First MSJ Seasonal Institute – Kyoto 2008
Probabilistic Approach to Geometry
Heat kernel estimates

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Probabilistic Approach to Geometry

Heat kernel estimates, I

$$\lim_{t \to 0} \left(-t \log \mathbf{P}_\mu (X_0 \in A \;\&\; X_t \in B) \right) = \frac{d(A,B)^2}{4}$$

This is the Hino-Ramirez version of Varadhan formula relating the probability that Brownian motion moves from A to B (left-hand side) to the distance between the two sets (right-hand side)

Plan of the 4 lectures

$$p(t,x,y) \simeq \frac{1}{V(x,\sqrt{t})} \exp\left(-\frac{d(x,y)^2}{t}\right).$$

(1) An overview of diffusive heat kernel upper bounds and two-sided bounds

(2-3) Manifolds with finitely many ends

(4) Heat kernels with Dirichlet boundary conditions.

The ubiquitous heat kernel

$$\frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{\|x-y\|^2}{4t}\right)$$

The various guises of the heat kernel $p(t, x, y)$:

1. The fundamental solution of the most basic parabolic PDE $(\partial_t - \Delta)u = 0$.
2. The kernel of the heat semigroup $e^{t\Delta}$.
3. The density of the distribution of the position at time t of the stochastic process driven by Δ.

$$u(t,x) = e^{t\Delta}u_0(x) = \int p(t,x,y)u_0(y)dy = \mathbb{E}_x(u_0(X_t)).$$

$$(\partial_t - \Delta)u = 0, \ \ u(0,x) = u_0(x).$$

The ubiquitous heat kernel

The various applications of the heat kernel/semigroup:

1. Smoothing/appoximation $e^{t\Delta} f \xrightarrow{t \to 0} f$
2. Defining/studying other objects:
   - Function spaces, e.g., Hardy spaces $H_1 = \{f \in L^1 : \sup_{t>0} |H_t f| \in L^1\}$.
   - Operators, e.g., $m(\Delta)$, Green function, Riesz transforms.
   - Spectral theory of $\Delta + V$.
   - Subordination: $e^{-t(-\Delta)^\alpha}$.
3. Large time behavior of the sample paths of BM: recurrence/transience, rate of escape
4. Capture some geometric properties: amenability, isoperimetry
5. The heat kernel measure $\mu_t^x(dy) = p(t,x,y)dy$, $L^2(\mu_t^x)$.

The ubiquitous heat kernel

Many different setups, many variants:

- In $\mathbb{R}^n$, $\Delta = \sum_1^n \left(\frac{\partial}{\partial x_i}\right)^2$
  - Uniformly elliptic operators: $\sum_{i,j} \frac{\partial}{\partial x_i} a_{i,j}(x) \frac{\partial}{\partial x_j}$
  - Heat equation in domains with Neumann/Dirichlet boundary conditions. Lower order terms, potentials.
  - Non linear variants: $\partial_t f - \mathrm{div}(|\nabla f|^{p-1} \nabla f) = 0$, $\partial_t f - \Delta f^m = 0$.
- On Riemannian manifolds: $\Delta = \mathrm{div}\,\mathrm{grad}$, $dd^* + d^*d$ (forms, tensors).
- On Lie groups: $\Delta = \sum X_i^2$, subRiemannian geometry.
- Dirichlet forms.
- Finsler geometry semigroups/ Aspects of mass transport

What are the fundamental questions/results?

- Asymptotic expansion as $t \to 0$:

$$p(t,x,y) \sim \frac{1}{(4\pi t)^{n/2}} e^{-d(x,y)^2/4t} \left(\sum_0^\infty A_k(x,y) t^k \right).$$

- Behavior as $t \to \infty$:
  - $p(t,x,y) \sim ?$, $\varphi(t) = \sup_x \{p(t,x,x)\} \sim ?$
  - Recurrence/transience; parabolicity/non-parabolicity;
  - Amenability, Isoperimetry;
  - Volume growth, other geometric invariants.
- Gaussian behavior: Varadhan's result

$$\lim_{t\to 0} 4t \log p(t,x,y) = -d(x,y)^2.$$

- Large scale space-time estimates: Aronson's estimate

$$p(t,x,y) \simeq t^{-n/2} e^{-c\|x-y\|^2/t}$$

for uniformly elliptic operators on $\mathbb{R}^n$.

On-diagonal behavior: the classical case

Let (M, g) be a complete (non-compact) Riemanian manifold. What controls the behavior of $p(t, x, x)$, $\sup_x\{p(t, x, x)\}$?

- The bound $\sup_x\{p(t, x, x)\} \leq Ct^{-n/2},\ t > 0$ is equivalent to Sobolev/Nash/Faber-Krahn/RCL inequalities:
  - Sobolev ($n > 2$): $\|f\|_{2n/(n-2)} \leq C\|\nabla f\|_2$.
  - Nash: $\|f\|_2^{2(1+2/n)} \leq C^2\|\nabla f\|_2^2\|f\|_1^{4/n}$.
  - Faber-Krahn $\lambda_D(\Omega) \geq c|\Omega|^{-2/n}$.
  - RCL ($n > 2$): $\mathcal{N}_-(-\Delta + V) \leq C\int V_-^{n/2} d\lambda$ where $\mathcal{N}_-(A)$ is the number of negative eigenvalues of A in L^2, ($n > 2$).

RCL=Rozenblum-Cwikel-Lieb; The equivalence with the Sobolev inequality, in a wider context, is a theorem of Varopoulos.

On-diagonal behavior

Consider a pair of positive monotone functions v, Λ with Λ decreasing continuous and related to v by

$$t = \int_0^{v(t)} \frac{ds}{s\Lambda(s)}$$

equivalently,

$$v'(t) = v(t)\Lambda(v(t)),\ v(0) = 0.$$

Assume that $u = v'/v$ satisfies $u(At) \geq au(t)$ for some $0 < a < 1 < A$.

Theorem (A. Grigor'yan)

$$\sup_x \{p(t,x,x)\} \leq \frac{C}{v(ct)} \iff \lambda_D(\Omega) \geq c\Lambda(c|\Omega|).$$

On-diagonal behavior and volume growth

Consider the volume growth condition $(V(x,r) = \mu(B(x,r))$

$$\inf_x\{V(x,t)\} \geq ct^d, \quad t > 1.$$

- This condition implies

$$\sup_x\{p(t,x,x)\} \leq Ct^{-d/(d+1)}, t > 1.$$

- If we add the condition that (this is called a pseudo-Poincaré inequality, $f_r(x) = V(x,r)^{-1} \int_{B(x,r)} f d\lambda$)

$$\|f - f_r\|_2 \leq Cr\|\nabla f\|_2, \quad f \in \mathcal{C}_c(M), \; r > 0$$

  then we get the much stronger result that

$$\sup_x\{p(t,x,x)\} \leq Ct^{-d/2}.$$

Flat Gaussian bounds

Consider a pair of positive monotone functions v, Λ with Λ decreasing continuous and related as before by

$$v'(t) = v(t)\Lambda(v(t)),\ v(0) = 0.$$

Assume that $u = v'/v$ satisfies $u(At) \geq au(t)$ for some $0 < a < 1 < A$.

Theorem (E.B. Davies, A. Grigor'yan, ...)

The condition

$$\lambda_D(\Omega) \geq c\Lambda(c|\Omega|),\ \ \Omega \subset M$$

implies

$$\left|\left(\frac{\partial}{\partial t}\right)^m p(t,x,y)\right| \leq \frac{C_{m,\epsilon}}{t^m v(ct)} \exp\left(-\frac{d(x,y)^2}{(4+\epsilon)t}\right)$$

Localized on-diagonal behavior

The following conditions are equivalent for a ball $B_0 = B(x_0, R_0)$:

- Localized Sobolev inequality (some $\alpha > 2$):

$$\left(\int_B |f|^{2\alpha/(\alpha-2)} d\lambda\right)^{(\alpha-2)/\alpha} \leq \frac{Cr^2}{V(x,r)^{2/\alpha}} \int_B (|\nabla f|^2 + r^{-2}|f|^2) d\lambda$$

$B = B(x, r)$, $f \in \mathcal{C}_c(B)$, $x \in B_0$, $r \in (0, R_0)$.

- Localized Faber-Krahn inequality (some $\alpha > 0$):

$$\lambda_D(U) \geq \frac{c}{r^2}\left(\frac{V(x,r)}{|U|}\right)^{2/\alpha}, \; U \subset B(x,r), \;\; x \in B_0, r \in (0, R_0).$$

- Doubling and on-diagonal upper bound (some $\alpha > 0$):

$$\frac{V(x,r)}{V(x,s)} \geq c\left(\frac{r}{s}\right)^{\alpha}, \;\; p(t,x,x) \leq \frac{C}{V(x,\sqrt{t})},$$

$x \in B_0$, $0 < t < R_0^2$, $0 < r < s < R_0$.

Doubling and Poincaré

Theorem (Grigor'yan, 91; LSC, 92)

For fixed $R \in (0, \infty]$, the following properties are equivalent:

(a) *The conjunction of*

- *The doubling property: $V(x, 2r) \leq DV(x, r)$, for all $r \in (0, R)$.*
- *The Poincaré inequality: For all $r \in (0, R)$, $B = B(x, r)$,*

$$\forall\ f \in Lip(B), \quad \int_B |f - f_B|^2 d\mu \leq Pr^2 \int_B |\nabla f|^2 d\mu.$$

(b) *The two-sided Gaussian bound: for all x, y, $t \in (0, \sqrt{R})$,*

$$p(t, x, y) \simeq \frac{1}{V(x, \sqrt{t})} \exp\left(-\frac{d(x, y)^2}{t}\right).$$

(c) *The parabolic Harnack inequality up to scale R.*

Examples with $R = \infty$

- Convex domains in Euclidean space.
- Complete Riemannian manifolds with $\mathrm{Ric} \geq 0$.
  Bishop-Gromov (Cheeger-Gromov-Taylor) and P. Buser, 1982.
  Li-Yau, 1986.
- Lie groups with polynomial volume growth.
  Gromov 1981, Varopoulos 1987.
- Quotients of any such space by an isometric group action.
- Spaces that are (measure) quasi-isometric to such a space.
  (Kanai, Coulhon, LSC)
- Coverings of compact manifolds which have polynomial volume growth
- and more ...

References

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