Collision-free motion planning on surfaces

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Outline

The motion planning problem

A *motion planning algorithm* for a mechanical system is a rule which assigns to a pair of states (A, B) of the system a continuous motion of the system starting at A and ending at B

X the configuration space of the system

PX the space of all continuous paths $\gamma\colon [0,1] \to X$

$\pi\colon PX \to X \times X, \quad \gamma \mapsto (\gamma(0), \gamma(1)),$ is a fibration

A motion planning algorithm is a section $s\colon X \times X \to PX$

(not necessarily continuous)

Proposition

$\exists$ a globally continuous section $s\colon X \times X \to PX$
$\iff$ X is contractible

Topological complexity

The *topological complexity* of a topological space X is the sectional category, or Schwarz genus, of the fibration $\pi\colon PX \to X \times X$

$$\mathrm{TC}(X) = \mathrm{secat}(\pi\colon PX \to X \times X)$$

$\mathrm{TC}(X)$: smallest integer k for which $X \times X$ has an open cover with k elements, over each of which π has a continuous section

Proposition

$\mathrm{TC}(X) = 1 \iff$ *X is contractible*

Solving the motion planning problem

Proposition (Farber)

If X is a Euclidean Neighborhood Retract, then $\mathrm{TC}(X)$ is equal to the smallest integer k so that there is a section $s\colon X \times X \to PX$ of the path space fibration and a decomposition

$$X \times X = F_1 \cup F_2 \cup \cdots \cup F_k, \quad F_i \cap F_j = \emptyset,$$

with F_i locally compact and $s|F_i\colon F_i \to PX$ continuous for each i

This gives a motion planning algorithm:

If $(A, B) \in X \times X$, $\exists!$ F_i with $(A, B) \in F_i$, and the path $s(A, B)$ is a continuous motion of the system starting at A and ending at B

Spheres

Example ($X = S^1$)

$F_1 = \{(x, -x) \mid x \in X\} \subset X \times X \qquad F_2 = X \times X \smallsetminus F_1$

$s|F_1 \colon F_1 \to PX$ counterclockwise path from x to $-x$

$s|F_2 \colon F_2 \to PX$ shortest geodesic arc from x to y

$\mathrm{TC}(S^1) = 2$

Example ($X = S^2$)

fix $e \in X$, ν a nowhere zero tangent vector field on $X \smallsetminus e$

$F_1 = \{(e, -e)\} \qquad F_2 = \{(x, -x) \mid x \neq e\} \qquad F_3 = \{(x, y) \mid x \neq -y\}$

$s|F_1 \colon F_1 \to PX$ any fixed path from e to $-e$

$s|F_2 \colon F_2 \to PX$ path x to $-x$ along semicircle tangent to $\nu(x)$

$s|F_3 \colon F_3 \to PX$ shortest geodesic arc from x to y

$\mathrm{TC}(S^2) \leq 3$

Main Theorem

Consider motion of n distinct particles in X condition: **no collisions**

i.e., motion in the configuration space of n distinct ordered points in X

$$F(X,n) = \{(x_1,\dots,x_n) \in X^{\times n} \mid x_i \neq x_j \text{ for } i \neq j\}$$

Focus on case $X = \Sigma_g$ an orientable surface ($g \geq 1$ for now)

Theorem (C.-Farber)

The topological complexity of the configuration space of n distinct ordered points on an orientable surface Σ_g of genus g is

$$\mathrm{TC}(F(\Sigma_g,n)) = \begin{cases} 2n+1 & \text{if } g = 1 \\ 2n+3 & \text{if } g \geq 2 \end{cases}$$

Requisite properties

- TC(X) depends only on the homotopy type of X
- upper bounds

$\mathrm{TC}(X) \leq 2\dim(X) + 1$ dim(X) the covering dimension of X

$\mathrm{TC}(X \times Y) \leq \mathrm{TC}(X) + \mathrm{TC}(Y) - 1$

- lower bound

$\mathrm{TC}(X) \geq \mathrm{zcl}(H^*(X)) + 1$ $H^*(X) = H^*(X;\mathbb{Q})$ unless otherwise noted

zcl($H^*(X)$) the zero-divisor cup length of $H^*(X)$

the cup length of $\ker\big[H^*(X) \otimes H^*(X) \xrightarrow{\cup} H^*(X)\big]$

Example ($X = S^2$ continued recall $\mathrm{TC}(S^2) \leq 3$)

If $0 \neq x \in H^2(S^2)$, then $(x \otimes 1 - 1 \otimes x)^2 = -2x \otimes x \neq 0$

$\mathrm{zcl}\, H^*(S^2) \geq 2 \implies \mathrm{TC}(S^2) \geq 3$ so $\mathrm{TC}(S^2) = 3$

Surfaces

Example ($X = T = S^1 \times S^1$)

product inequality $\implies \mathrm{TC}(T) \leq \mathrm{TC}(S^1) + \mathrm{TC}(S^1) - 1 = 3$

$a, b \in H^1(T)$ generators of $H^*(T)$

$\bar{a} = 1 \otimes a - a \otimes 1$, $\bar{b} = 1 \otimes b - b \otimes 1$ zero divisors in $H^*(T) \otimes H^*(T)$

$\bar{a}\bar{b} \neq 0 \implies \operatorname{zcl} H^*(T) \geq 2$ $\qquad$ $\mathrm{TC}(T) = 3$

Example ($X = \Sigma_g \quad g \geq 2$)

$\dim \Sigma_g = 2 \implies \mathrm{TC}(\Sigma_g) \leq 5$

$a, b, c, d \in H^1(\Sigma_g)$ generators of $H^*(\Sigma_g)$

$\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ zero divisors in $H^*(\Sigma_g) \otimes H^*(\Sigma_g)$ as above

$\bar{a}\bar{b}\bar{c}\bar{d} \neq 0 \implies \operatorname{zcl} H^*(\Sigma_g) \geq 4$ $\qquad$ $\mathrm{TC}(\Sigma_g) = 5$

Diagonal cohomology class

X an oriented real manifold $\dim X = m$

$\Delta \in H^m(X \times X)$ the cohomology class dual to the diagonal

For X closed with $\omega \in H^m(X)$ a fixed generator

$$\Delta = \sum (-1)^{|\beta_i|} \beta_i \times \beta_i^*$$

where $\{\beta_i\}$ and $\{\beta_i^*\}$ are dual bases for $H^*(X)$ satisfying

$$\beta_i \cup \beta_j^* = \delta_{i,j}\, \omega$$

$|\beta_i|$ degree of β_i

$\delta_{i,j}$ Kronecker symbol

Cohen-Taylor/Totaro spectral sequence

$p_i \colon X^{\times n} \to X$ $\quad p_{i,j} \colon X^{\times n} \to X \times X$ $\quad$ natural projections

$p_i(x_1, \ldots, x_n) = x_i$ $\quad p_{i,j}(x_1, \ldots, x_n) = (x_i, x_j)$ $\quad 1 \le i, j \le n \quad i \ne j$

inclusion $F(X, n) \hookrightarrow X^{\times n}$ determines Leray spectral sequence which converges to $H^*(F(X, n))$

initial term: quotient of the algebra $H^*(X^{\times n}) \otimes H^*(F(\mathbb{R}^m, n))$ by the relations $(p_i^*(x) - p_j^*(x)) \otimes \alpha_{i,j}$ for $i \ne j$ and $x \in H^*(X)$ where $\alpha_{i,j}$ generate $H^*(F(\mathbb{R}^m, n))$ (from famous Arnold, Cohen result)

first nontrivial differential: $\quad d\alpha_{i,j} = p_{i,j}^* \Delta$

Totaro theorem

Theorem

If X is a smooth, complex projective variety, the above spectral sequence degenerates immediately
The differential d above is the only nontrivial differential

Suppose X as above has real dimension m

Let $H = H^*(X^{\times n})$ and I the ideal in H generated by the elements

$$\Delta_{i,j} = p_{i,j}^*(\Delta) \in H^m(X^{\times n}) \qquad 1 \leq i < j \leq n$$

Proposition

H/I is a subalgebra of $H^*(F(X,n))$ $\qquad \mathrm{TC}(F(X,n)) \geq \mathrm{zcl}\, H/I + 1$

uses Totaro theorem and

Fact: If B is a subalgebra of A, then $\mathrm{zcl}\, A \geq \mathrm{zcl}\, B$

$\mathrm{TC}(F(T,n))$

Theorem

The topological complexity of the configuration space of n distinct ordered points on the torus T is $\mathrm{TC}(F(T,n)) = 2n+1$

$n = 1$: $F(T,1) = T = S^1 \times S^1 \implies \mathrm{TC}(F(T,1)) = 3$

$n \geq 2$: $F(T,n) \cong T \times F(T \smallsetminus \{\text{point}\}, n-1)$

$F(T \smallsetminus \{\text{point}\}, n-1)$ is a $K(G,1)$, G pure braid group of $T \smallsetminus \{\text{point}\}$

G iterated semidirect product of free groups (Fadell-Neuwirth bundles)

$\implies F(T \smallsetminus \{\text{point}\}, n-1) \simeq$ cell complex of dimension $n-1$

$\implies \mathrm{TC}(F(T \smallsetminus \{\text{point}\}, n-1)) \leq 2(n-1) - 1 = 2n-1$

product inequality:

$$\mathrm{TC}(F(T,n)) \leq \mathrm{TC}(T) + \mathrm{TC}(F(T \smallsetminus \{\text{point}\}, n-1)) - 1 = 2n+1$$

$\mathrm{TC}(F(T,n))$

remains to show that $\operatorname{zcl} H^*(F(T,n)) \geq 2n$

$a, b \in H^1(T)$ generators of $H^*(T)$

diagonal class in $H^2(T \times T)$ given by

$\Delta = ab \times 1 + 1 \times ab + b \times a - a \times b = (1 \times a - a \times 1)(1 \times b - b \times 1)$

$H_T = H^*(T^{\times n})$ an exterior algebra

generators a_i, b_i, $1 \leq i \leq n$, where $u_i = 1 \times \cdots \times u \times \cdots \times 1$

I_T ideal in H_T generated by $\Delta_{i,j} = p_{i,j}^* \Delta = (a_j - a_i)(b_j - b_i) \qquad i < j$

prior Proposition $\implies A_T = H_T / I_T$ subalgebra of $H^*(F(T,n))$

$\operatorname{zcl} H^*(F(T,n)) \geq \operatorname{zcl} A_T \implies$ enough to show that $\operatorname{zcl} A_T \geq 2n$

$\mathrm{TC}(F(T,n)) = 2n+1$

new H_T basis: $x_1 = a_1 \quad y_1 = b_1 \quad x_j = a_j - a_1 \quad y_j = b_j - b_1 \quad 2 \le j$

$\Delta_{1,j} = x_j y_j \qquad \Delta_{i,j} = x_j y_j - x_j y_i - x_i y_j + x_i y_i$ for $i > 1$

$I_T = \langle\, x_i y_i,\ x_k y_j + x_j y_k \,\rangle \quad 2 \le i \le n \quad 2 \le j < k \le n$ deg 2 gens

$A_T = H_T/I_T$ generated by x_i, y_i, $1 \le i \le n$, and has basis

$$\left\{x_1^{\epsilon_x} y_1^{\epsilon_y} x_J y_K \mid \epsilon_x, \epsilon_y \in \{0,1\},\ J, K \subset [2,n],\ \max J < \min K\right\}$$

$\bar{x}_j = x_j \otimes 1 - 1 \otimes x_j \qquad \bar{y}_j = y_j \otimes 1 - 1 \otimes y_j$ zero-divisors in $A_T \otimes A_T$

$$\prod_{j=1}^{n} \bar{x}_j \bar{y}_j = \pm\, y_1 y_2 \cdots y_n \otimes x_1 x_2 \cdots x_n + \text{other terms} \neq 0$$

$$\implies \operatorname{zcl} A_T \ge 2n \qquad \square$$

Remarks on A_T

- The subalgebra A_T is not isomorphic to $H^*(F(T,n))$

  the differential in the Cohen-Taylor/Totaro spectral sequence has nontrivial kernel

  but $\operatorname{zcl} A_T = \operatorname{zcl} H^*(F(T,n)) = 2n$

- The algebra $A_T = H_T/I_T$ is Koszul

  generating set $\{x_j y_j,\ x_j y_i + x_i y_j\}$ of I_T is a quadratic Gröbner basis (use the Buchberger criterion)

$\mathrm{TC}(F(\Sigma, n))$

Theorem

The topological complexity of the configuration space of n distinct ordered points on an orientable surface Σ of genus $g \geq 2$ is $\mathrm{TC}(F(\Sigma, n)) = 2n + 3$

$n = 1$: $F(\Sigma, 1) = \Sigma \implies \mathrm{TC}(F(\Sigma, 1)) = 5$

$n \geq 2$: $F(\Sigma, n)$ is a $K(G, 1)$, G pure braid group of Σ

Fadell-Neuwirth bundle $F(\Sigma, n) \to \Sigma$ has a section

$\implies G \cong \pi_1(F(\Sigma \smallsetminus \{\text{point}\}, n-1)) \rtimes \pi_1(\Sigma)$

$\implies G$ has cohomological dimension $n + 1$

$\implies F(\Sigma, n) \simeq$ cell complex of dimension $n + 1$

$\implies \mathrm{TC}(F(\Sigma, n)) \leq 2n + 3$

remains to show that $\mathrm{zcl}\, H^*(F(\Sigma, n)) \geq 2n + 2$

$\mathrm{TC}(F(\Sigma, n)) = 2n + 3$

$H^*(F(\Sigma, n))$ has a subquotient which contains the algebra A_T from the genus one case as a subalgebra

this, $\mathrm{zcl}\, A_T = 2n$, and computation in $H^*(\Sigma)$ can be used to show that $\mathrm{zcl}\, H^*(F(\Sigma, n)) \geq 2n + 2$ □

Remark

Compare with the topological complexity of the Cartesian product:

$$\mathrm{TC}(\Sigma^{\times n}) = 4n + 1 \qquad\qquad \mathrm{TC}(F(\Sigma, n)) = 2n + 3$$

Complexity of the collision-free motion planning problem for n distinct points is $\sim$ half the complexity of the problem when points can collide

Counterintuitive ...... $\mathrm{TC}(X)$ reflects only part of the *"true"* complexity of the motion planning problem

$\mathrm{TC}(F(S^2, n))$

Theorem

For $n \geq 3$, the topological complexity of the configuration space of n ordered points on the sphere is $\mathrm{TC}(F(S^2, n)) = 2n - 2$.

Proof uses:

$F(S^2, n) \simeq \mathrm{SO}(3) \times F(\mathbb{R}^2 \setminus \{\text{two points}\}, n - 3)$

Arrangements make an appearance! Yay!

$\mathrm{TC}(\mathrm{SO}(3)) = 4$ (Farber)

$\mathrm{TC}(F(\mathbb{R}^2 \setminus \{\text{two points}\}, n - 3)) = 2n - 5$ (Farber-Grant-Yuz)

$\mathrm{zcl}\, H^*(F(S^2, n); \mathbb{Z}_2) \geq 2n - 3$

Note: For $n \leq 2$, $F(S^2, n) \simeq S^2$ and $\mathrm{TC}(F(S^2, n)) = 3$

$\mathrm{TC}(F(\Sigma \smallsetminus \{m \text{ points}\}, n))$

Theorem

Let Σ be a surface of genus $g \geq 1$. For $m \geq 1$, the topological complexity of the configuration space of n ordered points on $\Sigma \smallsetminus \{m \text{ points}\}$ is $\mathrm{TC}(F(\Sigma \smallsetminus \{m \text{ points}\}, n)) = 2n + 1$.

Proof uses:

$$F(\Sigma \smallsetminus \{m \text{ points}\}, n) \simeq \text{cell complex of dimension } n$$

$$\implies \mathrm{TC}(F(\Sigma, n)) \leq 2n + 1$$

$$\operatorname{zcl} H^*(F(\Sigma \smallsetminus \{m \text{ points}\}, n); \mathbb{C}) \geq 2n \qquad \text{(used MHS for this)}$$

$\mathrm{TC}(F(S^2 \smallsetminus \{m \text{ points}\}, n))$

Theorem (Farber-Yuz, Farber-Grant-Yuz)

For $m \geq 1$, the topological complexity of the configuration space of n ordered points on $S^2 \smallsetminus \{m \text{ points}\}$ is

$$\mathrm{TC}(F(S^2 \smallsetminus \{m \text{ points}\}, n)) = \begin{cases} 1 & \text{if } m = 1 \text{ and } n = 1, \\ 2n-2 & \text{if } m = 1 \text{ and } n \geq 2, \\ 2n & \text{if } m = 2 \text{ and } n \geq 1, \\ 2n+1 & \text{if } m \geq 3 \text{ and } n \geq 1. \end{cases}$$

More arrangements! Yay!

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