A))^{\frac{2}{n}}$ for $A \in C_n$ is scaling invariant and is considered to be a function on C_n.

Proposition The volume corrected curvature $\kappa V(\Delta(A))^{\frac{2}{n}}$ is an analytic function on the set of Gram matrices $C_n = C_n^+ \cup C_n^0 \cup C_n^-$.

8 Nilpotent connections

X : complement of a hypersurface defined by f_j
E : trivial vector bundle over X with fiber V

$$\omega = \sum_j A_j \, d \log f_j$$

A_j : nilpotent endomorphism of V
Integrability condition

$$\omega \wedge \omega = 0$$

Theorem There is a one-to-one correspondence between unipotent representation of $\pi_1(X)$ and integrable nilpotent connection

$$\omega = \sum_j A_j \, d\log f_j.$$

Riemann Hilbert problem for unipotent monodromy (Aomoto, Hain)

Back to Schläfli function with $m = \left[\frac{n+1}{2}\right]$.
Define $f(I_{m-k}; z)$, $0 \leq k \leq m$ by

$$\sum \int_\gamma \omega(I_{m-1}, I_m) \cdots \omega(I_{m-k}, I_{m-k+1})$$

for $I_{m-k} \subset I_{m-k+1} \subset \cdots \subset I_m$,
where $I_p \subset I$ with $|I_p| = 2p$

γ : a path from a base point $\mathbf{x}_0 \in C_n^0$ to
$z \in X_{n+1}(\mathbf{C}) \setminus \mathcal{Z}$

W_k : the vector space over $\mathbf{C}$ spanned by $f(I_{m-k}; z)$ for all $I_{m-k} \subset I$

Put $W = W_0 \oplus W_1 \oplus \cdots \oplus W_m$.

Theorem Any $\varphi \in W$ satisfies the differential equation $d\varphi = \varphi\,\omega$ with

$$\omega = \sum_{I_{p-1} \subset I_p} A(I_{p-1}, I_p)\,\omega(I_{p-1}, I_p)$$

$A(I_{p-1}, I_p)$ is a nilpotent endomorphism of W.

Final remarks

(1) It is possible to analyze the asymptotic behavior of the volume function by means of the nilpotent connection in Theorem.

Hyperbolic volume function extends continuously on the boundary of C_n^-.
(conjectured by Milnor, due to Feng Luo ... by a different method)

(2) Study Gauss-Manin connection for

$$\int_{\Delta} e^{-\mu(x_1^2+\cdots+x_{n+1}^2)} f_1^{\lambda_1} \cdots f_m^{\lambda_m} dx_1 \cdots dx_{n+1}$$

where Δ is a chamber.

Monodromy in the case of discriminantal arrangement (fiber of $M_{n+\ell} \longrightarrow M_n$) with $\mu = 0$ corresponds to LKB representation of braid groups. Specialization $\lambda_j \longrightarrow 0$ gives information on volume functions.