First MSJ Seasonal Institute – Kyoto 2008
Probabilistic Approach to Geometry
Heat kernel estimates

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Probabilistic Approach to Geometry

Heat kernel estimates, IV

$$\lim_{t \to 0} \left(-t \log \mathbf{P}_\mu(X_0 \in A \;\&\; X_t \in B)\right) = \frac{d(A, B)^2}{4}$$

Dirichlet spaces of Harnack type

M is a locally compact metrizable space equipped with a Borel measure μ, finite on compact sets and positive on open sets, and a strictly local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$. Denote by d its intrinsic distance defined by:
$d(x,y) = \sup\{f(x) - f(y) : f \in \mathcal{F} \cap \mathcal{C}_c,\ d\Gamma(f,f) \leq d\mu\}$.
Assume: (A1) d is finite continuous and defines the toplogy of M, (A2) (M,d) is complete. We then get a complete length space.

Theorem (Sturm (1995))

The following are equivalent:

- *Volume doubling and Poincaré*
- *Two-sided heat kernel Gaussian bound* $p(t,x,y) \simeq \frac{1}{V(x,\sqrt{t})} e^{-d(x,y)^2/t}$.
- *Parabolic Harnack inequality*

Call a space with these equivalent properties a Dirichlet space of Harnack type.

Neumann and Dirichlet heat kernels in domains

Consider a domain U (say unbounded) in Euclidean space or in a Dirichlet space of Harnack type. We can define two heat semigroups. The Neumann heat semigroup $H_t^{N,U}$ and the Dirichlet heat semigroup $H_t^{D,U}$.

For which class of domains can one obtains two-sided heat kernel bounds?

The Neumann heat kernel in $U = \mathbb{R}^n_+ = \{x : x_n > 0\}$ equals

$$p_U^N(t,x,y) = \frac{1}{(4\pi t)^{n/2}} \left(e^{-\frac{\|y-x\|^2}{4t}} + e^{-\frac{\|y'-x\|^2}{4t}} \right)$$

The Dirichlet heat kernel in $U = \mathbb{R}^n_+ = \{x : x_n > 0\}$ equals

$$p_U^D(t,x,y) = \frac{1}{(4\pi t)^{n/2}} \left(e^{-\frac{\|y-x\|^2}{4t}} - e^{-\frac{\|y'-x\|^2}{4t}} \right)$$

Here, y' is the symmetric of y w.r.t. hyperplane $\{x_n = 0\}$.

Neumann and Dirichlet heat kernels in domains

The Neumann heat kernel in $U = \mathbb{R}^n_+ = \{x : x_n > 0\}$ equals

$$\begin{aligned} p_U^N(t,x,y) &= \frac{1}{(4\pi t)^{n/2}} \left(e^{-\frac{\|y-x\|^2}{4t}} + e^{-\frac{\|y'-x\|^2}{4t}} \right) \\ &\simeq \frac{1}{t^{n/2}} e^{-\frac{\|y-x\|^2}{t}} \end{aligned}$$

The Dirichlet heat kernel in $U = \mathbb{R}^n_+ = \{x : x_n > 0\}$ equals

$$\begin{aligned} p_U^D(t,x,y) &= \frac{1}{(4\pi t)^{n/2}} \left(e^{-\frac{\|y-x\|^2}{4t}} - e^{-\frac{\|y'-x\|^2}{4t}} \right) \\ &\simeq \frac{x_n y_n}{t^{n/2}(x_n + \sqrt{t})(y_n + \sqrt{t})} e^{-\frac{\|y-x\|^2}{4t}}. \end{aligned}$$

Inner uniform domains

A domain U is uniform if for any two points x,y in U there exists a curve joining x to y of length at most $C_0 d(x,y)$ and such that, for any z on the curve,

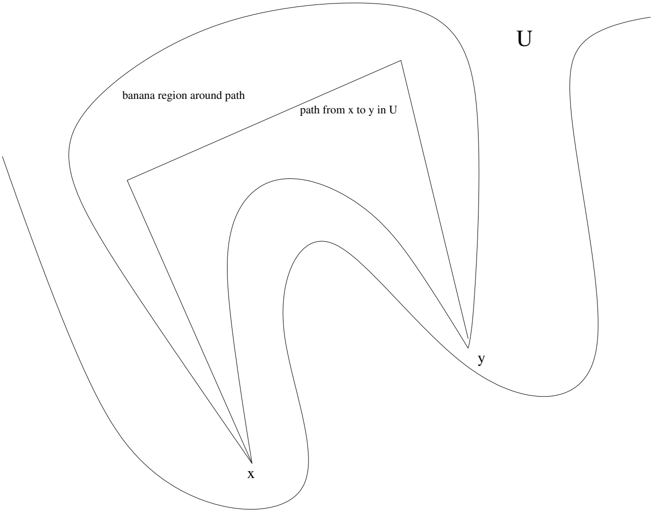
$$\mathrm{dist}(z, U^c) \geq c_0 \frac{d(x,z)d(y,z)}{d(x,y)}.$$

Let d_U be the inner distance in U.
Let $\widetilde{U}$ be the completion of (U, d_U).

A domain U is inner uniform if for any two points x,y in U there exists a curve joining x to y of length at most $C_0 d_U(x,y)$ and such that, for any z on the curve,

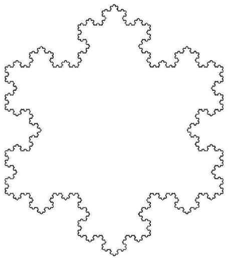
$$\mathrm{dist}(z, U^c) \geq c_0 \frac{d_U(x,z)d_U(y,z)}{d_U(x,y)}.$$

Inner uniform domains



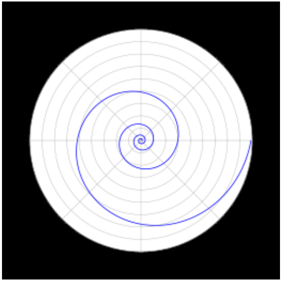
Examples of uniform domains

- Domain above the graph of a Lipschitz function Φ
  $U = \{x = (x_1, \ldots, x_n) : \Phi(x_1, \ldots, x_{n-1}) < x_n\}$
- The inside and outside of the snow flake



Examples of inner uniform domains

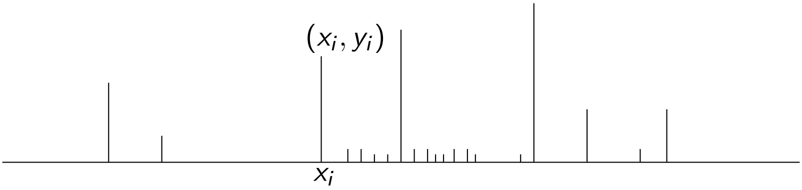
- Complement of any convex set (e.g., the outside of a parabola)
- Complement of the logarithmic spiral



Slited planes

Given a (finite or) countable family $\mathbf{f} = \{(x_i, y_i))\} \subset \mathbb{R}^2_+$ of points in the upper-half plane, let $\mathbb{R}^2_{+\mathbf{f}}$ be the upper-half plane with the vertical segments $s_i = \{z = (x_i, y) : 0 < y \leq y_i\}$ deleted.

It is easy to check that the domain $\mathbb{R}^2_{+\mathbf{f}}$ is inner uniform if and only if there is a constant $c > 0$ such that for any pair (i, j), $|x_i - x_j| \geq c \min\{y_i, y_j\}$. Such domains are never uniform if there is at least one non trivial slit.



Half-spaces in the Heisenberg group

Let $M = \mathbb{H}_3 = \{(x, y, z) : x, y, x \in \mathbb{R}\}$ be the Heisenberg group:
$(x, y, z) \cdot (x', y, z') = (x + x', y + y', z + z' + (1/2)(yx' - xy'))$
The Lie algebra is spanned by X, Y, Z, $[X, Y] = Z$.

The canonical sub-Riemannian geometry is associated with the sub-Laplacian $X^2 + Y^2$ and Dirichlet form $\int(|Xf|^2 + |Yf|^2)d\mu$. This space is of Harnack type with $V(r) = cr^4$.

A. Greshnov (2001) shows that

- The lateral half-spaces $U_> = \{(x, y, z) \in \mathbb{R}^3 : x > 0\}$ (of course, x can be replaced by y),
- The upper half-space $U_+ = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$,

are uniform in $\mathbb{H}_3$ equipped with its canonical sub-Riemanian metric.

The Neumann kernel in inner uniform domains

Theorem (P. Gyrya, LSC, 2007)

Let M be a Dirichlet space of Harnack type with a carré du champ operator. Let U be an inner uniform domain. Then the Dirichlet form

$\mathcal{E}_U^N(f,f) = \int \Gamma(f,f)d\mu, \; f \in \mathcal{F}_U^N$

$\mathcal{F}_U^N = \left\{ f \in L^2(U,\mu) \cap \mathcal{F}_{\mathrm{loc}}(U) : \int_U \Gamma(f,f)d\mu < \infty . \right\}.$

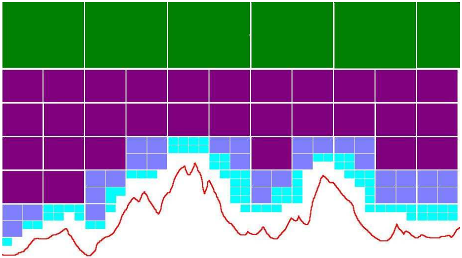
is a strictly local regular Dirichlet form on $\widetilde{U}$ which is of Harnack type and satisfies

$$p_U^N(t,x,y) \simeq \frac{1}{V(x,\sqrt{t})} \exp\left(-\frac{d_U(x,y)^2}{t} \right).$$

The global volume and inner volume are comparable $V(x,r) \simeq V_U(x,r)$.

The Neumann kernel: sketch of proof

(1) The doubling condition is easily seen to be satisfied.
(2) Check Poincaré using a (two step) Whitney covering argument.
(3) Doubling and Poincaré imply regularity of the Dirichlet form (e.g., Hajlasz)



Dirichlet heat kernel

What about the Dirichlet heat kernel?
When can we obtain two-sided bounds?

The function h

On a non-compact domain U, we seek a function $h > 0$, harmonic and vanishing at the boundary.

In the upper-half space $\mathbb{R}^n_+$, this is $h(x) = x_n$.

If U is non-compact and inner uniform in a Dirichlet space of Harnack type then such a function h exists, is unique up to a positive multiplicative constant and is roughly increasing as a function of the distance to the boundary in the following sense:

There exists $a, C \in (0, \infty)$ such that $h(y) \leq Ch(x_r)$ for all $x \in U$, $r > 0$ and $y \in B_U(x, r)$ where x_r is any point $d_U(x, x_r) = r$ and $d(x_r, U^c) \geq ar$.

(See, Aikawa, Ancona, Gyrya/LSC).

The Doob transform I

Consider the Dirichlet forms:

$\mathcal{E}_{h^2}^{D,U}(f,f) = \int_U \Gamma(f,f) h^2 d\mu$, $f \in \mathcal{F}_{h^2}^{D}(U) =$ closure of $\mathcal{F}_c(U)$.

$\mathcal{E}_{h^2}^{N,U}(f,f) = \int_U \Gamma(f,f) h^2 d\mu$,

$f \in \mathcal{F}_{h^2}^{N}(U) = \{f \in L^2(h^2 d\mu) \cap \mathcal{F}_{\text{loc}}(U) : \int_U \Gamma(f,f) h^2 d\mu < \infty\}$.

and the form $\mathcal{E}_U^{h^2}(f,f) = \int_U \Gamma(hf,hf) d\mu$, $f \in h^{-1}\mathcal{F}^D(U)$.

If h is harmonic positive in U and vanishes at the boundary and if U is non-compact inner uniform then:

(a) all these forms are equal to a strictly local regular Dirichlet form satisfying (A1)-(A2) on $\widetilde{U}$.

(b) they are of Harnack type.

Sketch of proof

By definition, $\mathcal{E}_{h^2}^{D,U}(f,f) = \int_U \Gamma(f,f) h^2 d\mu$, $f \in \mathcal{F}_{h^2}^D(U) =$ closure of $\mathcal{F}_c(U)$ is a strictly local regular Dirichlet form on U.

A simple computation shows that it equals $\mathcal{E}_U^{h^2}(f,f) = \int_U \Gamma(hf,hf) d\mu$, $f \in h^{-1}\mathcal{F}^D(U)$, when h has the assumed properties.

One then shows that $\mathrm{Lip}_c(\widetilde{U})$ is a core for both $\mathcal{E}_{h^2}^{D,U}(f,f)$ and $\mathcal{E}_{h^2}^{N,U}(f,f)$. Regularity on $\widetilde{U}$ and (A1)-(A2) follow.

Finally, one repeats the proof of Poincaré with the weight h^2. For that, one needs to check the doubling property for $h^2 d\mu$.

The Doob transform II

Because $\mathcal{E}_U^{h^2}(f,f) = \int_U \Gamma(hf,hf)d\mu$, $f \in h^{-1}\mathcal{F}^D(U)$ is a Dirichlet form of Harnack type, its heat kernel satisfies

$$p_U^{h^2}(t,x,y) \simeq \frac{1}{\sqrt{V_{U,h^2}(x,\sqrt{t})V_{U,h^2}(x,\sqrt{t})}} \exp\left(-\frac{d_U(x,y)^2}{t}\right).$$

but, also, we have $p_U^D(t,x,y) = h(x)h(y)p_U^{h^2}(t,x,y)$. Hence

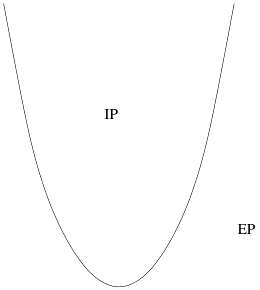
$$p_U^D(t,x,y) \simeq \frac{h(x)h(y)}{\sqrt{V_{U,h^2}(x,\sqrt{t})V_{U,h^2}(x,\sqrt{t})}} \exp\left(-\frac{d_U(x,y)^2}{t}\right).$$

$V_{U,h^2}(x,r) \simeq V(x,r)h(x_r)^2$

The exterior of a parabola

Consider the exterior of the parabola $y = x^2$ in the plane, that is, $V = \{x = (x_1, x_2) = x_2 < x_1^2\}$. Then

$$h(x) = \sqrt{2\left(\sqrt{x_1^2 + (1/4 - x_2)^2} + 1/4 - x_2\right)} - 1.$$



The exterior of a parabola

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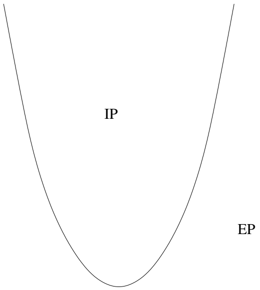

$$P_U^D(t,x,y) \simeq \frac{h(x)h(y)}{th(x_{\sqrt{t}})h(y_{\sqrt{t}})} \exp\left(-\frac{d_U(x,y)^2}{t}\right).$$

In particular, for any fixed $x, y \in U$,
$p_U^D(t,x,y) \simeq t^{-3/2}$ as t tends to infinity.
(compare with the upper-half plane for which the corresponding behavior given earlier is in t^{-2}).

Probability & Geometry

Thank you!