# Random walks on complex hyperplane arrangements and self-organizing libraries

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Sapporo, August 10, 2009

## Introduction: Tsetlin's library

- Shelf with $n$ numbered books

- Choose book $i$ with probability $w_i$, move it to front

Markov chain: States = permutations = $S_n$

$$\text{Transition probabilities} = \begin{cases} w_i & \text{for "book-move"} \\ 0 & \text{otherwise} \end{cases}$$

Studied also in CS: "dynamic file management", "cache management", ...

Much known: stationary distribution, eigenvalues of transition matrix $P_w$, ...

**Theorem.** *(Donnelly, Kapoor-Reingold, Phatarfod, 1991)*
*For Tsetlin's library:*

- *The eigenvalues* $\lambda_E$ *of* $P_w$ *are indexed by subsets* $E \subseteq [1, \ldots, n]$, *and* $\lambda_E = \sum_{i \in E} w_i$
- *The multiplicity of* $\lambda_E$ *is the number of derangements of* $n - |E|$ *elements.*

# Introduction

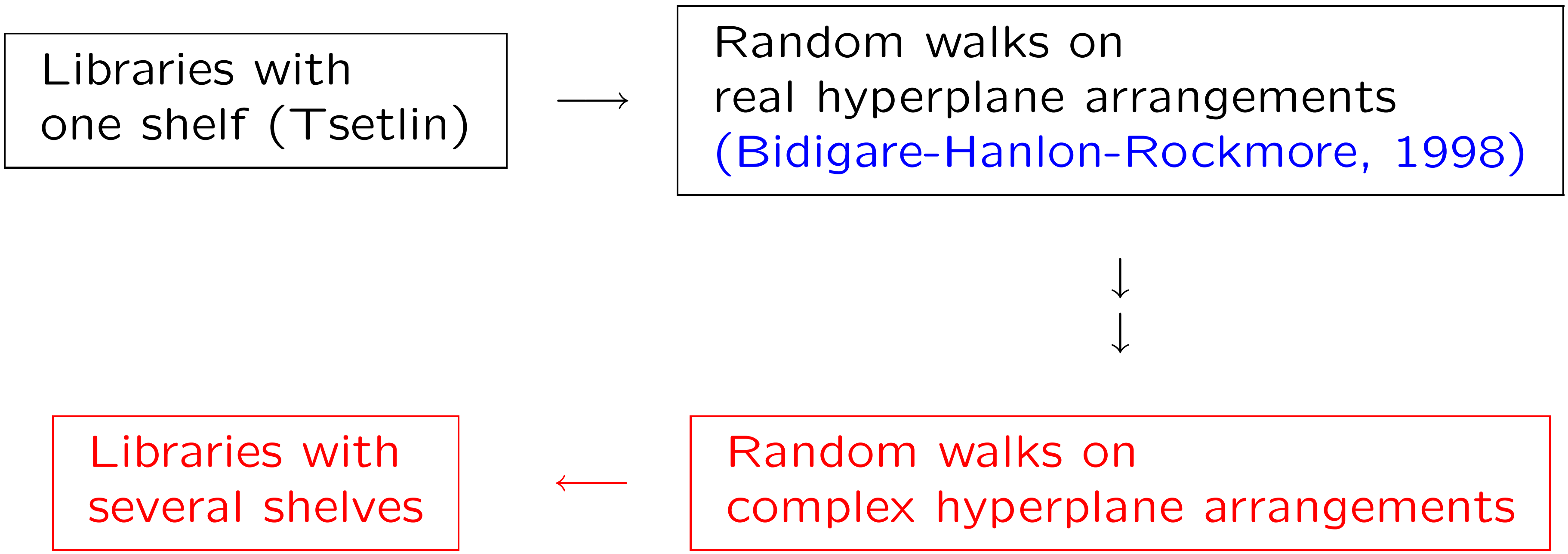
| | | |
|---|---|---|
| Libraries with one shelf ($\sim$ Tsetlin), "random-to-front" | $\longrightarrow$ | Random walks on complement of real hyperplane arrangements (Bidigare-Hanlon-Rockmore, 1998) |

Random-to-front shuffle:

$$4\ 7\ 2\ 1\ 5\ 3\ 6 \quad \Longrightarrow \quad 7\ 1\ 5\ 4\ 2\ 3\ 6$$

What about complex hyperplane arrangements?

## Introduction



## Introduction: Overview

Random walks on semigroups (Ken Brown, 2000)

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| | | |
|---|---|---|
| Random walks on $\mathbb{R}$-arrangements * | Random walks on $\mathbb{C}$-arrangements | Random walks on greedoids |
| ↓ | ↓ | |
| Libraries with one shelf (Tsetlin) | Libraries with several shelves | |

* Bidigare-Hanlon-Rockmore (1998), Brown-Diaconis (1999)

## Introduction: Library with several shelves

- $k$ shelves with $n$ numbered books, $n_j$ books on shelf $j$
- Choose set of books $E \subseteq [n]$ with probability $w_E$
- Move chosen books to front of resp. shelf, in induced order
  AND
  Move affected shelves to top, in induced order

Markov chain: States = permutations = $S_{n_1} \times \ldots \times S_{n_k} \times S_k$

$$\text{Transition probabilities} = \begin{cases} w_E & \text{for "book-borrow" move} \\ 0 & \text{otherwise} \end{cases}$$

# Introduction

Example. Let $n = 3$ and $\pi = (1, 2 \mid 3)$.

Four library configurations. Transition matrix $P_w$:

| | 1 2<br>3 | 2 1<br>3 | 3<br>1 2 | 3<br>2 1 |
|---|---|---|---|---|
| 1 2<br>3 | $w_1 + w_{1,2} + w_{1,3}$ | $w_1 + w_{1,3}$ | $w_1 + w_{1,2}$ | $w_1$ |
| 2 1<br>3 | $w_2 + w_{2,3}$ | $w_2 + w_{1,2} + w_{2,3}$ | $w_2$ | $w_2 + w_{1,2}$ |
| 3<br>1 2 | $w_3$ | 0 | $w_3 + w_{1,3}$ | $w_{1,3}$ |
| 3<br>2 1 | 0 | $w_3$ | $w_{2,3}$ | $w_3 + w_{2,3}$ |

Example (cont'd). Let $n = 3$ and $\pi = (1, 2 \mid 3)$. Transition matrix $P_w$ (previous slide) has four eigenvalues, all of multiplicity one:

$$\begin{cases} \varepsilon_1 = 0 \\ \varepsilon_2 = w_{1,3} + w_{2,3} \\ \varepsilon_3 = w_3 + w_{1,2} \\ \varepsilon_4 = 1 \end{cases}$$

## Introduction

**Theorem.** *(Eigenvalues for the $k$-shelf library walk.)*
*Let $\pi$ be the partition of $\{1,\ldots,n\}$ into $k$ blocks according to placement on shelves.*

1. *For each pair of unordered partitions $(\alpha,\beta)$such that $\alpha \leq \pi \leq \beta$ (i.e., $\beta$ refines $\pi$ and $\pi$ refines $\alpha$) there is an eigenvalue*

$$\varepsilon_{(\alpha,\beta)} = \sum w_E,$$

   *the sum extending over all $E \subseteq [n]$ such that $E$ is a union of blocks from $\beta$ and the union of shelves containing some element of $E$ is a union of blocks from $\alpha$.*

2. *The multiplicity of $\varepsilon_{(\alpha,\beta)}$ is*

$$\prod (p_i - 1)! \prod (q_j - 1)!$$

   *where $(p_1, p_2, \ldots)$ are the block sizes of $\beta$ and $(q_1, q_2, \ldots)$ the block sizes of $\alpha$ modulo $\pi$.*

3. *These are all the eigenvalues.*

## Questions

1. Is the detour via complex geometry really needed?

– No, not if one wants only the $k$-shelf library application, which needs no geometry at all.

2. Why stop at $k$ shelves?

– No need to do that. One can have several library rooms, each with a certain number of shelves each carrying books, such that rooms, shelves and books are permuted at each step.
– Or, several library buildings, . . . several planets, and so on ....

# 1. Random walks on semigroups

Def: An LRB (left regular band): semigroup $\Sigma$ satisfying

$$\begin{cases} x^2 = x & \text{for all } x \in \Sigma \\ xyx = xy & \text{for all } x, y \in \Sigma \end{cases}$$

There are *two posets* related to an LRB semigroup $\Sigma$.

**Proposition 1.** *Define a relation "$\leq$" on $\Sigma$ by*

$$x \leq y \quad \Leftrightarrow \quad xy = y \tag{1}$$

*This is a partial order relation.*

So, an LRB semigroup is also a poset.
The identity element $e$ is the unique minimal element.
The maximal elements form a left ideal:

$$x \in \Sigma, \;\; y \in \max(\Sigma) \; \Rightarrow \; xy \in \max(\Sigma)$$

**Proposition 2.** *Let* $\Sigma$ *be an LRB semigroup. Then there exists a unique finite lattice* $\Lambda$ *and an order-preserving and surjective map*

$$\text{supp} : \Sigma \to \Lambda \tag{2}$$

*such that for all* $x, y \in \Sigma$*:*

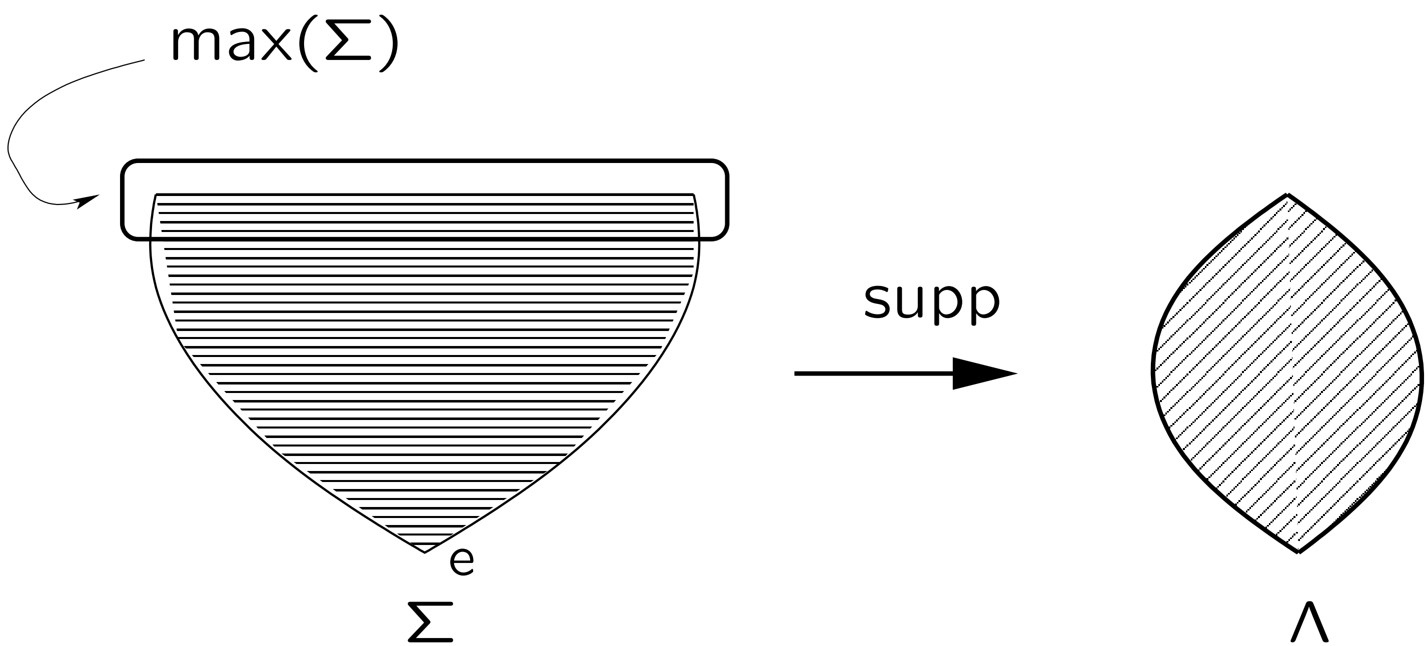
*1.* $\text{supp}(xy) = \text{supp}(x) \vee \text{supp}(y)$

*2.* $\text{supp}(x) \leq \text{supp}(y) \quad \Leftrightarrow \quad yx = y$

We call $\Lambda$ the *support lattice* and supp the *support map*.

# 1. Random walks on semigroups

## "The picture"




semigroup

support lattice

## 1. Random walks on semigroups

Random walk on max(Σ): Probability distribution $\{w_x\}$ on Σ.

STEP: $y \mapsto xy$, where $y \in$ max(Σ) and

$x \in \Sigma$ is chosen according to $w$.

Let $P_w$ be the transition matrix of the random walk on max(Σ):

$$P_w(c,d) = \sum_{x\,:\,xc=d} w_x$$

for $c, d \in$ max(Σ).

## 1. Random walks on semigroups

Two fundamental theorems of Brown (2000),

on *eigenvalues of $P_w$* resp. *stationarity*,

generalizing work of Bidigare 97, Bidigare-Hanlon-Rockmore 99, and Brown-Diaconis 98.

## 1. Random walks on semigroups

**Theorem 1.** (*Eigenvalues*)

1. *The matrix* $P_w$ *is diagonalizable.*

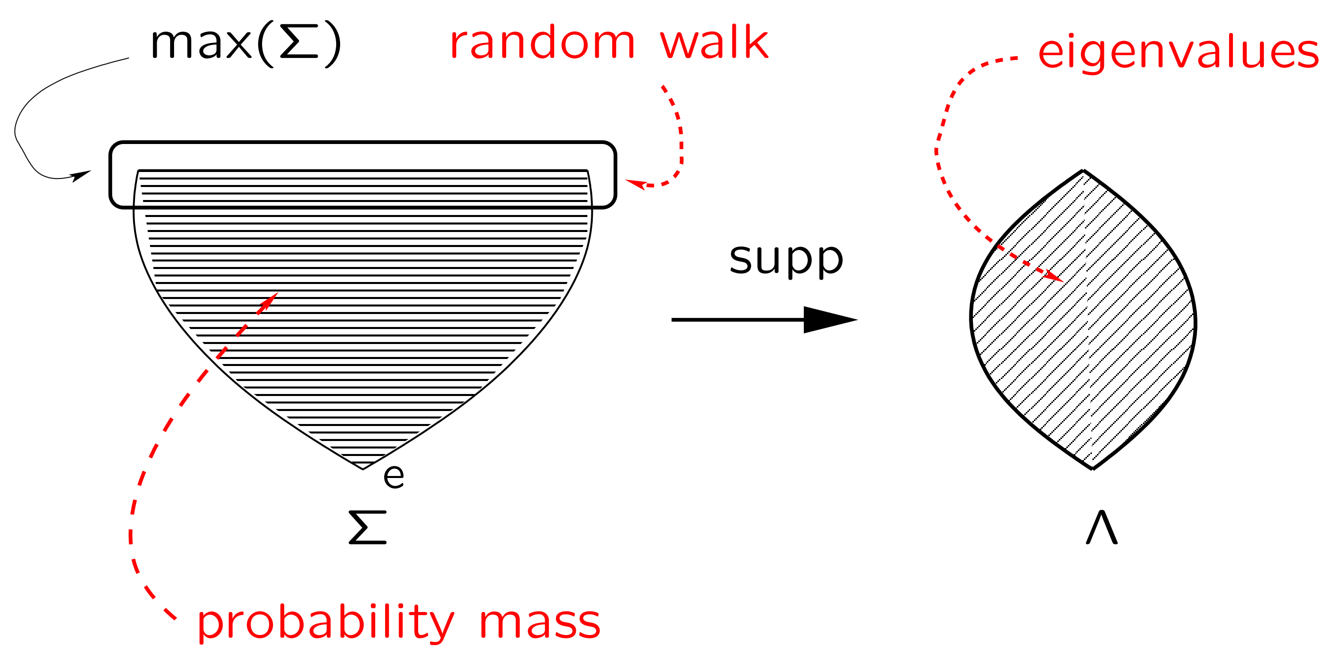
2. *For each* $X \in \Lambda$ *there is an eigenvalue* $\varepsilon_X = \sum_{y\,:\,\mathrm{supp}(y) \leq X} w_y$ *.*

3. *The multiplicity of the eigenvalue* $\varepsilon_X$ *is* $m_X = \sum_{Y\,:\,Y \geq X} \mu_\Lambda(X, Y) c_Y$*, where* $c_Y \stackrel{\mathrm{def}}{=} |\max(\Sigma_{\geq y})|$*, for any* $y \in \mathrm{supp}^{-1}(Y)$*.*

4. *These are all the eigenvalues.*

# 1. Random walks on semigroups

## "The picture"



**Theorem 2.** *(Stationarity)*
*Suppose that* $\Sigma$ *is generated by* $\{x \in \Sigma : w_x > 0\}$. *Then the random walk on* $\max(\Sigma)$ *has a unique stationary distribution* $\pi$.

Also provided:

- Algorithm how to sample an element distributed from $\pi$.
- Measure of convergence to $\pi$.

— stationarity will not be further discussed in this talk

## 2. Real hyperplane arrangements

$\mathcal{A} = \{H_1, \ldots, H_t\}$ arrangement
$\ell_1, \ldots, \ell_t$ linear forms on $\mathbb{R}^d$
$H_i = \{x : \ell_i(x) = 0\} \subseteq \mathbb{R}^d$ hyperplane

$L_{\mathcal{A}} = \{$intersections of $H_i$'s$\}$ ordered by reverse inclusion
— intersection lattice
Complement — convex cones "regions" or "chambers"

**Theorem.** *(Zaslavsky 1975)*

$$\#\ \textit{regions} = \sum_{x \in L_{\mathcal{A}}} |\mu(\widehat{0}, x)|$$

## — Where is the semigroup?

Encode position of point $x \in \mathbb{R}^d$ with respect to $\mathcal{A}$.

Sign vector (position vector): $\sigma(x) = \{\sigma_1, \ldots, \sigma_t\} \in \{+, -, 0\}^t$

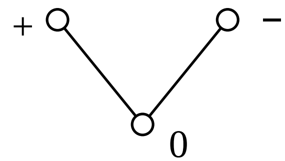
$$\sigma_i = \begin{cases} 0, & \text{if } \ell_i(x) = 0 \\ +, & \text{if } \ell_i(x) > 0 \\ -, & \text{if } \ell_i(x) < 0 \end{cases}$$

Combinatorics of sign vectors $\longrightarrow$ oriented matroid theory

## 2. Real hyperplane arrangements

Face semilattice:

$F_{\mathcal{A}} = \sigma(\mathbb{R}^d) \subseteq \{+,-,0\}^t$ — ordered componentwise by




Note: maximal el'ts of $F_{\mathcal{A}} \leftrightarrow$ regions

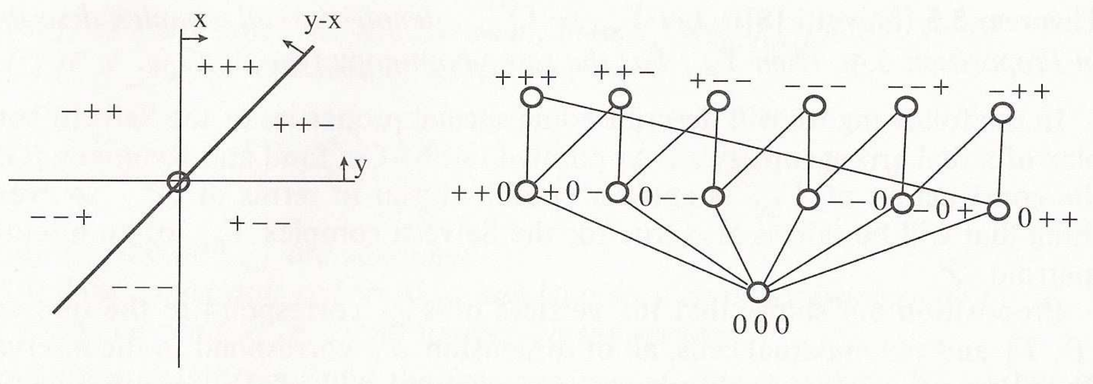



FIGURE 5.1. Face poset $\mathscr{L}$ of an arrangement of three lines in $\mathbb{R}^2$.

## 2. Real hyperplane arrangements

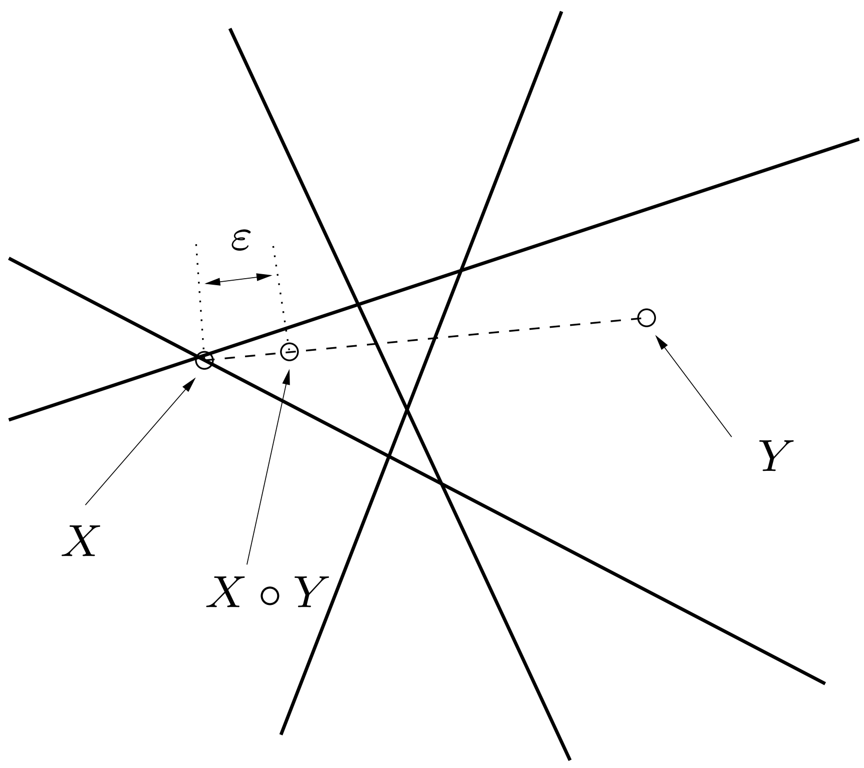
Fact: $F_{\mathcal{A}}$ describes cell structure of regular CW-decomposition of the unit sphere in $\mathbb{R}^d$ (the cell decomposition induced by the hyperplanes)

Composition: $X \circ Y \in \{+, -, 0\}^t$ defined by

$$(X \circ Y)_i = \begin{cases} X_i, & \text{if } X_i \neq 0 \\ Y_i, & \text{if } X_i = 0 \end{cases}$$

- associative, idempotent, unit element $= (0, \ldots, 0)$
- $X, Y \in F_{\mathcal{A}} \;\Rightarrow\; X \circ Y \in F_{\mathcal{A}}$

# 2. Real hyperplane arrangements



**Proposition 3.** $(F_{\mathcal{A}}, \circ)$ *is LRB semigroup with support lattice* $L_{\mathcal{A}}^{\mathsf{op}}$*. The support map*

$$\textit{supp} : F_{\mathcal{A}} \to L_{\mathcal{A}}^{\mathsf{op}}$$

*sends cell* $\sigma$ *to linear span* $\overline{\sigma}$*. (Equivalently, sends sign-vector to the set of positions of its zeroes.)*

Consequence: Theory of random walks on $\mathbb{R}$-arrangements

Probability distribution $w$ on $F_{\mathcal{A}}$ $\Rightarrow$ Random walk on $C_{\mathcal{A}}$

STEP: Choose $X \in F_{\mathcal{A}}$ according to measure $w$. Then, from current region $C \in C_{\mathcal{A}}$ move to $X \circ C$.

**Theorem 3.** *(Bidigare-Hanlon-Rockmore, Brown-Diaconis)*
*(a) Transition matrix is diagonalizable.*
*(b) For each $F \in L_{\mathcal{A}}$ there is an eigenvalue*

$$\lambda_F = \sum_{X:\mathrm{supp}X \subseteq F} w(X)$$

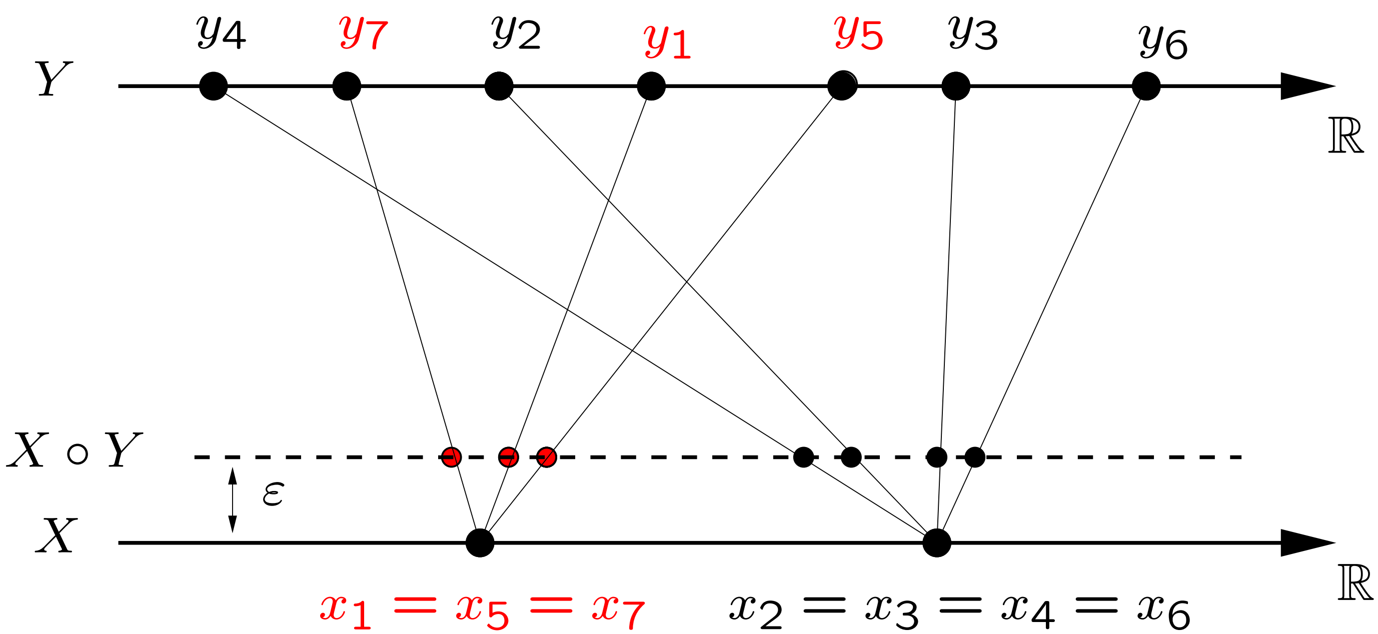
*(c) The multiplicity of $\lambda_F$ is $|\mu(\hat{0}, F)|$.*
*(d) These are all the eigenvalues.*

**Corollary 1.** *(Zaslavsky's formula)*

$$\#\ \text{regions} = \text{size of matrix } = \sum_{F \in L_{\mathcal{A}}} |\mu(\hat{0}, F)|$$

Special case, random-to-front shuffle:

4 7 2 1 5 3 6 $\longrightarrow$ 7 1 5 4 2 3 6



## 3. Complex hyperplane arrangements

Arrangement $\mathcal{A} = \{H_1, \ldots, H_t\}$
$\ell_1, \ldots, \ell_t$ linear forms on $\mathbb{C}^d$
$H_i = \{x : \ell_i(x) = 0\} \subseteq \mathbb{C}^d$ hyperplane

$M_{\mathcal{A}} = \mathbb{C}^d \setminus \cup\mathcal{A}$ — complement ($2d$-dimensional manifold))

How define random walk? Where is semigroup?

## 3. Complex hyperplane arrangements

Encode position of point $x \in \mathbb{C}^d$ with respect to $\mathcal{A}$.

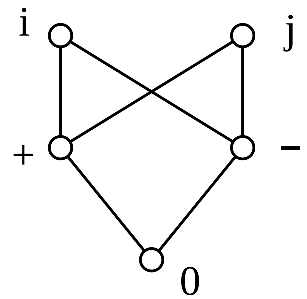
Sign vector (position vector): $\sigma(x) = \{\sigma_1, \ldots, \sigma_t\} \in \{0, +, -, i, j\}^t$

$$\sigma_i = \begin{cases} 0, & \text{if } \ell_i(x) = 0 \\ +, & \text{if } \Im(\ell_i(x)) = 0,\ \Re(\ell_i(x)) > 0 \\ -, & \text{if } \Im(\ell_i(x)) = 0,\ \Re(\ell_i(x)) < 0 \\ i, & \text{if } \Im(\ell_i(x)) > 0 \\ j, & \text{if } \Im(\ell_i(x)) < 0 \end{cases}$$

## 3. Complex hyperplane arrangements

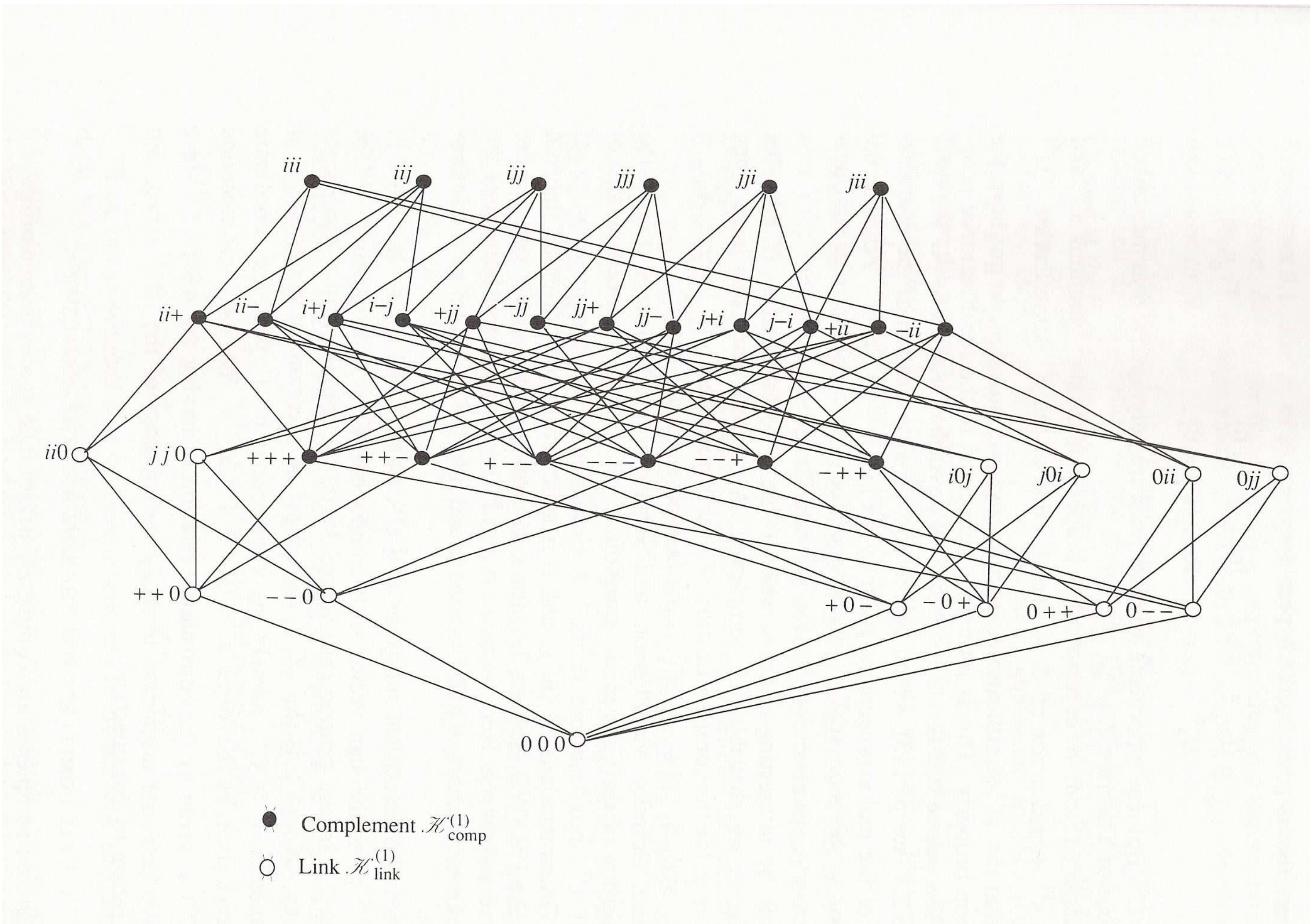
Face poset: $F_{\mathcal{A}} = \sigma(\mathbb{C}^d) \subseteq \{0, +, -, i, j\}^t$

— ordered componentwise by



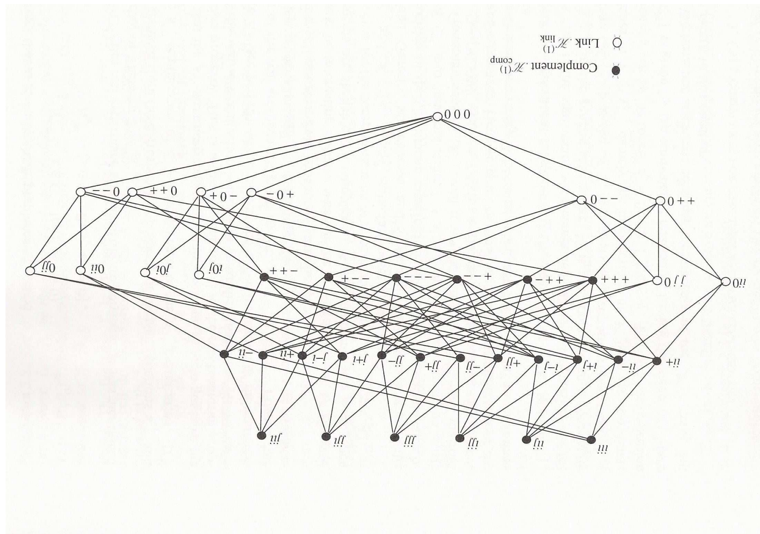

**Proposition 4.** *$F_{\mathcal{A}}$ is a ranked poset of length $2d$*

## 3. Complex hyperplane arrangements




Face poset of arrangement of 3 lines in $\mathbb{C}^2$

## 3. Complex hyperplane arrangements



**Theorem.** *(Bj-Ziegler 1992)*
*(a)* $F_{\mathcal{A}}$ *determines cell structure of regular CW-decomposition of the unit sphere in* $\mathbb{R}^{2d} \cong \mathbb{C}^d$
*(b)* $C_{\mathcal{A}} \stackrel{\text{def}}{=} F_{\mathcal{A}} \cap \{+, -, i, j\}^t$ *with opposite order determines cell structure of a regular CW complex having the homotopy type of the complement* $M_{\mathcal{A}}$.

Composition: $Z \circ W \in \{0, +, -, i, j\}^t$ defined by

$$(Z \circ W)_i = \begin{cases} Z_i, & \text{if } W_i \not> Z_i \\ W_i, & \text{if } W_i > Z_i \end{cases}$$

- associative, idempotent, unit element $= (0, \ldots, 0)$
- $X, Y \in F_{\mathcal{A}} \;\Rightarrow\; X \circ Y \in F_{\mathcal{A}}$
  (geometric reason: move $\varepsilon$ distance from $X$ toward $Y$)

## 3. Complex hyperplane arrangements

**Proposition 5.** *$(F_{\mathcal{A}}, \circ)$ is LRB semigroup.*

What is its support lattice?

# 3. Complex hyperplane arrangements

For $\mathbb{C}$-arrangements, notion of intersection lattice splits into two.

**1.** The *intersection lattice* $L_{\mathcal{A}}$: all intersections of subfamilies of hyperplanes $H_i$ ordered by reverse inclusion.

**2.** The *augmented intersection lattice* $L_{\mathcal{A},\,\mathsf{aug}}$: all intersections of subfamilies of the *augmented arrangement*

$$\mathcal{A}_{\mathsf{aug}} = \{H_1, \ldots, H_t, H_1^{\mathbb{R}}, \ldots, H_t^{\mathbb{R}}\}$$

ordered by reverse inclusion.

Here, $H_i^{\mathbb{R}} \stackrel{\mathsf{def}}{=} \{z \in \mathbb{C}^d : \Im(\ell_i(z)) = 0\}$ is a $(2d-1)$-dimensional real hyperplane in $\mathbb{C}^d \cong \mathbb{R}^{2d}$ containing $H_i$.

**Proposition 6.** *$(F_{\mathcal{A}}, \circ)$ is an LRB semigroup with support lattice $L^{\mathsf{op}}_{\mathcal{A},\mathsf{aug}}$. The support map*

$$\mathsf{supp} : F_{\mathcal{A}} \to L^{\mathsf{op}}_{\mathcal{A},\mathsf{aug}}$$

*sends a "sign vector" $Z \in F_{\mathcal{A}}$ to the intersection of all subspaces in $\mathcal{A}_{\mathsf{aug}}$ that contain the corresponding stratum.*

## 3. Complex hyperplane arrangements

Consequence: $\exists$ theory of random walks on $\mathbb{C}$-arrangements

Different versions exist

obtained by choosing various sub-LRB-semigroups of the complex sign vector semigroup $(F_{\mathcal{A}}, \circ)$

## 4. Complexified $\mathbb{R}$-arrangements

All forms $\ell_i(z)$ have $\mathbb{R}$-coefficients $\rightarrow \begin{cases} \text{real arrangement } \mathcal{A}^{\mathbb{R}} \\ \text{complex arrangement } \mathcal{A}^{\mathbb{C}} \end{cases}$

Construction specializes to that of M. Salvetti.

Fact: $F_{\mathcal{A}^{\mathbb{C}}}$ is determined by $F_{\mathcal{A}^{\mathbb{R}}}$, namely,

$$\begin{aligned} \phi : \mathrm{Int}(F_{\mathcal{A}^{\mathbb{R}}}) &\rightarrow F_{\mathcal{A}^{\mathbb{C}}} \\ [Y, X] &\mapsto X \circ iY \end{aligned}$$

is a poset isomorphism.

Here $\mathrm{Int}(F_{\mathcal{A}^{\mathbb{R}}}) \stackrel{\text{def}}{=}$ set of intervals in the real face poset $F_{\mathcal{A}^{\mathbb{R}}}$

# 4. Complexified $\mathbb{R}$-arrangements

Structure of $F_{\mathcal{A}^{\mathbb{C}}}$ in terms of intervals Int($F_{\mathcal{A}^{\mathbb{R}}}$)

Order:

$$[Y, X] \le [R, S] \iff \begin{cases} Y \le R \\ R \circ X \le S \end{cases}$$

Composition:

$$[Y, X] \circ [R, S] = [Y \circ R, Y \circ R \circ X \circ S]$$

## 5. The real braid arrangement

$\mathcal{A} = \{x_i - x_j \mid 1 \leq i < j \leq n\}$ in $\mathbb{R}^n$.

Intersection lattice $L_{\mathcal{A}} \cong \Pi_n$ (set partitions, refinement)

Ex: $(134 \mid 27 \mid 5 \mid 6) \leftrightarrow \begin{cases} x_1 = x_3 = x_4 \\ x_2 = x_7 \end{cases}$

Face semilattice $F_{\mathcal{A}} \cong \Pi_n^{\text{ord}}$ (ordered set partitions, coarsening)

Ex: $\langle 134 \mid 6 \mid 27 \mid 5 \rangle \leftrightarrow \begin{cases} x_1 = x_3 = x_4 \\ < x_6 \\ < x_2 = x_7 \\ < x_5 \end{cases}$

Complement: Regions $C_{\mathcal{A}} \cong S_n$ (permutations of $[n]$)

Composition in $F_{\mathcal{A}}$:

If $X = \langle\, X_1, \ldots, X_p \,\rangle$ and $Y = \langle\, Y_1, \ldots, Y_q \,\rangle$, $X_i, Y_j \subseteq [n]$, then

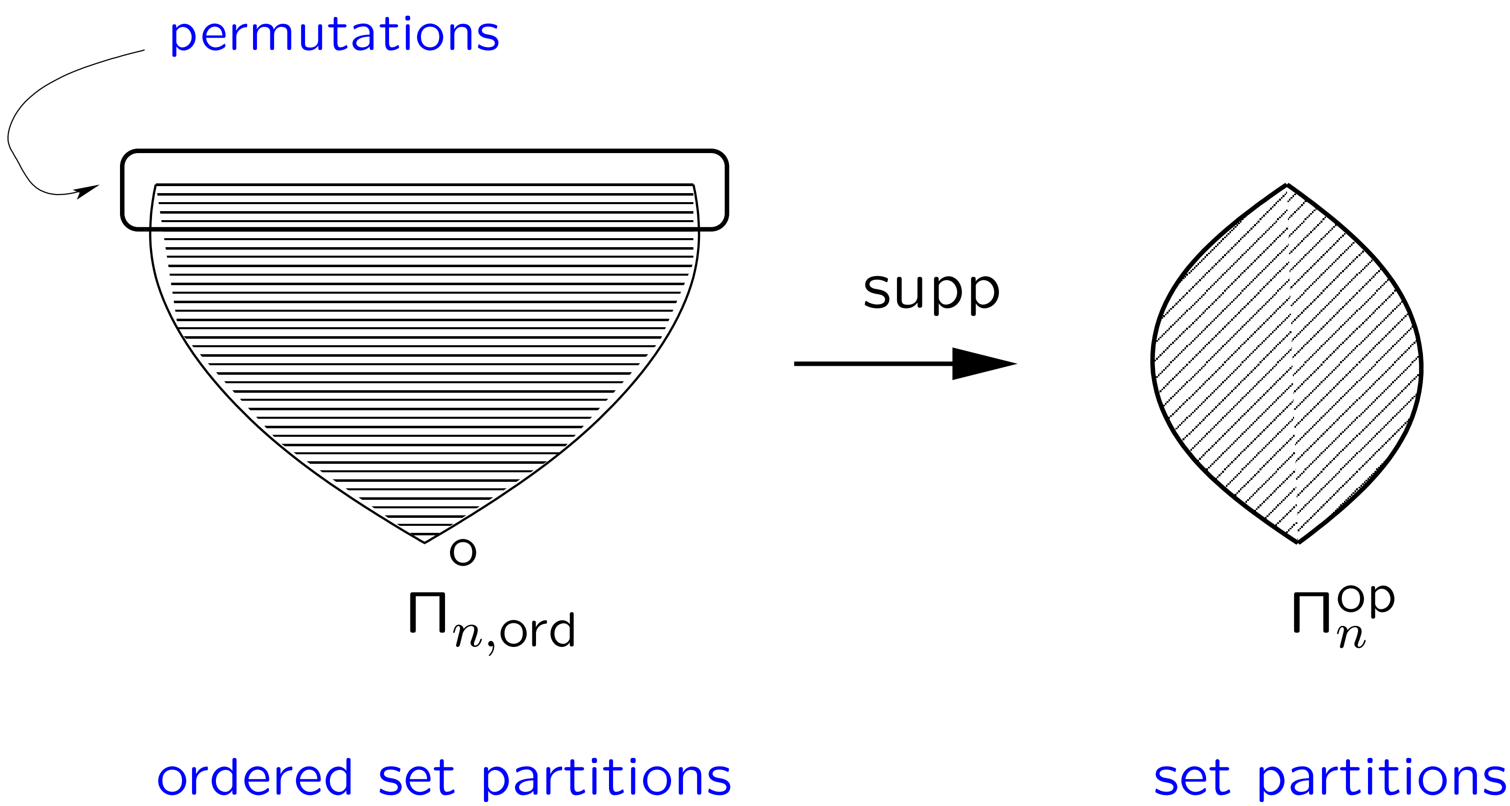
$$X \circ Y = \Big\langle\, X_i \cap Y_j \,\Big\rangle,$$

blocks ordered lexicographically by indices $(i, j)$

Ex: $\langle\, 257 \mid 3 \mid 146 \,\rangle \circ \langle\, 17 \mid 25 \mid 346 \,\rangle = \langle\, 7 \mid 25 \mid 3 \mid 1 \mid 46 \,\rangle$

## 5. The real braid arrangement

### "The picture"



## 6. The complex braid arrangement

$\mathcal{A} = \{x_i - x_j \mid 1 \le i < j \le n\}$ in $\mathbb{C}^n$.

Intersection lattice $L_{\mathcal{A}} \cong \Pi_n$
(set partitions)

Augmented intersection lattice $L_{\mathcal{A},\,\mathrm{aug}} \cong \mathrm{Int}(\Pi_n)$
(intervals of set partitions)

Face semilattice $F_{\mathcal{A}} \cong \mathrm{Int}(\Pi_n^{\mathrm{ord}})$
(intervals $[Y, X]$ in semilattice of ordered set partitions)

Complement $C_{\mathcal{A}} \cong$ intervals $[Y, X]$, $X$ maximal
$\leftrightarrow$ permutation $X$ divided into blocks $Y$

## 6. The complex braid arrangement

Complement $C_{\mathcal{A}} \cong$ intervals $[Y, X]$, $X$ maximal
$\leftrightarrow$ permutation $X$ divided into blocks $Y$

Block-divided permutations $\leftrightarrow$ library placements of books

So,

Complement $C_{\mathcal{A}} \leftrightarrow$ library placements of books

This is how we get random walk on library placements of books

## 7. Walks on complex hyperplane arrangements

Returning to earlier slide ...

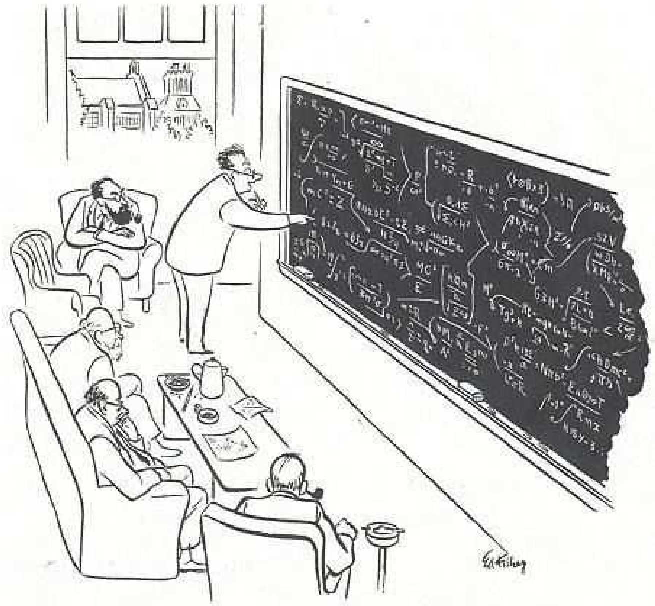
$\exists$ theory of random walks on $\mathbb{C}$-arrangements

Different versions exist

obtained by choosing various sub-LRB-semigroups of the complex sign vector semigroup $(F_{\mathcal{A}}, \circ)$

## 7. Walks on complex hyperplane arrangements

Working out the combinatorial details for one sub-LRB random walk on the complex braid arrangement ..........



*"Say, I think I see where we went off. Isn't eight times seven fifty-six?"*

...... we arrive at the description of eigenvalues of the transition matrix of the $k$-shelf library walk, as stated earlier.

## References


- Random walks, arrangements, cell complexes, greedoids, and self-organizing libraries, in *"Building Bridges"* (eds. M. Grötschel and G. O. H. Katona),
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Available at www.math.kth.se/$\sim$bjorner