Critical sets of master functions: resonance and logarithmic derivations

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Western

Credits:

- D. Cohen, G. D., M. Falk, A. Varchenko, *Critical points and resonance of hyperplane arrangements*, `arXiv:0907.0896v1`
- American Institute of Mathematics SQuaREs program: H. Schenck, M. Schultze, M. Wakefield, U. Walther.

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Outline

- History: master functions, zeroes of one-forms
- Geometry associated with critical sets:
  - logarithmic differential forms
  - another characterization of free arrangements
- Critical points and resonance
- The codimension of the critical set is not “combinatorial.”

History

Definition (Master functions)

Let $R = \mathbb{C}[x_1, \ldots, x_\ell]$, and $f_1, \ldots, f_n \in R$ be linear forms. Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$. The *master function* $\Phi_\lambda = \Phi_\lambda(f_1, \ldots, f_n)$ is defined by:

$$\Phi_\lambda = \prod_{i=1}^{n} f_i^{\lambda_i}.$$

Question: describe solutions to $\nabla \Phi_\lambda = 0$. Some history:

- hypergeometric functions: Aomoto, Kita; Orlik, Terao; Mukhin, Scherbak, Varchenko; Damon; Silvotti.
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Notation (Hyperplane arrangement)

$\mathcal{A}$ will always be an essential, central arrangement of n hyperplanes in ℓ-dimensional affine space. Let

$$M = \mathbb{C}^\ell - \bigcup_{i=1}^{n} H_i,$$
$$\mathbb{P}M = \mathbb{P}^{\ell-1} - \bigcup_i \overline{H}_i.$$

Then, for a point $\mathbf{x} \in \mathbb{C}^\ell$,

$$\nabla \Phi_\lambda(\mathbf{x}) = 0 \quad \Leftrightarrow \quad \nabla \log \Phi_\lambda(\mathbf{x}) = 0$$
$$\Leftrightarrow \quad \sum_{i=1}^{n} \lambda_i \frac{\mathrm{d}f_i}{f_i}\Big|_{\mathbf{x}} = 0.$$

Moral: the question amounts to characterizing the zero locus of a 1-form.

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Example (from $\mathfrak{sl}_2$ KZ-equations)

Consider the master function

$$\Phi_{\ell,n} = \prod_{i=1}^{\ell}\prod_{j=1}^{n}(x_i - z_j x_0)^{m_j/\kappa} \prod_{1\le p<q\le \ell}(x_p - x_q)^{2/\kappa},$$

where $\kappa \in \mathbb{C}^*$, $m_1, \ldots, m_n \in \mathbb{Z}_{\geq 0}$, and $z_1, \ldots, z_n \in \mathbb{C}$.
Varchenko: isolated critical points $\rightsquigarrow$ commuting Hamiltonians in Gaudin model for Bethe Ansatz.

The smallest example

Let $\omega_\lambda := \mathrm{d}\log\Phi_\lambda$.

Example

Consider the master function

$$\Phi_\lambda = x^{\lambda_1} y^{\lambda_2} (x - y)^{\lambda_3}.$$

Then

$$\omega_\lambda = \left(\frac{\lambda_1}{x} + \frac{\lambda_3}{x - y}\right)\mathrm{d}x + \left(\frac{\lambda_2}{y} - \frac{\lambda_3}{x - y}\right)\mathrm{d}y.$$

So $\omega_\lambda = 0$ when

$$\frac{\lambda_1}{x} + \frac{\lambda_3}{x - y} = 0,$$
$$\frac{\lambda_2}{y} - \frac{\lambda_3}{x - y} = 0,$$

equivalent (on M) to $\lambda_1 + \lambda_2 + \lambda_3 = 0$, $\lambda_1 y + \lambda_2 x = 0$.

Example ($x^{\lambda_1} y^{\lambda_2} (x-y)^{\lambda_3}$, continued)

So if $\lambda_1 + \lambda_2 + \lambda_3 = 0$, there is a single critical point at $[\lambda_2 : -\lambda_1] \in \mathbb{P}^1$.

- Let $\Sigma_\lambda = \Sigma_\lambda(\mathcal{A}) = \{\mathbf{x} \in M(\mathcal{A}) \colon \omega_\lambda(\mathbf{x}) = 0\}$ denote the critical set for a choice of weights λ.
- Equations are homogeneous: let $\mathbb{P}\Sigma_\lambda$ be the quotient in $\mathbb{P}^{\ell-1}$.

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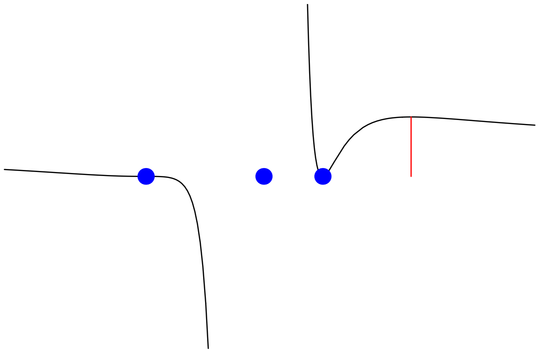

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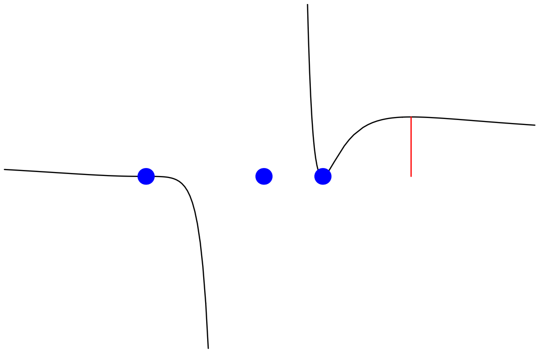

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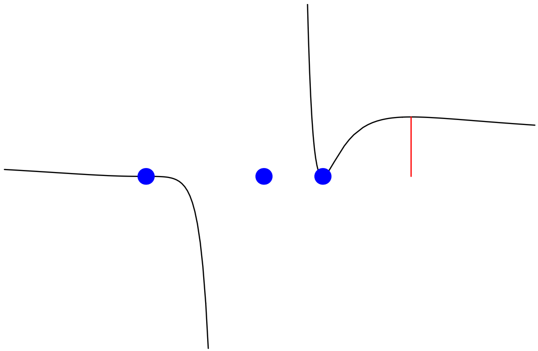

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The (projective) critical set is usually isolated

Theorem (Varchenko/$\mathbb{R}$; Orlik-Terao/$\mathbb{C}$ (1995))

For any arrangement $\mathcal{A}$, there is an open set $Y \subseteq \mathbb{C}^n$ such that: for all $\lambda \in Y$, $\mathbb{P}\Sigma_\lambda$ is isolated and nondegenerate. Moreover,

$$|\mathbb{P}\Sigma_\lambda| = \beta(\mathcal{A}) = (\widetilde{\chi}(\mathbb{P}M)).$$

A universal construction

Fixing the arrangement and allowing the weights to vary has good properties. Let $\Sigma = \Sigma(\mathcal{A})$ be the variety of all solutions to $\omega_\lambda = 0$ in $M \times \mathbb{C}^n$. One may project $\Sigma \subseteq M \times \mathbb{C}^n$ to either factor.

Theorem (Orlik-Terao '95)

The projection $\Sigma \to M$ gives Σ the structure of a rank $n - \ell$ vector bundle on M. In particular, Σ is smooth and has dimension n.

In equations: let $a_1, \ldots, a_n$ be coordinates in $\mathbb{C}^n$, and

$$\omega_{\underline{a}} = \sum_{i=1}^{n} a_i \mathrm{d}f_i / f_i.$$

Then $\Sigma = V(I')$, where

$$I' = (\langle \frac{\partial}{\partial x_i}, \omega_{\underline{a}} \rangle \colon 1 \leq i \leq \ell),$$

where $\langle \cdot \rangle$ denotes the duality pairing on the (co)tangent bundle of M.

Notation (Logarithmic derivations, forms)

- *Let $D(\mathcal{A})$ consist of those derivations $\theta\colon R \to R$ for which $\theta(f_i) \in (f_i)$ for all i, $1 \le i \le n$.*
- *$\Omega^p(*\mathcal{A})$: meromorphic p-forms, with (arbitrary) poles on the hyperplanes, $1 \le p \le \ell$.*
- *$\Omega^p(\mathcal{A})$: logarithmic p-forms: $\eta \in \Omega^p(*\mathcal{A})$ for which $Q\eta$ and $dQ/Q \wedge \eta$ are regular p-forms.*

Recall that $D(\mathcal{A})$ and $\Omega^1(\mathcal{A})$ are dual to each other. For certain arrangements, $D(\mathcal{A})$ is a free R-module. ($\mathcal{A}$ is a "free arrangement.")

Conjecture (Terao)

If $L(\mathcal{A}) \cong L(\mathcal{A}')$, then $\mathcal{A}$ and $\mathcal{A}'$ are both free, or both not free.

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Definition

Let

$$I = (\langle \theta, \omega_{\underline{a}} \rangle \colon \theta \in D(\mathcal{A})),$$

a homogeneous ideal of R.

(Compare: $I' = (\langle \partial/\partial x_i, \omega_{\underline{a}} \rangle \colon 1 \leq i \leq n)$, and $\Sigma = V(I')$.)

Proposition

For any arrangement $\mathcal{A}$, $V(I) = \overline{\Sigma} \subseteq \mathbb{C}^\ell \times \mathbb{C}^n$.

Main idea: show $I = (QI') : Q$.

Example ($xy(x-y)$ again)

$D(\mathcal{A})$ is generated by Euler derivation θ_1 and $\theta_2 = x^2\partial/\partial x + y^2\partial/\partial y$, so

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Remark

$\overline{\Sigma}(\mathcal{A})$ is not smooth. The components of $\overline{\Sigma} - \Sigma$ are indexed by irreducible subspaces in the intersection lattice. (Use de Concini-Procesi wonderful compactification.)

A codimension-1 critical set

Example (A_3)

Consider arrangement defined by $Q = xyz(x-y)(x-z)(y-z)$. Let $\Phi_1 = \big(x(y-z)\big)/\big(z(x-y)\big)$, $\Phi_2 = \big(y(x-z)\big)\big(z(x-y)\big)$.
Let $\Phi_{\lambda(a,b)} = \Phi_1^a \Phi_2^b$. From this morning's argument we get

$$\omega_\lambda = \big(ay(x-z) + bx(y-z)\big)\eta,$$

where $\eta = \mathrm{d}\log\Phi_1 - \mathrm{d}\log\Phi_2$. Then Σ_λ is given by $ay(x-z) + bx(y-z)$ as long as $a, b \neq 0$.

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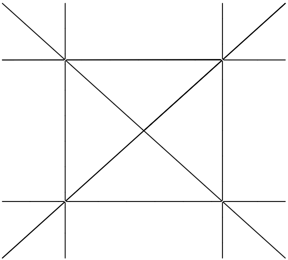

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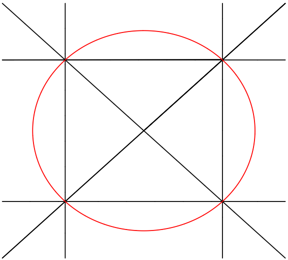

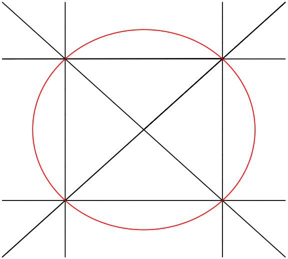

Example (A_3, continued)

On the other hand, $\overline{\Sigma}_{\lambda(1,0)}$ is given by $y(x-z)=0$, a union of lines.

So: $\overline{\Sigma}_\lambda$ is not closure of Σ_λ.

Another characterization of freeness

Since Σ is smooth, $\overline{\Sigma}$ is irreducible of codimension ℓ in $\mathbb{C}^\ell \times \mathbb{C}^n$.

Theorem (CDFV'09)

$\overline{\Sigma}(\mathcal{A})$ is a complete intersection if and only if $\mathcal{A}$ is free.

Definition

Say an arrangement $\mathcal{A}$ is *tame* if $\operatorname{pdim}_R \Omega^p(\mathcal{A}) \leq p$ for $1 \leq p \leq \ell$.

(Includes rank-3 arrangements, generic and free arrangements.)

Theorem (CDFV'09)

If $\mathcal{A}$ is tame, then $\overline{\Sigma}$ is arithmetically Cohen-Macaulay.

(C.I. $\Rightarrow$ Gorenstein $\Rightarrow$ a.C.M.)

- If $\overline{\Sigma}$ is arithmetically Cohen-Macaulay, is $\mathcal{A}$ tame?
- Are there (nonfree) arrangements $\mathcal{A}$ for which $\overline{\Sigma}$ is Gorenstein?

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Example

Let $\mathcal{A}$ be arrangement of 15 hyperplanes in $\mathbb{C}^4$ dual to the points

$$\{(b_1, b_2, b_3, b_4)\colon\ b_i \in \{0, 1\},\ \text{not all zero}\}.$$

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Resonance varieties

Let $A = H^{\cdot}(\mathbb{P}M, \mathbb{Z})$, the (deconed) Orlik-Solomon algebra of $\mathcal{A}$.
Identify $A^1 = \{\lambda \in \mathbb{C}^n \colon \sum_{i=1}^n \lambda_i = 0\}$.

Definition

The pth resonance variety $\mathcal{R}^p(\mathcal{A})$ of $\mathcal{A}$:

$$\mathcal{R}^p(\mathcal{A}) = \{\omega_\lambda \in A^1 : H^p(A, \omega_\lambda \wedge -) \neq 0\}.$$

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Orlik-Terao's set $Y \cap \mathcal{R}^{\ell-2} = \emptyset$, (but $Y \subseteq A^1$). Recall

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Compare with resonance varieties?
Positive results:

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A heuristic

Suppose that $\omega_\lambda \in \mathcal{R}^1(\mathcal{A})$. Then $\exists \eta \in A^1$ for which

$$\omega_\lambda \wedge \eta = 0, \qquad (1)$$

but $\eta \notin \mathbb{C} \cdot \omega_\lambda$. But (1) is true on the level of forms, so $\exists\ f, g$ polynomials for which

$$\begin{aligned} f\omega_\lambda &= g\eta; \\ \omega_\lambda &= g/f \cdot \eta. \end{aligned}$$

So $\Sigma_\lambda \supseteq V(g)$. (So, codimension 1.)

Resonance ⇒ high-dimensional critical sets

Theorem (CDFV'09)

Let $\mathcal{A}$ be an arrangement. Then $\mathcal{R}^1(\mathcal{A}) \subseteq \mathcal{S}^1(\mathcal{A})$.

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Methods:

- Complex of meromorphic forms resolves defining ideal of $\Sigma(\mathcal{A})$:

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  where S is polynomial ring in $\ell + n$ variables (coordinates and parameters).
- The complex is self-dual, so

$$H^p(\Omega^\cdot(*\mathcal{A}), \omega_\lambda) \cong \mathrm{Ext}^p_{S_Q}(S_Q/I', (R_\lambda)_Q),$$

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- If $\mathcal{A}$ is tame, (2) is exact. (In general?)
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Theorem

Suppose $\mathcal{A}$ is an arrangement with rank-2 flats X, X' for which

1. $|X|, |X'| \geq 3$;
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Then there exists a λ for which Σ_λ has codimension 1, but $H^1(A, \omega_\lambda) = 0$. (So $\mathcal{S}^1 \supsetneq \mathcal{R}^1$).

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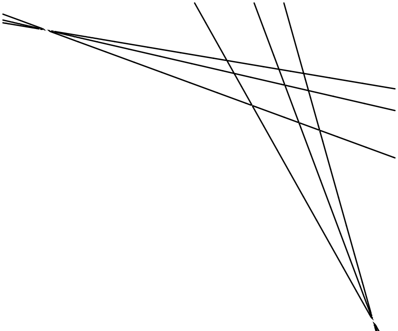

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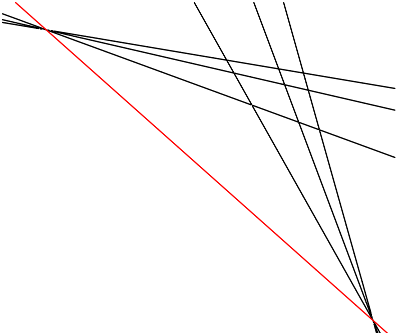

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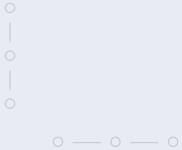

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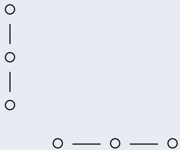

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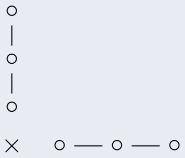

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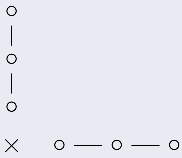

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some explanation

- Consider the map $\phi\colon R^n \to \Omega^1(\mathcal{A})$ given extending

$$\lambda \mapsto \omega_\lambda$$

R-linearly. Dual $\phi^*\colon D(\mathcal{A}) \to R^n$ by

$$\theta \mapsto \big(\theta(f_1)/f_1, \ldots, \theta(f_n)/f_n\big).$$

- Σ is the graph of $\widetilde{\ker\,\phi}$.
- Pick basis for linear space of relations amongst f_i's: say $\sum_{i=1}^n c_{ij} f_i = 0$ for $1 \le j \le n - \ell$.
- Let B be the $n \times (n-\ell)$ matrix with $B_{ij} = c_{ij} f_i$.
- Get exact sequences

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Example $((x+z)(2x+z)(3x+z)(y+z)(2y+z)(3y+z))$

Here,

$$B = \begin{pmatrix} x+z & -2(2x+z) & 3x+z & 0 & 0 & 0 \\ 0 & 0 & 0 & y+z & -2(2y+z) & 3y+z \\ * & * & & \cdots & & * \end{pmatrix}^t$$

Consider this mod z:

$$\overline{B} = \begin{pmatrix} x & -4x & 3x & 0 & 0 & 0 \\ 0 & 0 & 0 & y & -4y & 3y \\ * & * & & \cdots & & * \end{pmatrix}^t$$

Kernels are saturated, so for any λ in span of $\lambda_1 = (1, -4, 3, 0, 0, 0)$, $\lambda_2 = (0, 0, 0, 1, -4, 3)$, zero locus of ω_λ contains line $z = 0$.

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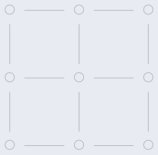
Example ("Ziegler arrangements")

Consider arrangements of 9 lines in $\mathbb{P}^2$ with six triple points:



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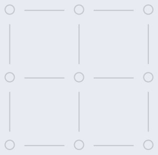

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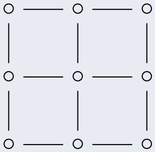

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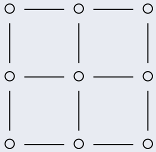

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As before, $\mathcal{S}^1$ contains some $\mathbb{P}^1$'s coming from $\overline{\Sigma}_\lambda$'s which are projective lines.

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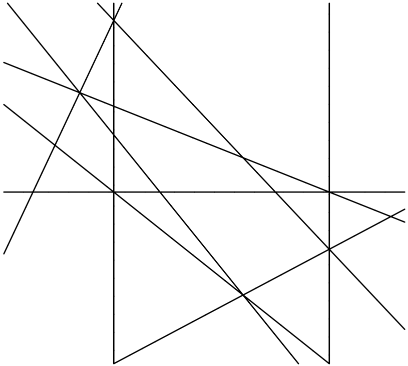

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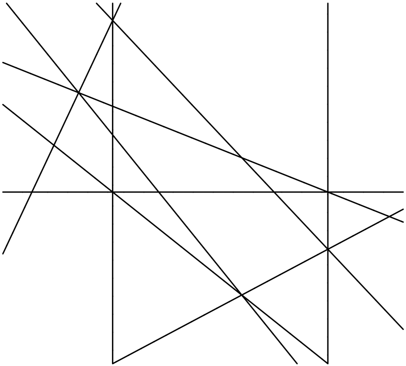

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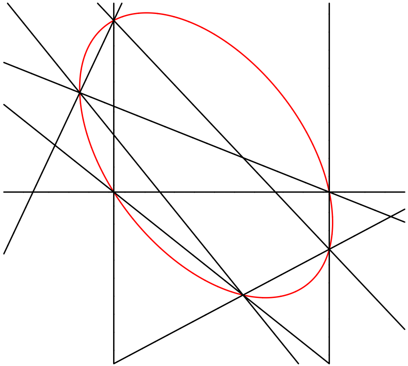

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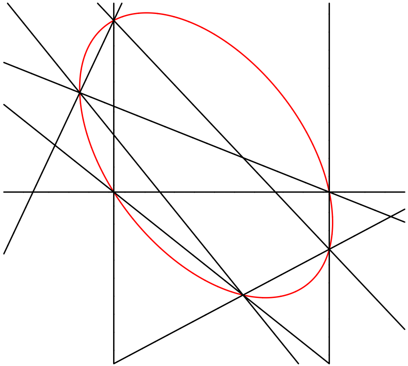

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Explicitly?

The Ziegler matroid realization is given by

$$Q = xyz(x+y+z)(s_1x+z)(x+s_2y+z)f_7(s_3x+s_3y+z)(s_4x+y),$$

where

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together with some open conditions. The six multiple points lie on a conic iff $s_1(1-s_3) = (s_2-1)s_3s_4$. In that case, let λ be any nonzero multiple of

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The critical set of the corresponding master function Φ_λ consists of the conic,

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