Iterated integrals and fundamental groups of the complements of hyperplane arrangements

Toshitake Kohno

The University of Tokyo

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Notation

$\mathcal{A} = \{H_1, \cdots, H_m\}$: arrangement of affine hyperplanes in the complex vector space $\mathbf{C}^n$
Consider the complement

$$M(\mathcal{A}) = \mathbf{C}^n \setminus \bigcup_{H \in \mathcal{A}} H$$

Let f_j be a defining linear form for the hyperplane H_j, $1 \leq j \leq m$ and put

$$\omega_j = \frac{1}{2\pi\sqrt{-1}} d \log f_j$$

A : Orlik-Solomon algebra
$\overline{B}^*(A)$: reduced bar complex for the Orlik-Solomon algebra

Bar complex and fundamental group

Let J be the augmentation ideal of the group algebra of the fundamental group of $M = M(\mathcal{A})$.

Theorem

For the reduced bar complex for the Orlik-Solomon algebra the iterated integral map gives an isomorphism

$$\mathcal{F}^{-k} H^0(\overline{B}^*(A)) \cong \mathrm{Hom}(\mathbf{Z}\pi_1(M, \mathbf{x}_0)/J^{k+1}, \mathbf{C})$$

Here $\mathcal{F}^{-k}$ denoted the filtration on the 0-dimensional cohomology of the reduced bar complex induced by the length of the iterated integrals of logarithmic forms.

Holonomy Lie algebra

Define the holonomy Lie algebra as a quotient of free Lie algebra by

$$\mathfrak{h}(M) = \mathcal{L}(X_1, \cdots, X_m)/\mathfrak{a}$$

where $\mathfrak{a}$ is an ideal generate by

$$[X_{j_p}, X_{j_1} + \cdots + X_{j_k}], \quad 1 \leq p < k$$

for maximal family of hyperplanes $\{H_{j_1}, \cdots, H_{j_k}\}$ such that

$$\mathrm{codim}_{\mathbf{C}}(H_{j_1} \cap \cdots \cap H_{j_k}) = 2$$

Universal holonomy map

We put

$$\omega = \sum_{j=1}^{m} \omega_j X_j.$$

Then there is a universal holonomy map

$$\Theta_0 : \pi_1(M, \mathbf{x}_0) \longrightarrow \mathbf{C}\langle\langle X_1, \cdots, X_m \rangle\rangle / \mathfrak{a}$$

defined by

$$\Theta_0(\gamma) = 1 + \sum_{k=1}^{\infty} \int_\gamma \underbrace{\omega \cdots \omega}_{k}$$

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This induces an isomorphism

$$\mathbf{C}\widehat{\pi}_1(M, \mathbf{x}_0) \cong \mathbf{C}\langle\langle X_1, \cdots, X_m \rangle\rangle / \mathfrak{a}$$

By taking the primitive part we have an isomorphism between nilpotent completion of the fundamental group and the holonomy Lie algebra over $\mathbf{Q}$.

Lower central series and the holonomy Lie algebra

Let

$$\pi_1(M) = \Gamma_1 \supset \Gamma_2 \supset \cdots \subset \Gamma_k \supset \cdots$$

be the lower central series defined by

$$\Gamma_{k+1} = [\Gamma_1, \Gamma_k]$$

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Then there is an isomorphism of graded Lie algebra

$$\bigoplus_{k \geq 1} [\Gamma_k / \Gamma_{k+1}] \otimes \mathbf{Q} \cong \mathrm{gr}\, \mathfrak{h}(M) \otimes \mathbf{Q}$$

This can also be shown by Sullivan's theory of minimal models and the mixed Hodge structure on M for the complement of a hypersurface.

Because of the integrability of the connection

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we have the following linear representations of the fundamental group.

Univerlsal holonomy map and representation of π_1

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Theorem

For any representation of the holonomy Lie algebra $r : \mathfrak{h}(M) \to \mathrm{End}(V)$ there is a linear representation of the fundamental group

$$\pi_1(M, \mathbf{x}_0) \longrightarrow \mathrm{GL}(V)$$

obtained by substituting the representation r to the universal holonomy homomorphism.

Hyperlogarithms

We denote by $C^\infty(\widetilde{M})$ the space of smooth functions on the universal covering of M. Let $F_k(M)$ be the image of the iterated integral for paths wth fixed starting point gives a map

$$\mathcal{F}^{-k}H^0(\overline{B}^*(A)) \longrightarrow C^\infty(\widetilde{M}).$$

There is an increasing filtration of functions

$$\mathbf{C} = F_0(M) \subset F_1(M) \subset \cdots \subset F_k(M) \subset \cdots$$

called hyperlogarithms and we have

$$dF_{k+1}(M) \subset F_k(M) \otimes A^1, \quad k = 0, 1, 2, \cdots$$

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Theorem

If the fundamental group is residually torsion free nilpotent, then the hyperlogarithms separates points on the universal covering of the complement $M(\mathcal{A})$.

Homotopy invairiants

We have the following condition for an iterated integral of logarithmic forms to be homotopy invariant.

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Theorem

Linear combinations of iterated integrals of the logarithmic forms ω_j, $1 \leq j \leq m$ *on the complement of hyperplane arrangement*

$$\sum_{j_1 \cdots j_k} a_{j_1 \cdots j_k} \int_\gamma \omega_{j_1} \cdots \omega_{j_k}, \quad a_{j_1 \cdots j_k} \in \mathbf{C}$$

depends only on the homotopy class of based loop γ *if and only if the correspondence* $X_{j_1} \cdots X_{j_k} \mapsto a_{j_1 \cdots j_k}$ *defines a linear map* $U\mathfrak{h}(M) \otimes \mathbf{C} \longrightarrow \mathbf{C}$

Riemann-Hilbert problem for unipotent monodromy

The following statement was first obtained by Aomoto (see also the work of Hain).

Theorem

Let

$$\rho : \pi_1(M, \mathbf{x}_0) \longrightarrow \mathrm{GL}(V)$$

be a unipotent representation of the fundamental group of the complement of hyperplane arrangement. Then there exists an integrable connection

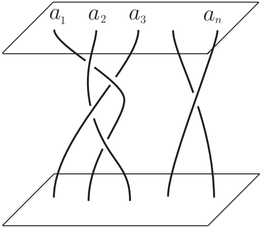
$$\omega = \sum_{j=1}^{\ell} A_j \omega_j, \quad A_j \in \mathrm{End}(V_j)$$

such that each A_j is nilpotent and the monodromy representation of ω coincides with ρ.

Configuration space of ordered distinct n points

$$\mathrm{Conf}_n(X) = \{(x_1, \cdots, x_n) \in X^n \ ; \ x_i \neq x_j \text{ if } i \neq j\}$$

$M_n = \mathrm{Conf}_n(\mathbf{C})$ is the complement of the braid arrangement. An element of $\Omega\mathrm{Conf}_n(\mathbf{C})$ is considered to be a pure braid.



In the case $m \geq 3$, there is an isomorphism

$$H^*(B\mathrm{Conf}_n(\mathbf{R}^m)) \cong H^*_{DR}(\Omega\mathrm{Conf}_n(\mathbf{R}^m))$$

Bar complex for configuration spaces

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In the case $m = 2$,

$$H^0(B\mathrm{Conf}_n(\mathbf{C}))$$

is isomorphic to the space of finite type invariants for P_n.

Acyclicity of the bar complex

The following vanishing holds for the cohomology of the bar complex of the complement of the braid arrangement $M_n = \mathrm{Conf}_n(\mathbf{C})$.

Theorem

We have

$$H^j(B^*(M_n)) \cong 0, \quad j \neq 0.$$

The acyclicity holds for fiber-type arrangements.

Singular braids

We discuss the application of our construction to finite type invariants for braids.

Consider singular braids with transverse double points.
We replace the double points $p_1, \cdots, p_k$ by positive or negative crossings according as $\epsilon_j = \pm 1$ and we denote by $\beta_{\epsilon_1 \cdots \epsilon_k}$ the obtained braid.

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For a function $v : B_n \longrightarrow \mathbf{K}$ we define its extension on singular braids with transverse double points by

$$\tilde{v}(\beta) = \sum_{\epsilon_j = \pm 1, 1 \leq j \leq k} \epsilon_1 \cdots \epsilon_k \; v(\beta_{\epsilon_1 \cdots \epsilon_k}).$$

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There is an increasing sequence of singular braids

$$B_n \subset S_1(B_n) \subset \cdots \subset S_k(B_n) \subset \cdots$$

where $S_k(B_n)$ is the set of singular n-braids with at most k transverse double points.

Finite type invariants for braids

v is of finite type of order k if and only if its extension $\tilde{v}$ vanishes on $S_m(B_n)$ for $m > k$.

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$V_k(B_n)_{\mathbf{K}}$: the space of order k invariants for B_n with values in $\mathbf{K}$ where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$.
There is an increasing sequence of vector spaces.

$$V_0(B_n)_{\mathbf{K}} \subset V_1(B_n)_{\mathbf{K}} \subset \cdots \subset V_k(B_n)_{\mathbf{K}} \subset \cdots$$

$V(B_n)_{\mathbf{K}} = \bigcup_{k \geq 0} V_k(B_n)_{\mathbf{K}}$: the space of finite type invariants for B_n with values in $\mathbf{K}$.
In a similar way we define $V(P_n)_{\mathbf{K}}$, the space of finite type invariants for the pure braid group P_n with values in $\mathbf{K}$.

Bar complex and finite type invariants

For the bar complex there is a filtration defined by

$$\mathcal{F}^{-k}B^*(\mathrm{Conf}_n(\mathbf{C})) = \bigoplus_{q \leq k} B^{-q,p}(\mathrm{Conf}_n(\mathbf{C})), \quad k = 0, 1, 2, \cdots$$

This induces a filtration $\mathcal{F}^{-k}H^0(B^*(\mathrm{Conf}_n(\mathbf{C})))$, $k \geq 0$, on the cohomology of the bar complex.
The above iterated integral defines a map

$$\mathcal{I} : H^0(B^*(\mathrm{Conf}_n(\mathbf{C}))) \longrightarrow \mathrm{Hom}(\mathbf{Z}P_n, \mathbf{K}).$$

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Theorem

The iterated integral map $\mathcal{I}$ gives the isomopshisms

$$\mathcal{F}^{-k}H^0(B^*(\mathrm{Conf}_n(\mathbf{C}))) \otimes \mathbf{K} \cong V_k(P_n)_{\mathbf{K}},$$
$$H^0(B^*(\mathrm{Conf}_n(\mathbf{C}))) \otimes \mathbf{K} \cong V(P_n)_{\mathbf{K}}.$$

Algebra of horizontal chord diagrams

$\mathcal{A}_n$: the algebra over $\mathbf{Z}$ generated by X_{ij}, $1 \leq i \neq j \leq n$, with the relations :

$$X_{ij} = X_{ji}$$
$$[X_{ik}, X_{ij} + X_{jk}] = 0 \quad i, j, k \text{ distinct,}$$
$$[X_{ij}, X_{k\ell}] = 0 \quad i, j, k, \ell \text{ distinct.}$$

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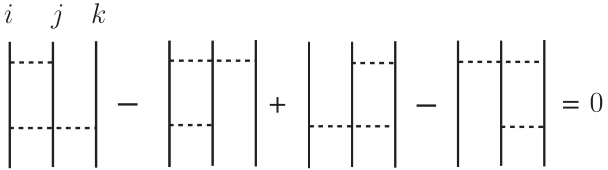
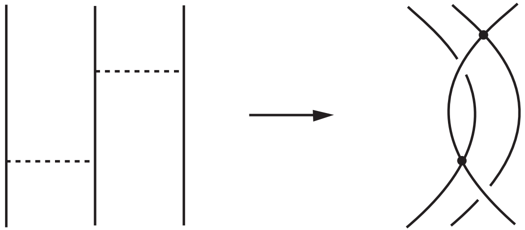
$\mathcal{A}_n$: graded algebra with $\deg X_{ij} = 1$.
$\mathcal{A}_{n,k}$: the degree k part of $\mathcal{A}_n$.



$X_{ik}X_{ij}$ $\quad X_{ij}X_{ik}$ $\quad X_{ik}X_{jk}$ $\quad X_{jk}X_{ik}$

Singular braids and horizontal chord diagrams

Horizontal chord diagrams are considered to be models for singular pure braids by contracting chords.



Extension by symmetric group

There is a direct sum decomposition $\mathcal{A}_n = \oplus_{k \geq 0} \mathcal{A}_{n,k}$. The algebra $\widehat{\mathcal{A}}_n$ is defined to be the direct product

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$\mathbf{Z}S_n$: the group algebra of the symmetric group S_n over $\mathbf{Z}$. We define the semi-direct product $\mathcal{A}_n \rtimes \mathbf{Z}S_n$ by the relation

$$X_{ij} \cdot \sigma = \sigma \cdot X_{\sigma(i)\sigma(j)}$$

for $\sigma \in S_n$.

The element R

The symbol $t_{ij} \in S_n$ stands for the permutation of i-th and j-th letters.

The element $R \in (\widehat{\mathcal{A}}_2 \otimes \mathbf{Q}) \rtimes \mathbf{Z}S_2$ is defined by

$$R = t_{12} \exp\left(\frac{1}{2} X_{12}\right).$$

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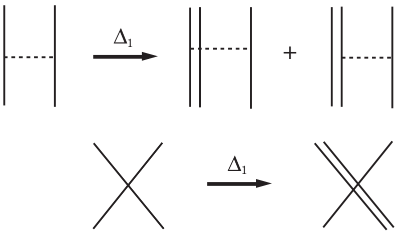
Doubling operation Δ :



Axiomatic definition of Drinfel'd associator

A Drinfel'd associator Φ is an element of $\widehat{\mathcal{A}}_3 \otimes \mathbf{C}$ satisfying the following properties.

- (strong invertibility)

$$\varepsilon_1(\Phi) = \varepsilon_2(\Phi) = \varepsilon_3(\Phi) = 1$$

- (skew symmetry)

$$\Phi^{-1} = t_{13} \cdot \Phi \cdot t_{13}$$

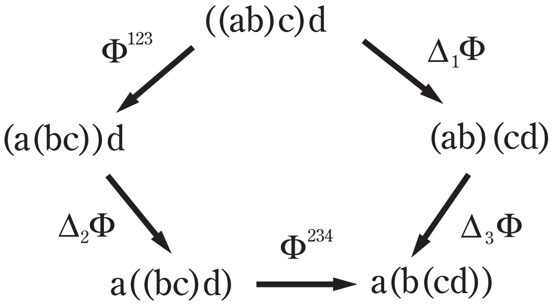
- (pentagon relation)

$$(\Phi \otimes id) \cdot (\Delta_2 \Phi) \cdot (id \otimes \Phi) = (\Delta_1 \Phi) \cdot (\Delta_3 \Phi) \quad \text{in} \quad \widehat{\mathcal{A}}_4 \otimes \mathbf{C}.$$

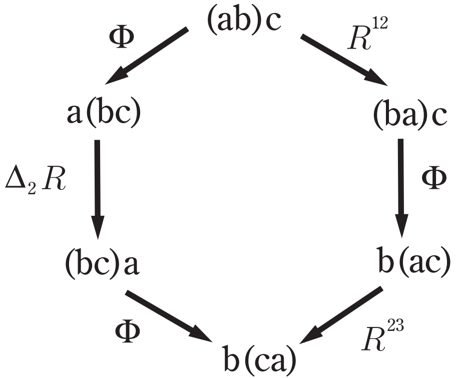
- (hexagon relation)

$$\Phi \cdot (\Delta_2 R) \cdot \Phi = (R \otimes id) \cdot \Phi \cdot (id \otimes R)$$

Pentagon relation



Hexagon relation



Drinfel'd associator and KZ equation

The original Drinfel'd associator was introduced for the purpose of describing the monodromy representation of the KZ equation. It is an element in the ring of non-commutative formal power series $\mathbf{C}[[X, Y]]$ describing a relation of the solutions $G_0(z)$ and $G_1(z)$ of the differential equation

$$G'(z) = \frac{1}{2\pi\sqrt{-1}} \left(\frac{X}{z} + \frac{Y}{z-1} \right) G(z) \tag{0.1}$$

with the asymptotic behavior

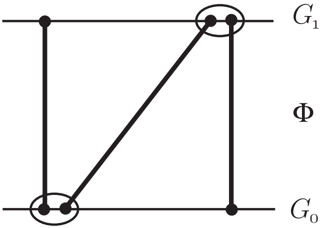
$$G_0(z) \sim z^{\frac{1}{2\pi\sqrt{-1}} X}, \quad z \longrightarrow 0$$
$$G_1(z) \sim (1-z)^{\frac{1}{2\pi\sqrt{-1}} Y}, \quad z \longrightarrow 1.$$

We set

$$G_0(z) = G_1(z) \Phi_{KZ}(X, Y)$$

Graphical representation of Drinfel'd associator

$\Phi_{KZ}(X_{12}, X_{23})$ satisfies the axiomatic properties for an associator.



Drinfel'd associator over $\mathbf{Q}$

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An explicit rational associator up to degree 4 terms is of the form

$$\Phi(X,Y) = 1 - \frac{\zeta(2)}{(2\pi i)^2}[X,Y] - \frac{\zeta(4)}{(2\pi i)^4}[X,[X,[X,Y]]] - \frac{\zeta(4)}{(2\pi i)^4}[Y,[Y,[X,Y]]] - \frac{\zeta(3,1)}{(2\pi i)^4}[X,[Y,[X,Y]]] + \frac{1}{2}\frac{\zeta(2)^2}{(2\pi i)^4}[X,Y]^2 + \cdots$$

with $X = X_{12}$, $Y = X_{23}$, where $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$ and $\zeta(3,1) = \pi^4/360$.

Universal holonomy homomorphism of braids over $\mathbf{Q}$

Set $R_{j,j+1} = t_{j,j+1} \exp\left(\frac{1}{2} X_{j,j+1}\right)$.

Theorem

For the generators σ_j, $1 \leq j \leq n-1$, of the braid group B_n we put

$$\Theta(\sigma_j) = \Phi_j \cdot R_{j,j+1} \cdot \Phi_j^{-1}, \quad 1 \leq j \leq n-1.$$

Here Φ_j is defined by means of a rational Drinfel'd associator by the formulae

$$\Phi_j = \Phi\left(\sum_{i=1}^{j-1} X_{ij},\ X_{j,j+1}\right), \quad j > 1$$

and $\Phi_1 = 1$. Then, Θ defines an injective homomorphism

$$\Theta : B_n \longrightarrow (\widehat{\mathcal{A}}_n \otimes \mathbf{Q}) \rtimes \mathbf{Z} S_n.$$

Theorem

We have the following multiplicative isomorpshisms for finite type invariants over the field of rational numbers.

$$V(P_n)_{\mathbf{Q}} \cong \mathrm{Hom}(\mathcal{A}_n, \mathbf{Q}),$$
$$V(B_n)_{\mathbf{Q}} \cong \mathrm{Hom}(\mathcal{A}_n \rtimes \mathbf{Z}S_n, \mathbf{Q}).$$

These finite type invariants are complete in the sense that they separate braids.

Generators for $H_*(\Omega \mathrm{Conf}_n(\mathbf{R}^m)), m \geq 3$

Description of the homology of the loop spaces of configuration space in the case $m \geq 3$

$$\gamma_{ij} : S^{m-1} \to \mathrm{Conf}_n(\mathbf{R}^m),\ 1 \leq i < j \leq n,$$

$\gamma_{ij}(S^{m-1})$ has the linking number 1 with the diagonal set Δ_{ij} defined by $x_i = x_j$.

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S^{m-1} is considered to be the suspension of its equator, i.e., $S^{m-1} = \Sigma S^{m-2}$.

$$\alpha_{ij} : S^{m-2} \to \Omega \mathrm{Conf}_n(\mathbf{R}^m),\ 1 \leq i < j \leq n.$$

$X_{ij} = [\alpha_{ij}]$ generate $H_*(\Omega \mathrm{Conf}_n(\mathbf{R}^m))$.

Relations among X_{ij}

There is a relation:

$$[X_{ij}, X_{ik} + X_{jk}] = 0$$

for $i < j < k$. Here $\deg X_{ij} = m - 2$ and we consider the Lie bracket in a graded sense.

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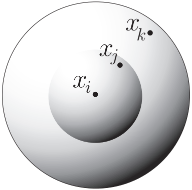
Why such relations among X_{ij}?



$$\varphi : S^{m-1} \times S^{m-1} \to \mathrm{Conf}_n(\mathbf{R}^m)$$
$$A \mapsto X_{ij}, \quad B \mapsto X_{ik} + X_{jk}$$
$[A, B] = 0$ holds in $H_*(\Omega(S^{m-1} \times S^{m-1}))$.

Theorem (Cohen and Gitler)

$H_*(\Omega \mathrm{Conf}_n(\mathbf{R}^m)), m \geq 3$ is isomorphic to the enveloping algebra of the graded Lie algebra generated by X_{ij}, $i \neq j$ with relations:

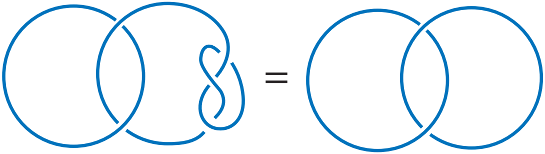
$$[X_{ij}, X_{kl}], \quad i, j, k, l \text{ distinct}$$

$$[X_{ij}, X_{ik} + X_{jk}], \quad i < j < k$$

(infinitesimal pure braid relations)

Here we set $X_{ij} = (-1)^{m-2} X_{ji}$ for $i > j$.

Link homotopy is a link theory “modulo knot theory”.



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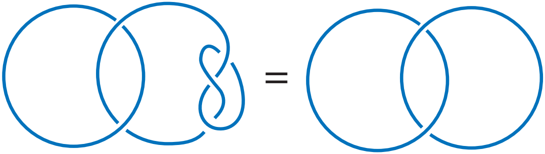

The homology of the loop space of configuration space is a “universal” source for constructing link homotopy invariants.

Construction of link homotopy invariants

Let L be an n-component link. There is an induced map

$$f : T^n \to \mathrm{Conf}_n(\mathbf{R}^3).$$

Put $\overline{T}^n = T^n/(1\text{-skeleton})$

$$\Omega\overline{f} : \Omega\overline{T}^n \to \Omega\mathrm{Conf}_n(\mathbf{R}^3)$$

Theorem

Let $\sigma \in H_{n-1}(\Omega\overline{T}^n)$ be an indecomposable element. Then

$$\nu : H^{n-1}(\Omega\mathrm{Conf}_n(\mathbf{R}^3)) \to \mathbf{Z}$$

defined by

$$\nu(\omega) = \langle \Omega\overline{f}^* \omega, \sigma \rangle$$

is a well-defined link homotopy invariant modulo values on decomposable elements in $H_{n-1}(\Omega\overline{T}^n)$.