Freeness of Arrangements and Arrangement Bundles

Amanda Hager

Department of Mathematics
University of Iowa

August 9, 2009

Introduction

- Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a hyperplane arrangement in $\mathbb{C}^\ell$, with defining polynomial $Q = Q(\mathcal{A})$.
- Let $M = M(\mathcal{A}) := \mathbb{C}^\ell - \cup H_i$.
- Let $p : \mathbb{C}^\ell \to \mathbb{C}^k$ be a forgetful map, the restriction of which to M is a fiber bundle projection where the base space and the fiber are arrangement complements.
- Question: If we have information about the base and the fiber, what is known about the total space?

We already know some things. The homology groups and the cohomology ring factor. We also have a relationship between the fundamental groups. We know that the Poincaré polynomial factors.

What about $D(\mathcal{A})$?

Module of Derivations

- Let $S = S(\mathbb{C}^{\ell *})$ be the symmetric algebra on $\mathbb{C}^{\ell *} \cong \mathbb{C}[x_1, \ldots, x_\ell]$.
- Let $Der_{\mathbb{C}} S := \{\theta : S \to S \mid \theta \ \mathbb{C}\text{-linear, satisfies Leibniz rule}\}$.
- Let $D(\mathcal{A}) := \{\theta \in Der_{\mathbb{C}} S \mid \theta(Q) \in QS\}$ be the *module of $\mathcal{A}$-derivations*.
- $\mathcal{A}$ is *free* if $D(\mathcal{A})$ is a free S-module.

We may write $\theta = f_1 \frac{\partial}{\partial x_1} + \cdots + f_\ell \frac{\partial}{\partial x_\ell}$, where $f_1, \ldots, f_\ell \in S$. Given a collection $\theta_1, \ldots, \theta_m \in D(\mathcal{A})$, where $\theta_j = \sum_i f_{ij} \frac{\partial}{\partial x_i}$, we can always form the coefficient matrix:

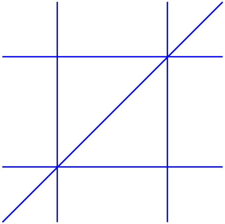
$$\begin{bmatrix} f_{11} & \ldots & f_{1m} \\ \vdots & \ddots & \vdots \\ f_{\ell 1} & \ldots & f_{\ell m} \end{bmatrix}$$

Recall *Saito's Criterion*: If $\{\theta_1, \ldots, \theta_\ell\} \in D(\mathcal{A})$ and the determinant of the above matrix is Q (up to multiplication by nonzero constant), then we have a basis for $D(\mathcal{A})$.

Example (Braid Arrangement A_3)

$$Q = (x - y)(x - z)(x - w)(y - z)(y - w)(z - w)$$

The map $p(x, y, z, w) = (x, y)$ results in a fiber bundle. The base arrangement is the diagonal in $\mathbb{C}^2$. The fiber is a discriminantal arrangement:



$$p^{-1}(x_0, y_0) = \{(x_0, y_0, z, w) \mid z \neq x_0, y_0; w \neq x_0, y_0; w \neq z\}$$

Example (Braid Arrangement A_3)

$$Q = (x-y)(x-z)(x-w)(y-z)(y-w)(z-w)$$

One basis for $D(A_3)$ is given by:

$$\begin{bmatrix} 1 & x & x^2 & x^3 \\ 1 & y & y^2 & y^3 \\ 1 & z & z^2 & z^3 \\ 1 & w & w^2 & w^3 \end{bmatrix}$$

Another basis is given by:

$$\left[\begin{array}{cc:cc} 1 & 0 & 0 & 0 \\ 1 & x-y & 0 & 0 \\ \hdashline 1 & x-z & (x-z)(y-z) & 0 \\ 1 & x-w & (x-w)(y-w) & (x-w)(y-w)(z-w) \end{array}\right]$$

$L(\mathcal{A})$ and Modularity

Let $L = L(\mathcal{A})$ be the set of nonempty intersections of hyperplanes of $\mathcal{A}$, partially ordered by reverse inclusion.
If $\mathcal{A}$ is central, then $L(\mathcal{A})$ is a geometric lattice with two operations, *meet* and *join*:

$$X \wedge Y = \cap\{H \in \mathcal{A} \,|\, X \cup Y \subset H\}$$
$$X \vee Y = X \cap Y$$

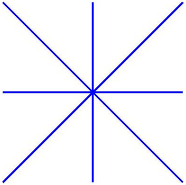
If $X, Y \in L$, then $X + Y \subset X \wedge Y$.
If $X \in L$ and $\forall Y \in L$ we have $X + Y = X \wedge Y$, then X is *modular*.

If an element of L is modular, we may project onto its orthogonal complement to obtain a fiber bundle.

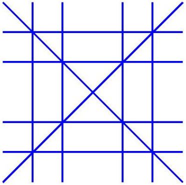
Example (in $\mathbb{C}^4$)

$$Q = xy(x^2 - y^2)(x^2 - z^2)(x^2 - w^2)(y^2 - z^2)(y^2 - w^2)(z^2 - w^2)$$

Let $p : \mathbb{C}^4 \to \mathbb{C}^2$ be defined by $p(x, y, z, w) = (x, y)$. Then p is a fiber bundle projection, and the base and fiber arrangements are as follows:



Base Arrangement



Fiber Arrangement

Example (in $\mathbb{C}^4$)

Basis for $D(B)$:

$$\begin{bmatrix} x & 0 \\ y & y(x^2-y^2) \end{bmatrix}$$

Basis for $D(F)$:

$$\begin{bmatrix} z(x_0^2-z^2)(y_0^2-z^2) & w(x_0^2-z^2)(y_0^2-z^2) \\ w(x_0^2-w^2)(y_0^2-w^2) & z(x_0^2-w^2)(y_0^2-w^2) \end{bmatrix}$$

Basis for $D(\mathcal{A})$:

$$\left[\begin{array}{cc|cc} x & 0 & 0 & 0 \\ y & y(x^2-y^2) & 0 & 0 \\ \hline z & z(x^2-z^2) & z(x^2-z^2)(y^2-z^2) & w(x^2-z^2)(y^2-z^2) \\ w & w(x^2-w^2) & w(x^2-w^2)(y^2-w^2) & z(x^2-w^2)(y^2-w^2) \end{array}\right]$$

Example (in $\mathbb{C}^4$)

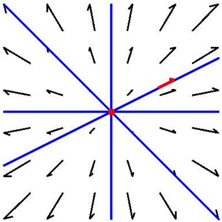
$$Q = xy(x^2 - y^2)(x^2 - z^2)(x^2 - w^2)(y^2 - z^2)(y^2 - w^2)(z^2 - w^2)$$

What makes this example interesting?

- The projection corresponds to the intersection of the first four hyperplanes.
- This lattice element is modular of rank 2, or codimension 2.
- There are no other rank 2 modular elements.
- There are no rank 3 modular elements.
- The fiber and base are *inductively free*, which is stronger than free.

Concluding Remarks/Questions

We may view $\theta \in D(\mathcal{A})$ as a vector field tangent to the arrangement. If $X \in L(\mathcal{A})$ and $v \in M(\mathcal{A}^X)$ (the complement of the restriction), then $\theta_v \in T_v M(\mathcal{A}^X)$.



$$Q = xy(x+y)(x-2y)$$
$$\theta = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$
$$\theta_{(2,1)} = 2\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$
$$\theta_{(0,0)} = 0$$

Concluding Remarks/Questions

- Does base free, fiber free $\Rightarrow$ total space free?
- Does base free, fiber inductively free $\Rightarrow$ total space free/inductively free?
- Even if $\mathcal{A}$ is not free, does the presence of modular elements help us to decompose $D(\mathcal{A})$ in a manageable way?
- Given the interpretation of derivations as vector fields, what geometric or topological information can we get from $D(\mathcal{A})$?

Thank you!