シュワルツ微分の幾何と解析
Geometry and Analysis of Schwarzian Derivative

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Introduction

Contents

What is the Schwarzian derivative?

Let $f(z)$ be a non-constant meromorphic function of one complex variable. We define

- the pre-Schwarzian derivative: $T_f = \dfrac{f''}{f'}$
- the Schwarzian derivative:

$$S_f = (T_f)' - \frac{1}{2}(T_f)^2 = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

Traditionally, S_f is denoted by $\{f(z), z\}$.
Note that T_f and S_f are meromorphic functions.

Basic properties of pre-Schwarzian derivative

Since $T_f = [\log f']'$, we have the following:

- $T_f = 0$ iff f is an affine transformation of the form $f(z) = az + b$ $(a \neq 0)$.
- $T_{g \circ f} = (T_g \circ f) \cdot f' + T_f$
- In particular,

  $$T_{g \circ f} = T_f, \quad T_{f \circ g} = (T_f \circ g) \cdot g'$$

  for an affine transformation g.
- $T_f(z)$ is analytic at $z = z_0$ iff f is a locally univalent holomorphic at z_0.

Basic properties of Schwarzian derivative

- $S_f = 0$ iff f is a Möbius transformation.
- $S_{g\circ f} = (S_g \circ f) \cdot (f')^2 + S_f$
- In particular,

$$S_{g\circ f} = S_f, \quad S_{f\circ g} = (S_f \circ g) \cdot (g')^2$$

  for a Möbius transformation g. By using them, $S_f(z)$ can be defined when $f(z) = \infty$ or $z = \infty$.
- $S_f(z)$ is analytic at $z = z_0$ iff f is locally univalent meromorphic at z_0.

Examples

Let us look at a couple of examples. First, for $w = e^z$, we have $w' = e^z$ so that $T_w = (\log w')' = 1$. Hence,

$$S_{\exp} = -\frac{1}{2}.$$

We next consider $w = z^\alpha$ for a constant $\alpha \neq 0$. Then, $w' = \alpha z^{\alpha-1}$ and thus $T_w = (\alpha - 1)/z$. Finally, we get

$$\{w, z\} = -\frac{\alpha - 1}{z^2} - \frac{1}{2} \cdot \frac{(\alpha - 1)^2}{z^2} = \frac{1 - \alpha^2}{2z^2}.$$

Other important examples:

$$\{z/(1-z)^2, z\} = \frac{-6}{(1-z^2)^2}, \quad \{\log z, z\} = \frac{1}{2z^2}.$$

Early History

- J. L. Lagrange (1779) Related computations
- E. Kummer (1836) in Gauss hypergeometric equation
- H. A. Schwarz (1869) defined S_f to construct a "triangle function"
- H. A. Schwarz (1873) solved the Fuchs problem on hypergeometric functions
- A. Cayley (1880) named "Schwarzian derivative"
- H. A. Schwarz (1890) mentioned the work of Lagrange in the notes on his treatise II.

H. A. Schwarz

Karl Hermann Amandus Schwarz
(From Wikipedia)

Schwarz's discovery

dung geradlinig begrenzter Polygone angegebenen ganz analoge Schlussfolgerung zum Ziele.

Wenn die von Kreisbogen begrenzte Figur in der Ebene (u) durch die Function

$$u' = \frac{C_1 u + C_2}{C_3 u + C_4}$$

auf eine Ebene (u') abgebildet wird, so ist die entsprechende Figur in der Ebene (u') ebenfalls von Kreisbogen begrenzt, unter denen sich auch geradlinige Strecken befinden können.

Um ein für alle diese Abbildungen, welche aus einander durch Transformation mittelst reciproker Radii vectores abgeleitet werden können, zugleich geltendes Resultat zu erhalten, eliminire man die Constanten C.

Es ist

$$\frac{d}{dt}\log\frac{du'}{dt} = \frac{d}{dt}\log\frac{du}{dt} - 2\frac{C_3}{C_3u+C_4}\frac{du}{dt}$$

$$\frac{d^2}{dt^2}\log\frac{du'}{dt} = \frac{d^2}{dt^2}\log\frac{du}{dt} + 2\frac{C_3^2}{(C_3u+C_4)^2}\left(\frac{du}{dt}\right)^2 - 2\frac{C_3}{C_3u+C_4}\frac{d^2u}{dt^2}$$

Bezeichnet man die Function cf. Band 71, p

$$\frac{d^2}{dt^2}\log\frac{du}{dt} - \tfrac{1}{2}\left(\frac{d}{dt}\log\frac{du}{dt}\right)^2 \text{ mit } \Psi(u,t),$$

so folgt hieraus, dass $\Psi(u',t) = \Psi(u,t)$ ist, dass mithin der Ausdruck Ψ von den Constanten C unabhängig ist.

H. A. Schwarz, *Ueber einige Abbildungsaufgaben*

Journal für die reine und angewandte Mathematik, **70** *(1869), 105?120.*

Computations due to Schwarz

According to Schwarz, put

$$u' = \frac{C_1 u + C_2}{C_3 u + C_4}$$

for a function $u = u(t)$. Then,

$$T_{u'} = \frac{d}{dt}\log\frac{du'}{dt} = \frac{d}{dt}\left[\log\frac{du}{dt} + \log\frac{C_1C_4 - C_2C_3}{(C_3 z + C_4)^2}\right]$$
$$= \frac{d^2u}{dt^2}\Big/\frac{du}{dt} - 2\frac{C_3}{C_3u + C_4}\frac{du}{dt} = T_u - 2\frac{C_3}{C_3u + C_4}\frac{du}{dt}$$

and

$$\frac{d}{dt}T_{u'} = \frac{d}{dt}T_u + 2\frac{C_3^2}{(C_3u + C_4)^2}\frac{du}{dt} - 2\frac{C_3}{C_3u + C_4}\frac{d^2u}{dt^2}.$$

Hence,

$$S_{u'} = \frac{d}{dt}T_{u'} - \frac{1}{2}T_{u'}^2 = \frac{d}{dt}T_u - \frac{1}{2}T_u^2 = S_u.$$

Differential operator annihilating Möbius maps

The Schwarzian derivative can be thought of as the simplest (non-linear) differential operator which annihilates all the Möbius transformations. Recalling that $w = (az+b)/(cz+d)$ satisfies

$$w' = \frac{ad-bc}{(cz+d)^2},$$

we observe

$$0 = \frac{d^2}{dz^2}(w')^{-1/2} = -\frac{w'''/w' - \frac{3}{2}(w''/w')^2}{2(w')^{1/2}}.$$

More generally, for a function $w = f(z)$, we let $\varphi(z) = 1/\sqrt{f'(z)} = (w')^{-1/2}$ and compute

$$\varphi''(z) = \frac{d}{dz}\left[-\frac{1}{2}w''(w')^{-3/2}\right] = -\frac{\varphi(z)}{2}S_f(z).$$

Lagrange's paper

ou bien (étant indifférent quelle que soit la variable sous les signes des fonctions) on aura plus simplement ces deux-ci

$$\frac{\varphi''(z)}{\varphi(z)} = k, \quad \frac{\Phi''(z)}{\Phi(z)} = k,$$

savoir

$$\frac{d^2\varphi(z)}{\varphi(z)\,dz^2} = k, \quad \frac{d^2\Phi(z)}{\Phi(z)\,dz^2} = k,$$

qui sont intégrables par les règles connues.

On aura donc, en intégrant,

$$\varphi(z) = \mathrm{M}e^{z\sqrt{k}} + \mathrm{N}e^{-z\sqrt{k}}, \quad \Phi(z) = \mathrm{P}e^{z\sqrt{k}} + \mathrm{Q}e^{-z\sqrt{k}},$$

M, N, P, Q étant des coefficients quelconques positifs ou négatifs, réels ou imaginaires.

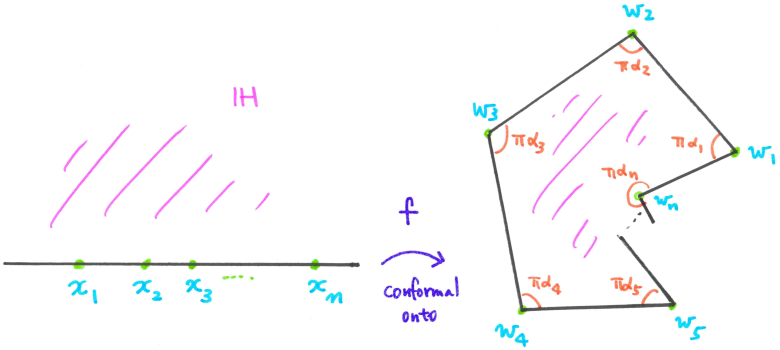
Or

$$\varphi(z) = \frac{1}{\sqrt{f'(z)}}, \quad \Phi(z) = \frac{1}{\sqrt{\mathrm{F}'(z)}};$$

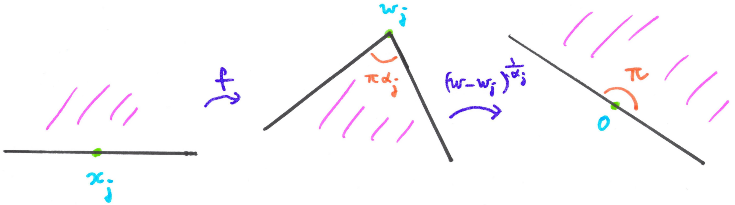
J. L. Lagrange, *Sur la construction des cartes géographiques*, 1779.

Conformal mappings and Schwarzian

Schwarz-Christoffel mapping



Near the vertex



By Schwarz reflection principle, $g(z) := (f(z) - w_j)^{1/\alpha_j}$ extends to a conformal mapping near $z = x_j$. Since $f(z) = w_j + g(z)^{\alpha_j}$, one gets

$$T_f(z) = (\alpha_j - 1)\frac{g'(z)}{g(z)} + T_g(z) = \frac{\alpha_j - 1}{z - x_j} + O(1) \quad z \to x_j.$$

Schwarz-Christoffel formula

Since f behaves like $1/z$ at ∞, we have $T_f(z) = -2/z + O(1/z^2)$ as $z \to \infty$. We thus obtain

$$T_f(z) = \sum_{j=1}^{n} \frac{\alpha_j - 1}{z - x_j},$$

with

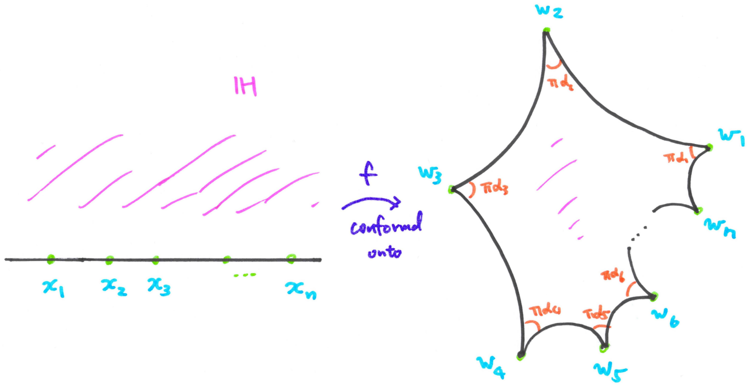
$$\sum_{j=1}^{n} (\alpha_j - 1) = -2.$$

By integrating the above, we finally get

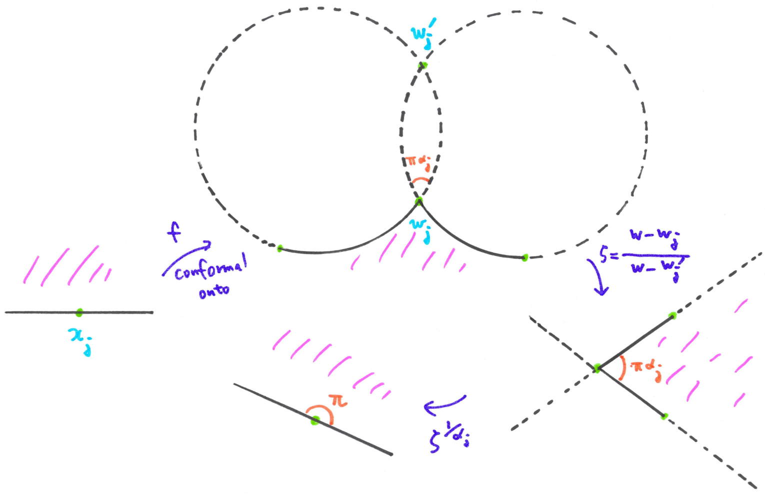
$$f'(z) = C \prod_{j=1}^{n} (z - x_j)^{\alpha_j - 1},$$

where C is a non-zero constant.

Conformal mapping onto curvilinear polygon



Near the vertex



Local analysis

By the Schwarz reflection principle, $g(z) := (L \circ f(z))^{1/\alpha_j}$ extends to a conformal mapping near $z = x_j$, where $L(w) = (w - w_j)/(w - w'_j)$. Since $f(z) = L(g(z)^{\alpha_j})$ and $\{z^\alpha, z\} = (1 - \alpha^2)/2z^2$, one gets

$$S_f(z) = (1 - \alpha_j^2)\frac{g'(z)^2}{2g(z)^2} + S_g(z) = \frac{1 - \alpha_j^2}{2(z - x_j)^2} + \frac{\beta_j}{z - x_j} + O(1)$$

as $z \to x_j$. We can also see that $\beta_j \in \mathbb{R}$.
Since the point at infinity is an ordinary point, we can observe that $S_f(z) = O(1/z^4)$ as $z \to \infty$.

The form of Schwarzian

$$S_f(z) = \sum_{j=1}^{n} \left[\frac{1-\alpha_j^2}{2(z-x_j)^2} + \frac{\beta_j}{z-x_j} \right],$$

where β_j are real numbers with

$$\sum_{j=1}^{n} \beta_j = 0, \quad \sum_{j=1}^{n} (2x_j\beta_j + 1 - \alpha_j^2) = 0,$$
$$\sum_{j=1}^{n} (x_j^2\beta_j + x_j(1-\alpha_j^2)) = 0.$$

Problems:

1. Determination of the parameters β_j
2. Solve the equation $S_f = \varphi$ for a given φ!

We consider the latter problem in the next section.

Inverse problem

ODE of Sturm-Liouville type

Let $\varphi(z)$ be a holomorphic function defined on a plane domain Ω. Find a (possibly multiple-valued) meromorphic function $f(z)$ on Ω such that $S_f = \varphi$. Recall that $\omega = 1/\sqrt{f'}$ satisfies

$$\frac{\omega''}{\omega} = -\frac{1}{2}S_f = -\frac{\varphi}{2}.$$

Consider the ODE

$$2y'' + \varphi y = 2\frac{d^2y}{dz^2} + \varphi(z)y = 0$$

and let y_0, y_1 be two linearly independent solutions. Then the Wronskian $W = W(y_0, y_1) = y_0 y_1' - y_0' y_1$ satisfies $W' = 0$. We may assume that $W \equiv 1$. Then the function $f = y_1/y_0$ satisfies

$$f' = \frac{W}{y_0^2} = y_0^{-2} \quad \Leftrightarrow \quad y_0 = 1/\sqrt{f'},$$

and thus $S_f = \varphi$.

Monodromies

When Ω is not simply connected, solutions to the ODE $2y'' + 2\varphi y = 0$ are not necessarily single-valued.
Let y_0 and y_1 be local solutions to the ODE around $z_0 \in \Omega$ with $W \equiv 1$. Then y_j continues analytically along a curve $\gamma \in \pi_1(\Omega, z_0)$ to $\tilde{y}_j$ near z_0. We can write $\tilde{y}_1 = Ay_1 + By_0$ and $\tilde{y}_0 = Cy_1 + Dy_0$ for some constants A, B, C, D. Since $W(\tilde{y}_0, \tilde{y}_1) = W(y_0, y_1) = 1$, we have $AD - BC = 1$. In this way, we obtain the monodromy homomorphism

$$\rho_\varphi : \pi_1(\Omega, z_0) \to \mathrm{SL}(2, \mathbb{C}),$$

which describes the monodromy for φ

$$\rho_\varphi : \pi_1(\Omega, z_0) \to \mathrm{PSL}(2, \mathbb{C}); \gamma \mapsto \frac{Ay_1 + By_0}{Cy_1 + Dy_0} = \frac{Af + B}{Cf + D}.$$

Fuchsian group

Let Ω be a domain in $\widehat{\mathbb{C}} = \mathbb{C}P^1$ with $\#\partial\Omega \geq 3$ (or, more generally, a hyperbolic Riemann surface). Then, by the Uniformization Theorem, there is an analytic universal covering projection $\pi : \mathbb{H} \to \Omega$. The covering transformation group

$$\Gamma = \{\gamma \in \mathrm{Aut}(\mathbb{H}) = \mathrm{PSL}(2, \mathbb{R}) : \pi \circ \gamma = \pi\}$$

is a (torsion-free) Fuchsian group. Unfortunately, it is almost impossible to obtain an explicit form of π, like Riemann mapping functions.

Poincaré's idea

Even though the inverse mapping of the covering projection π is multiple-valued in general, the branches are related by Möbius transformations $\gamma \in \Gamma$. Therefore, the Scwarzian derivative $\Psi_\Omega = S_{\pi^{-1}}$ is (single-valued) analytic function on Ω.
When Ω is an n-times punctured sphere with $n \geq 3$, the form of Ψ_Ω is determined to some extent:

$$\Psi_\Omega(z) = S_{\pi^{-1}}(z) = \sum_{j=1}^{n} \left[\frac{1}{2(z - z_j)^2} + \frac{\beta_j}{z - z_j} \right],$$

where β_j are complex constants, called accessory parameters. It is, however, not easy to determine the parameters β_j when $n \geq 4$.

Remark

For instance, when $\varphi = \Psi_\Omega$, the above monodromy yields an isomorphism $\pi_1(\Omega, z_0) \to \Gamma$ (up to conjugate). A slight perturbation $\rho_{\Psi_\Omega+\varphi}$ gives a holomorphic deformation of the Fuchsian group Γ.
Even though we do not have exact forms of the solutions, we can compute the monodromy by numerical integration.
Such an approach has been used to visualize the Bers embedding of Teichmüller spaces of dimension 1 (Komori-S. -Wada-Yamashita).
Also, this method can be extended to "uniformize" a Riemann furface with conical singularities.

Nehari's theorem

Let f be a nonconstant meromorphic function on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

- If f is univalent, then

$$(1 - |z|^2)^2 |S_f(z)| \le 6, \quad |z| < 1.$$

- Conversely, if

$$(1 - |z|^2)^2 |S_f(z)| \le 2, \quad |z| < 1,$$

  then f is univalent.

The constants 6 and 2 are sharp.

Proof of Nehari's Theorem

A crusial fact is that the quantity $H(z,f) = (1-|z|^2)^2|S_f(z)|$ is invariant under the pull-back by analytic automorphisms of $\mathbb{D}$. Namely,

$$H(g(z), f\circ g) = H(z,f) \text{ for } g(z) = \frac{\alpha z+\beta}{\bar\beta z+\bar\alpha},\ |\alpha|^2-|\beta|^2=1.$$

The first assertion at the origin is equivalent to $|a_3 - a_2^2| \le 1$ for $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$, which follows from the Bieberbach-Gronwall area theorem.

Proof of Nehari's Theorem, 2

For the second assertion, put $\varphi = S_f$ so that

$$|\varphi(z)| \le \frac{2}{(1-|z|^2)^2}.$$

Suppose, to the contrary, that $f(z_0) \neq f(z_1)$ for $z_0 \neq z_1 \in \mathbb{D}$. By the invariance, we may assume that $z_0 = 0, 0 < z_1 < 1$ and $f = y_1/y_0$ with $2y_j'' + \varphi y_j = 0,\ y_0(0) = y_1'(0) = 0, y_0'(0) = y_1(0) = 1$. We note that the function $F(z) = \mathrm{arctanh} z = \frac{1}{2}\log\frac{1+z}{1-z}$ satisfies

$$S_F(z) = \frac{2}{(1-z^2)^2}.$$

Then a comparison theorem for the solutions to an ODE of Sturm-Liouville type, we have $y_1(x) \neq 0$ for $0 < x < 1$, which is a contradiction.

Bers embedding of the Teichmüller space

Quasiconformal mapping

A(n orientation-preserving) homeomorshism $f : \Omega \to \Omega_1$ between plane domains is called quasiconformal (or qc for short) if f belongs to the Sobolev class $W^{1,2}_{\mathrm{loc}}(\Omega)$ and satisfies

$$|\partial_{\bar z} f(z)| \le k|\partial_z f(z)| \quad \text{a.e. on } \Omega$$

for a constant $k < 1$. It is known that $f_z = \partial_z f \neq 0$ a.e. for a qc map f. The quotient

$$\mu = \mu_f = \frac{\partial_{\bar z} f}{\partial_z f}$$

is a measurable function on Ω with $\|\mu\|_\infty \le k$ and called the Beltrami coefficient of f. The mapping f is sometimes called μ-conformal. For conformal mappings g, h, we have

$$\mu_{g\circ f\circ h} = \mu_f \circ h \cdot \frac{\overline{h'}}{h'}.$$

Thus the notion of quasiconformality extends to Riemann surfaces.

Measurable Riemann mapping theorem

Measurable Riemann mapping theorem

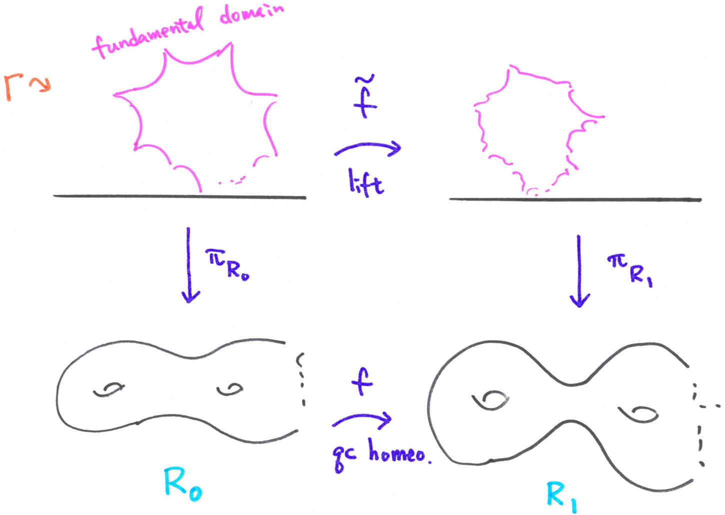
Let $\mu \in L^\infty(\mathbb{C})$ with $\|\mu\|_\infty < 1$. Then there exists a unique quasiconformal homeomorphism $f : \mathbb{C} \to \mathbb{C}$ with $\mu_f = \mu$ and normalized so that $f(0) = 0, f(1) = 1$.

Such a map will be denoted by w^μ.
Let Γ be a subgroup of $\mathrm{Möb} = \mathrm{Aut}(\widehat{\mathbb{C}})$ acting on a domain $\Omega \subset \widehat{\mathbb{C}}$. Set

$$M(\Omega, \Gamma) = \{\mu \in L^\infty(\Omega) : \mu \circ \gamma \cdot \bar{\gamma}'/\gamma' = \mu \text{ a.e.}\}.$$

Then μ-conformal mapping f conjugates Γ into an analytic automorphism groups of $f(\Omega)$. In particular, if $\Omega = \widehat{\mathbb{C}}$, then $f\Gamma f^{-1} \subset \mathrm{Möb}$.

Lift of a qc homeo



Simultaneous uniformization

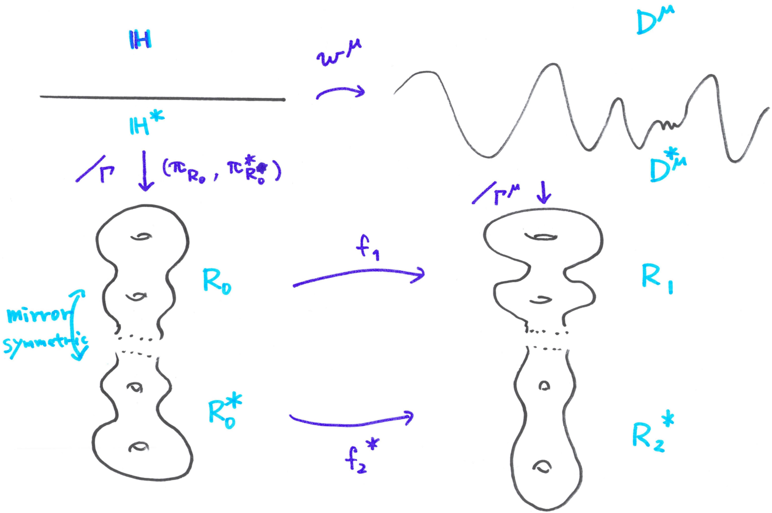
Let R_0 be a fixed hyperbolic Riemann surface and $\pi_{R_0} : \mathbb{H} \to R_0$ be an analytic universal covering projection. We consider quasiconformal deformations of R_0. Let $f_1 : R_0 \to R_1$ be a quasiconformal homeomorphism and μ_1 the lift of the Beltrami tensor of f. Namely, μ_1 is the Beltrami coefficient of a lift $\tilde{f}_1 : \mathbb{H} \to \mathbb{H}$ relative to π_{R_0} and π_{R_1}. Then μ_1 is Γ-invariant $(-1,1)$ form on $\mathbb{H}$. Similarly, we consider $f_2 : R_0 \to R_2$.

We now define

$$\mu = (\mu_1, \mu_2) = \begin{cases} \mu_1(z) & \text{if } z \in \mathbb{H}, \\ \mu_2^*(z) = \overline{\mu_2(\bar{z})} & \text{if } z \in \mathbb{H}^*. \end{cases}$$

Then the μ-conformal mapping $w = w^\mu$ deforms Γ into a Möbius group Γ^μ, which acts on $D^\mu = w(\mathbb{H})$ and $D^{\mu,*} = w(\mathbb{H}^*)$. By construction, $D^\mu/\Gamma^\mu \cong R_1$ and $D^{\mu,*}/\Gamma^\mu \cong R_2^*$, where R_2^* is the mirror image of R_2.

Simultaneous uniformization (scheme)



Teichmüller space

Let Γ be a Fuchsian group acting on $\mathbb{H}$. For $\mu \in M(\mathbb{H}, \Gamma)$, it is known that the mapping $w_\mu = w^{(\mu,\mu)}$ has the property $w_\mu(\bar{z}) = \overline{w_\mu(z)}$. In particular, $w_\mu(\mathbb{H}) = \mathbb{H}$ and $\Gamma_\mu = w_\mu \Gamma w_\mu^{-1}$ is again a Fuchsian group. (In other words, w_μ uniformizes a pair of mirror symmetric surfaces.) Two Beltrami coefficients $\mu_1, \mu_2 \in M(\mathbb{H}, \Gamma)$ are called Teichmüller equivalent if $w_{\mu_1} = w_{\mu_2}$ on $\mathbb{R}$. The set of Teichmüller equivalence classes $[\mu]$ for $\mu \in M(\mathbb{H}, \Gamma)$ is called the Teichmüller space of Γ and will be denoted by $T(\Gamma)$. The distance

$$d(\mu, \nu) = \operatorname*{ess\,sup}_{z \in \mathbb{H}} \operatorname{arctanh} \left| \frac{\mu(z) - \nu(z)}{1 - \bar{\nu}(z)\mu(z)} \right|$$

induces a distance on $T(\Gamma)$, called the Teichmüller distance. Unfortunately, the correspondence $\mu \mapsto w_\mu$ is not complex analytic because the definition of (μ, μ) involves the conjugate $\overline{\mu(\bar{z})}$. Thus the above construction does not give us a complex structure directly.

Bers' idea

To get rid of complex conjugation, Bers considered the map $w^{(\mu,0)}$ for $\mu \in M(\mathbb{H}, \Gamma)$. For simpicity, we write $w^\mu = w^{(\mu,0)}$ through the understanding that μ is extended to the lower half-plane $\mathbb{H}^*$ as 0. Note that the map w^μ uniformizes the Riemann surface $f_\mu(R_0) = \mathbb{H}/\Gamma_\mu$ and R_0^*.

Proposition

Let $\mu_1, \mu_2 \in M(\mathbb{H}, \Gamma)$. The following conditions are equivalent:

1. $w_{\mu_1} = w_{\mu_2}$ on $\mathbb{R}$.
2. $w^{\mu_1} = w^{\mu_2}$ on $\mathbb{R}$.
3. $w^{\mu_1} = w^{\mu_2}$ on $\mathbb{H}^*$.
4. $S_{w^{\mu_1}} = S_{w^{\mu_2}}$ on $\mathbb{H}^*$.

Bers embedding

We define

$$\Phi[\mu](z) = S_{w^\mu}(z), \quad z \in \mathbb{H}^*.$$

Hence, μ_1 is Teichmüller equivalent to μ_2 iff $\Phi[\mu_1] = \Phi[\mu_2]$ on $\mathbb{H}^*$. Let $B_2(\mathbb{H}^*, \Gamma)$ be the complex Banach space consisting of holomorphic quadratic differentials φ for Γ with finite norm

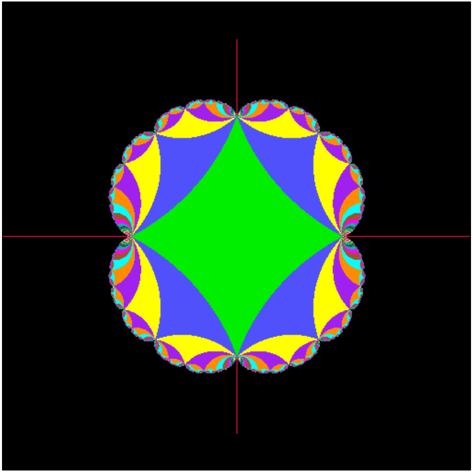
$$\|\varphi\|_2 = \sup_{z \in \mathbb{H}^*} (1 - |z|^2)^2 |\varphi(z)| < \infty.$$

Recall that φ is called a quadratic differential for Γ (or an automorphic form for Γ of weight 4) if $\varphi \circ \gamma \cdot (\gamma')^2 = \varphi$ for all $\gamma \in \Gamma$.

Theorem (Bers embedding, circa 1961)

The mapping Φ induces a topological embedding $\beta : T(\Gamma) \to B_2(\mathbb{H}^*, \Gamma)$. Moreover, the image of β is a domain contained in the open ball $\|\varphi\|_2 < 6$.

Picture of Bers embedding



Computer graphics of the Bers enbedding of the Teichmüller space of a once-punctured squre torus, due to Yasushi Yamashita

Generalizations

Attempts for higher dimensions or manifolds

- M. Yoshida (around 1980) via projective geometry
- Ahlfors (1989) Schwarzian for curves in $\mathbb{R}^n$ and diffeomorphisms of $\mathbb{R}^n$
- Osgood-Stowe (1992) Schwarzian for conformal maps between Riemanian manifolds
- Carne (1990), Wada (1998) approach via Clifford algebras
- FitzGerald-Gong (1993) in several complex variables
- Molzon-Mortenson (1996) maps between complex mfds with complex projective connections
- Kobayashi-Wada (2000) relation with concircular geometry
- Many more: Sato, Sasaki, Ovsienko-Tabachnikov, Chuaqui-Gevirtz, and so on.

Higher-order Schwarzians

- Aharonov (1969), followed up by Harmelin (1982) (called "Aharonov invariants")
- Tamanoi (1995) more later
- Schippers (2000) extension of $S_{f^{-1}}$
- Kozlovski and Sands (2009) a dynamical approach
- Chuaqui, Gröhn and Rättyä (2011) via the ODE $y^{(n)} + \varphi(z)y = 0$
- Seong-A Kim and Sugawa (2011) "invariant" and "projective" Schwarzians

In this talk, we will introduce the projective Schwarzians based on Tamanoi's one.

Another interpretation of Schwarzian

Let $f(z)$ be a locally univalent meromorphic function. Assume that $f(z) \neq \infty$ for a fixed point z. Then one can find a Möbius transformation $M_z(w) = (aw + b)/(cw + d)$ such that

$$M_z(0) = f(z),\ M_z'(0) = f'(z),\ M_z''(0) = f''(0).$$

Then $M_z(w)$ is the best Möbius approximation of $f(z + w)$ in a sense. Observe that

$$M_z^{-1}(f(z + w)) = z + \frac{S_f(z)}{3!}w^3 + O(w^4)$$

as $w \to 0$. Note also

$$M_z^{-1}(t) = \frac{f'(z)(t - f(z))}{\frac{1}{2}f''(z)(t - f(z)) + f'(z)^2}.$$

Tamanoi's Schwarzian derivatives

Tamanoi (1995):

$$W = M_z^{-1}(f(\zeta)) = \frac{f'(z)(f(\zeta) - f(z))}{\frac{1}{2}f''(z)(f(\zeta) - f(z)) + f'(z)^2}$$
$$= \sum_{n=0}^{\infty} S_n[f](z) \frac{(\zeta - z)^{n+1}}{(n+1)!}.$$

$S_n[f]$ is called (Tamanoi's) Schwarzian derivative of virtual order n.

First several ones

$$S_0[f] = 1$$
$$S_1[f] = 0$$
$$S_2[f] = S_f$$
$$S_3[f] = S_f'$$
$$S_4[f] = S_f'' + 4S_f^2$$
$$S_5[f] = S_f''' + 13S_f S_f'.$$

Representation in terms of $S_f^{(k)}$

Define a sequence of polynomials $T_n = T_n(x_2, \cdots, x_n)$ of $n-1$ indeterminates inductively by $T_0 = 1, T_1 = 0, T_2 = x_2$, and

$$T_n = \sum_{k=2}^{n-1} \frac{\partial T_{n-1}}{\partial x_k} \cdot x_{k+1} + \frac{x_2}{2} \sum_{k=1}^{n-1} \binom{n}{k} T_{k-1} T_{n-k-1}, \quad n \geq 3.$$

Then, for instance, $T_3 = x_3,\ T_4 = x_4 + 4x_2^2$ and $T_5 = x_5 + 13x_2x_3$.

Lemma

$S_n[f] = T_n(S_f, S_f', \ldots, S_f^{n-2}), \quad n \geq 3.$

Schwarzian on Riemann suface?

Recalling the relation

$$S_{g\circ f} = (S_g \circ f) \cdot (f')^2 + S_f,$$

we immediately recognize that the Schwarzian derivative is not well-defined for a holomorphic map between Riemann surfaces. On the other hand, if we consider finer structure than complex one, then the Schwarzian can be defined. Indeed, the Schwarzian derivative of a (non-constant) holomorphic map between Riemann surfaces with projective structure can be defined as a holomorphic quadratic differential. This idea, however, breaks down when we try to extend it to higher-order Schwarzians. We propose one possible way to accomplish it below, based on the joint work with S.-A Kim.

Setting

Let $f : \Omega \to \Omega'$ be a nonconstant holomorphic map between Riemann surfaces with projective structures. If the source domain Ω is equipped with conformal metric ρ, we can define a kind of invariant Schwarzian derivatives of higher order, called projective Schwarzian derivatives. (We do not need a conformal metric on Ω'.)
For simplicity, we will consider only plane domains (with standard projective structures) in the sequel.

Covariant derivatives

Let $\varphi = \varphi(z)dz^n$ be an n-differential on Ω. Then its covariant derivative in z-direction w.r.t. the Levi-Civita connection of ρ is defined by

$$\Lambda_\rho(\varphi) = \left[\partial\varphi - 2n(\partial \log \rho)\varphi\right] dz^{n+1}.$$

We define $\mathfrak{D}^n_\rho f$ by

$$\mathfrak{D}^n_\rho f dz^n = \Lambda^{n-2}_\rho(S_f(z)dz^2), \quad n \geq 2.$$

By naturality, for a Möbius transformation h, we have

$$\mathfrak{D}^n_{h^*\rho}(f \circ h) = (\mathfrak{D}^n_\rho f) \circ h \cdot (h')^n.$$

Definition of projective Schwarzians

Define $V_\rho^n f$ $(n \geq 2)$ by

$$V_\rho^n f = T_n(\mathfrak{D}_\rho^2 f, \ldots, \mathfrak{D}_\rho^n f).$$

Note that $V_\rho^n f = S_n[f]$ when $\rho = |dz|$.

Univalence criteria with $V^3 f$

From now on, we suppose that $\Omega = \mathbb{D} = \{|z| < 1\},\ \rho = |dz|/(1-|z|^2)$ and $\Omega' = \widehat{\mathbb{C}}$ with standard projective structure. We simply write $V^n f = V_\rho^n f$ and $V_f = V^3 f$ for a nonconstant meromorphic function $f : \mathbb{D} \to \widehat{\mathbb{C}}$. A straightforward computation yields

$$V_f(z) = (S_f)'(z) - \frac{4\bar{z}}{1-|z|^2} S_f(z).$$

Theorem (S.-A Kim and S. 2011)

Let f be a non-constant meromorphic function on the unit disk $\mathbb{D}$. If f is univalent in $\mathbb{D}$, then $\|V_f\|_3 \le 16$. The number 16 is sharp. Conversely, if $\|V_f\|_3 \le 3/2$, then f is univalent in $\mathbb{D}$.