Homotopy types of the complements of hyperplane arrangements, local system homology and iterated integrals

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August 2009

Plan

- **Part 1** : Homotopy types of the complements of hyperplane arrangements, Local system homology and Morse theory

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- **Part 2** : Iterated integrals of logarithmic forms and hyperplane arrangements

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- **Part 3** : Volumes of hyperbolic simplices, Functions on the moduli space of arrangements

Introduction

$\mathcal{A} = \{H_1, \cdots, H_m\}$: arrangement of affine hyperplanes in the complex vector space $\mathbf{C}^n$

Consider the complement

$$M(\mathcal{A}) = \mathbf{C}^n \setminus \bigcup_{H \in \mathcal{A}} H$$

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Some topological properties of $M(\mathcal{A})$ are determined combinatorially by the intersection lattice $L(\mathcal{A})$

- Betti numbers
- Cohomology ring (isomorphic to Orlik-Solomon algebra)

Homotopy type of $M(\mathcal{A})$ is not determined by $L(\mathcal{A})$.

Real arrangements and complexification

Let $V_{\mathbf{R}} = \mathbf{R}^n$ be a real vector space.
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Real arrangements and complexification

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$V_{\mathbf{R}} \setminus \bigcup_{H \in \mathcal{A}}$ consists of finitely many connected components called chambers.
Consider the complement of the complexified real arrangement

$$M(\mathcal{A}) = \mathbf{C}^n \setminus \bigcup_{H \in \mathcal{A}} H \otimes \mathbf{C}$$

Some classical formulae

$ch(A)$: the set of chambers
$bch(A)$: the set of bounded chambers

[Zaslavsky]

$$|ch(A)| = \sum_{i=0}^{n} b_i(M(\mathcal{A}))$$

$$|bch(A)| = |\sum_{i=0}^{n} (-1)^i b_i(M(\mathcal{A}))|$$

Salvetti complex

The real hyperplane arrangement $\mathcal{A}$ determines a stratification S (facet decomposition) of $\mathbf{R}^n$.

$$F_1 > F_2 \Longleftrightarrow \overline{F_1} \supset F_2$$

For a flag $F = (F_{j_0} < \cdots < F_{j_p})$ one associated a dual simplex $\sigma(F)$.

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For a facet F the dual cell is defined by

$$D(F) = \bigcup \sigma(F^i < F^{i-1} < \cdots < F^0)$$

with $F^i = F$ and $\operatorname{codim} F^j = j$.

Salvetti complex (continued)

Let $\pi : M(\mathcal{A}) \longrightarrow \mathbf{R}^n$ be the projection corresponding the the real part.
A facet decomposition of $M(\mathcal{A})$ is given by

$$\bigcup_F \pi^{-1}(F)$$

The associated dual complex is called the Salvetti complex $S(\mathcal{A})$, which is an n dimensional CW complex.

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Theorem (Salvetti)

The inclusion

$$S(\mathcal{A}) \longrightarrow M(\mathcal{A})$$

is a homotopy equivalence.

Related problems

- Cohomology of Artin groups with coefficients in local systems (De Concini, Procesi, Salvetti)
- $K(\pi, 1)$ problem for $M(\mathcal{A})$: When is the universal covering of $S(\mathcal{A})$ contractible?

– Deligne : Simplicial arrangement is $K(\pi, 1)$. This class contains complexified Coxeter arrangements (see also Brieskorn-Saito)

– Falk's $K(\pi, 1)$ test
– Description of the universal covering of $S(\mathcal{A})$ (Paris)
– Remarkable progress in unitary reflection arrangements (Bessis)
– Discrete Morse theory

Local system homology

$\mathcal{L}$: rank 1 local system over $M(\mathcal{A})$.
Study the local system homology $H_*(M(\mathcal{A}), \mathcal{L})$.
Compare with $H_*^{lf}(M(\mathcal{A}), \mathcal{L})$, homology with locally finite (possibly infinite) chains.

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associating the imaginary part provides a complex to compute the homology $H_*^{lf}(M(\mathcal{A}), \mathcal{L})$.
The image of the natural map

$$H_*(M(\mathcal{A}), \mathcal{L}) \longrightarrow H_*^{lf}(M(\mathcal{A}), \mathcal{L})$$

is generated by bounded chambers.

Vanishing theorem

For a complex arrangement $\mathcal{A}$ choose a smooth compactification $i : M(\mathcal{A}) \longrightarrow X$ with normal crossing divisors.

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Assume that the local system $\mathcal{L}$ is generic in the following sense:
There is an isomorphism

$$i_* \mathcal{L} \cong i_! \mathcal{L}$$

where i_* is the direct image and $i_!$ is the extension by 0. This means that the monodromy of $\mathcal{L}$ along any divisor at infinity is not equal to 1.

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Theorem

If the local system is generic in the above sense, then there is an isomorphism

$$H_*(M(\mathcal{A}), \mathcal{L}) \cong H_*^{lf}(M(\mathcal{A}), \mathcal{L})$$

We have $H_j(M(\mathcal{A}), \mathcal{L}) = 0$ *for any* $j \neq n$.

Vanishing theorem (continued)

In case of complexified real arrangement the above $H_*^{lf}(M(\mathcal{A}), \mathcal{L})$ is spanned by bounded chambers.
For the proof of vanishing theorem we use the following: In general we have

$$H^*(X, i_*\mathcal{L}) \cong H^*(M(\mathcal{A}), \mathcal{L}), \quad H^*(X, i_!\mathcal{L}) \cong H_c^*(M(\mathcal{A}), \mathcal{L})$$

where H_c denote the cohomology with compact supports.
There is a Poincaré duality isomorphism:

$$H_k^{lf}(M(\mathcal{A}), \mathcal{L}) \cong H^{2n-k}((M(\mathcal{A}), \mathcal{L})$$
$$H_k(M(\mathcal{A}), \mathcal{L}) \cong H_c^{2n-k}((M(\mathcal{A}), \mathcal{L}).$$

Related work on cohomology

A cohomological counterpart for the vanishing theorem for Orlik-Solomon algebra and Aomoto complex has been studied by:

Esnault-Viehweg, Aomoto, Orlik, Kita,
Schechtman-Terao-Varchenko, Yuzvinsky, D. Cohen, Suciu ....

There are also works on characteristic and resonance varieties.

The case of discriminantal arrangement was studied systematically by Varchenko and Schechtman in relation with the solution of KZ equation by hypergeometric integrals. There is "resonance at infinity" in the case of conformal field theory.

Minimality

Theorem (Dimca-Papadima, Randell)

Let $\mathcal{A}$ be an arrangement of affine hyperplanes in $\mathbf{C}^n$. Then the complement $M(\mathcal{A})$ is homotopy equivalent to an n-dimensional minimal CW complex, i.e., the number of k-dimensional cells equals to $b_k(M(\mathcal{A}))$ for any $k \geq 0$.

Minimality (continued)

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As a corollary of minimality we have

$$\dim H_k(M(\mathcal{A}), \mathcal{L})) \leq b_k(M(\mathcal{A}))$$

which was shown by D. Cohen by a different method.

Iterated integrals - an overview

In this part we describe the application of the theory of iterated integrals to hyperplane arrangements.
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- Iterated integrals of 1-forms provide "non-commutative" information on the fundamental group.

Remark: Kontsevich integral can be considered as a generalization of iterated integrals on braids for the case of knots.

Iterated integals of 1-forms

$\omega_1, \cdots, \omega_k$: differential 1-forms on a smooth manifold M
$\gamma : [0,1] \to M$ a smooth path
pull-back $\gamma^* \omega_i = f_i(t) dt$, $1 \le i \le k$

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The iterated integral of the 1-forms $\int_\gamma \omega_1 \omega_2 \cdots \omega_k$ is defined as

$$\int_{0 \leq t_1 \leq \cdots \leq t_k \leq 1} f_1(t_1) f_2(t_2) \cdots f_k(t_k) \; dt_1 dt_2 \cdots dt_k.$$

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example. Dilogarithm is expressed as the iterated integral of $\omega_1 = \frac{dz}{1-z}$ and $\omega_0 = \frac{dz}{z}$ as

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-z)}{z}\ dz.$$

Definition of iterated integals of differential forms

$\omega_1, \cdots, \omega_k$: differential forms on M
ΩM : loop space M

$$\Delta_k = \{(t_1, \cdots, t_k) \in \mathbf{R}^k \ ; \ 0 \leq t_1 \leq \cdots \leq t_k \leq 1\}$$

$$\varphi : \Delta_k \times \Omega M \to \underbrace{M \times \cdots \times M}_{k}$$

defined by $\varphi(t_1, \cdots, t_k; \gamma) = (\gamma(t_1), \cdots, \gamma(t_k))$

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The iterated integral of $\omega_1, \cdots, \omega_k$ is defined as

$$\int \omega_1 \cdots \omega_k = \int_{\Delta_k} \varphi^*(\omega_1 \times \cdots \times \omega_k)$$

The expression

$$\int_{\Delta_k} \varphi^*(\omega_1 \times \cdots \times \omega_k)$$

is the integration along fiber with respect to the projection $p : \Delta_k \times \Omega M \to \Omega M$.

differential form on the loop space ΩM
with degree $p_1 + \cdots + p_k - k$.

Differentiation on loop spaces

As a differential form on the loop space $d\int \omega_1 \cdots \omega_k$ is

$$\sum_{j=1}^{k} (-1)^{\nu_{j-1}+1} \int \omega_1 \cdots \omega_{j-1} d\omega_j \ \omega_{j+1} \cdots \omega_k$$

$$+ \sum_{j=1}^{k-1} (-1)^{\nu_j+1} \int \omega_1 \cdots \omega_{j-1} (\omega_j \wedge \omega_{j+1}) \omega_{j+2} \cdots \omega_k$$

where $\nu_j = \deg \omega_1 + \cdots + \deg \omega_j - j$.

Bar complex

$\mathcal{E}^{*-1}(M)$: the differential graded algebra whose degree j part is

$$\mathcal{E}^{j+1}(M), \quad j > 0$$

$$\mathcal{E}^1(M)/d\mathcal{E}^0(M), \quad j = 0$$

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The tensor algebra $T\mathcal{E}^{*-1}(M)$ is equipped with the structure of a graded algebra. We set

$$B^{-k,p}(M) = \left[\bigotimes^k \mathcal{E}^{*-1}(M) \right]^{p-k}$$

where the right hand side stands for the degree $p - k$ part.

The structure of a double complex

For a differential q form φ we set $J\varphi = (-1)^q\varphi$. The differential $d' : B^{-k,p}(M) \longrightarrow B^{-k,p+1}(M)$ is defined by

$$d'(\varphi_1 \otimes \cdots \otimes \varphi_k) = \sum_{i=1}^{k} (-1)^i \, J\varphi_1 \otimes \cdots \otimes J\varphi_{i-1} \otimes d\varphi_i \otimes \varphi_{i+1} \otimes \cdots \otimes \varphi_k.$$

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The differential $d'' : B^{-k,p}(M) \longrightarrow B^{-k+1,p}(M)$ is defined by

$$\begin{aligned} &d''(\varphi_1 \otimes \cdots \otimes \varphi_k) \\ &\quad = \sum_{i=1}^{k} (-1)^{i-1} \, J\varphi_1 \otimes \cdots \otimes [(J\varphi_i) \wedge \varphi_{i+1}] \otimes \cdots \otimes \varphi_k. \end{aligned}$$

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$\oplus_{k,p} B^{-k,p}(M)$ has a structure of a double complex.
The associated total complex is denoted by $B^*(M)$ and is called the bar complex of the de Rham complex of M.

Chen's main theorem

There is a cochain map from the bar complex to the de Rham complex of the loop space

$$I : B^*(M) \longrightarrow \mathcal{E}^*(\Omega M)$$

given by the iterated integral

$$I(\omega_1 \otimes \cdots \otimes \omega_k) = \int \omega_1 \cdots \omega_k$$

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Theorem (K. T. Chen)

If M is simply connected, then the above map I induces an isomorphism

$$H^*(B^*(M)) \cong H^*(\Omega M).$$

Bar complex for Orlik-Solomon algebra

Let A be the Orlik-Solomon algebra.
Define the reduced complex by

$$\overline{A}^q = \begin{cases} 0, & q < 0 \\ A^{q+1}, & q \geq 0 \end{cases}$$

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Define the reduced complex by

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The reduced bar complex of the Orlik-Solomon algebra is the tensor algebra defined by

$$\overline{B}^*(A) = \bigoplus_{k \geq 0} \left(\bigotimes^k \overline{A} \right)$$

Bar complex for Orlik-Solomon algebra (continued)

The coboundary operator for the bar complex is

$$d(\varphi_1 \otimes \cdots \otimes \varphi_k) = \sum_{j=1}^{k-1} (-1)^{\nu_j + 1} \varphi_1 \otimes \cdots \otimes (\varphi_j \wedge \varphi_{j+1}) \otimes \cdots \otimes \varphi_k$$

with $\varphi_j \in \overline{A}^{q_j}$ and $\nu_j = q_1 + \cdots + q_j$.

Bar complex for Orlik-Solomon algebra (continued)

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with $\varphi_j \in \overline{A}^{q_j}$ and $\nu_j = q_1 + \cdots + q_j$.
There is a natural filtration defined by

$$\mathcal{F}^{-k}(\overline{B}^*(A)) = \bigoplus_{\ell \le k} (\bigotimes^{\ell} \overline{A})$$

Comparison theorem

Theorem

For the complement of hyperplane arrangement the integration map $\mathcal{I} : \overline{B}^*(A) \longrightarrow B^*(M(\mathcal{A}))$ *induces an isomorphism*

$$H^*(\overline{B}^*(A)) \cong H^*(B^*(M(\mathcal{A})))$$

Bar complex and fundamental group

Theorem

For the reduced bar complex for the Orlik-Solomon algebra there is an isomorphism

$$\mathcal{F}^{-k} H^0(\overline{B}^*(A)) \cong \mathrm{Hom}(\mathbf{Z}\pi_1(M, \mathbf{x}_0)/J^{k+1}, \mathbf{C})$$