ABP最大値原理について

小池　茂昭　（東北大学）

2016年3月19日　（於）筑波大学

ABP

A. D. Aleksandrov,　I. Y. Bakelman,　C. Pucci

1．序

$\Omega \subset \mathbf{R}^n$: bounded

$$\exists C_0 = C_0(n) > 0$$

such that

$$-\triangle u \le f \quad \text{in } \Omega \qquad \text{(classical subsolution)}$$

$$\Downarrow$$

$$\max_{\overline{\Omega}} u \le \max_{\partial\Omega} u + C_0 \mathrm{diam}(\Omega)^2 \|f^+\|_{L^\infty(\Omega)}$$

Assume $\operatorname{diam}(\Omega) = 1$ and $0 \in \Omega$.

Consider $v(x) := u(x) + \mu|x|^2$.

$$-\triangle v \le f - 2\mu n < 0$$

provided $\mu > \frac{\|f^+\|_\infty}{2n}$.

$$\max_{\overline{\Omega}} v = \max_{\partial\Omega} v$$

Otherwise, $v(x_0) = \max_{\overline{\Omega}} v$ for $x_0 \in \Omega$,

$$0 \le -\triangle v(x_0) < 0 \qquad \text{矛盾}$$

$$v(x) = u(x) + \mu|x|^2$$

Hence, $\max_{\overline{\Omega}} v = \max_{\partial\Omega} v$

$$\max_{\overline{\Omega}} u \le \max_{\overline{\Omega}} v = \max_{\partial\Omega} v$$

$$\le \max_{\partial\Omega} u + \mu \max_{\partial\Omega} |x|^2$$

$$\max_{\partial\Omega} u \le \max_{\partial\Omega} u + \frac{\text{diam}(\Omega)^2}{2n} \|f^+\|_\infty$$

2. ABP最大値原理（楕円型）

ABP最大値原理

$$\exists C_1 = C_1(n) > 0$$

such that

$$-\triangle u \le f \quad \text{in } \Omega$$

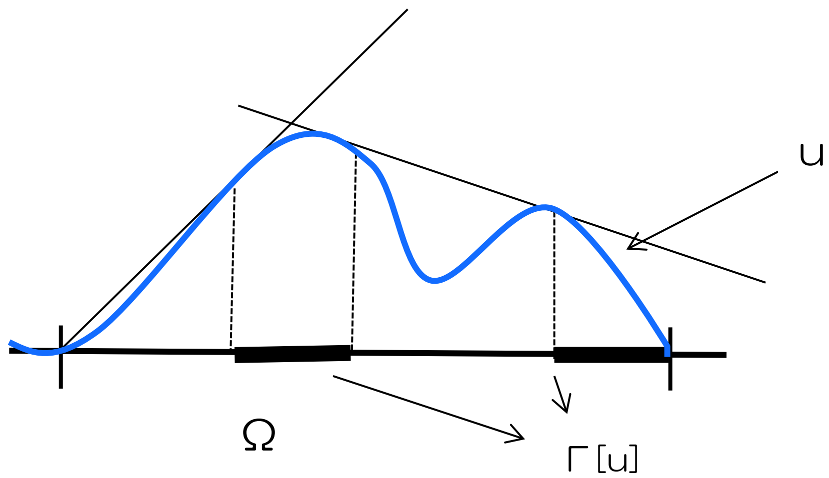
$$\Downarrow$$

$$\max_{\overline{\Omega}} u \le \max_{\partial\Omega} u + C_1 \mathrm{diam}(\Omega) \| f^+ \|_{L^n(\Gamma[u])}$$

$$\Gamma_r[u] := \{x \in \Omega \mid \exists p \in B_r \text{ s. t. } u(y) \leq u(x) + \langle p, y - x \rangle \ (\forall y \in \Omega)\}$$

$$B_r := \{p \in \mathbf{R}^n \mid |p| < r\}$$

$$\Gamma[u] := \bigcup_{r>0} \Gamma_r[u] \qquad \text{upper contact set} \quad (\text{上接集合})$$



Sketch of proof

Change of variables

$$\max_{\overline{\Omega}} u - \max_{\partial\Omega} u =: r \geq 0$$

If $r = 0$, then the proof is done with any $C_1 > 0$.

Assume $r > 0$.

Claim: $\quad B_r = Du(\Gamma_r[u])$

$$\boldsymbol{B_r} \supset \boldsymbol{Du}(\boldsymbol{\Gamma_r}[\boldsymbol{u}])$$

$$\because$$

$\boldsymbol{x} \in \boldsymbol{\Gamma_r}[\boldsymbol{u}] \Rightarrow \exists \boldsymbol{p} \in \boldsymbol{B_r}$ such that

$$\boldsymbol{u}(\boldsymbol{y}) \leq \boldsymbol{u}(\boldsymbol{x}) + \langle \boldsymbol{p}, \boldsymbol{y} - \boldsymbol{x} \rangle \quad (\forall \boldsymbol{y} \in \boldsymbol{\Omega})$$

$\therefore \boldsymbol{p} = \boldsymbol{Du}(\boldsymbol{x})$ i.e. $\boldsymbol{Du}(\boldsymbol{x}) \in \boldsymbol{B_r}$

$$B_r \subset Du(\Gamma_r[u])$$

$$\because$$

$\forall p \in B_r$ fixed

$\exists x \in \overline{\Omega}$ (Not $x \in \Omega$!) such that

$$\max_{y \in \overline{\Omega}} \{u(y) - \langle p, y \rangle\} = u(x) - \langle p, x \rangle$$

i.e. $u(y) \leq u(x) + \langle p, y - x \rangle \quad (\forall y \in \Omega)$

If $x \in \partial\Omega$, $\max_{\overline{\Omega}} u < \max_{\partial\Omega} u + r\text{diam}(\Omega)$.

$$\Downarrow$$

Recall diam(Ω)=1, $r = \max_{\overline{\Omega}} u - \max_{\partial\Omega} u$

a contradiction

$$\Downarrow$$

Hence, $x \in \Omega$. $\Rightarrow p = Du(x)$

i.e. $p \in Du(\Gamma_r[u])$

Change of variables formula

$$\int_{T(A)} g(\xi)\#(T^{-1}\xi)d\xi = \int_A g(T(x))|\mathrm{det}(DT(x))|dx$$

$$\#(T^{-1}\xi) \geq 1, \quad T := Du, \quad A := \Gamma_r[u], \quad g \equiv 1$$

$$\Downarrow$$

$$\int_{B_r} d\xi = \int_{Du(\Gamma_r[u])} d\xi \leq \int_{\Gamma_r[u]} |\mathrm{det} D^2 u| dx$$

$$\omega_n r^n = \int_{B_r} d\xi \leq \int_{\Gamma_r[u]} |\mathrm{det} D^2 u| dx$$

$$x \in \Gamma_r[u] \Rightarrow u(y) \leq u(x) + \langle Du(x), y - x \rangle$$

i.e. $D^2 u(x) \leq O$

λ_k: eigen-value of $D^2 u(x) \leq 0$

$$|\mathrm{det} D^2 u(x)| = (-\lambda_1) \times \cdots \times (-\lambda_n)$$

$$\int_{\Gamma_r[u]} |\mathrm{det} D^2 u| dx \leq \int_{\Gamma_r[u]} \left(\frac{-\triangle u}{n} \right)^n dx$$

$\Uparrow$

Arithmetic mean-Geometric mean inequality

(相加相乗平均)

$$\omega_n r^n \leq \frac{1}{n^n} \int_{\Gamma_r[u]} (f^+)^n dx$$

$\therefore$

$$\max_{\overline{\Omega}} u - \max_{\partial\Omega} u = r \leq \frac{1}{n\omega_n^{1/n}} \|f^+\|_{L^n(\Gamma_r[u])} \quad \square$$

ABP最大値原理

$$\exists C_2 = C_2(n, \|b\|_n) > 0$$

such that

$$-\triangle u + \langle b, Du \rangle \leq f \quad \text{in } \Omega$$

$$\Downarrow$$

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u + C_2 \mathrm{diam}(\Omega) \|f^+\|_{L^n(\Gamma[u])}$$

Apply Change of variables formula to $g(\xi) := \frac{1}{\kappa + |\xi|^n}$ $\quad (\kappa \geq 0)$

$$\int_{B_r} \frac{1}{\kappa + |\xi|^n} d\xi \leq \int_{\Gamma_r[u]} \frac{|\det D^2 u|}{\kappa + |Du|^n} dx \leq \frac{1}{n^n} \int_{\Gamma_r[u]} \frac{(-\triangle u)^n}{\kappa + |Du|^n} dx$$

Here, $\kappa := \|f^+\|^n_{L^n(\Gamma[u])}$

$$(a+b)^n \leq 2^{n-1}(a^n + b^n) \ (\forall a, b \geq 0)$$

$$\leq \frac{2^{n-1}}{n^n} \left(\int_{\Gamma_r[u]} \frac{|b|^n |Du|^n}{\cancel{\kappa} + |Du|^n} dx + \int_{\Gamma_r[u]} \frac{(f^+)^n}{\kappa + |\cancel{D}\cancel{u}|^n} \right) dx$$

$$\leq \frac{2^{n-1}}{n^n} (\|b\|^n_n + 1)$$

$$\mathsf{LHS} = n\omega_n \Big[\log(\kappa + s^n)\Big]_0^r = n\omega_n \log\left(1 + \frac{r^n}{\kappa}\right)$$

$$\log\left(1 + \frac{r^n}{\kappa}\right) \leq C(\|b\|_n^n + 1)$$

$$\Downarrow$$

$$1 + \frac{r^n}{\kappa} \leq \mathsf{exp}\left\{C(\|b\|_n^n + 1)\right\}$$

$$\Downarrow$$

$$r^n \leq \kappa \mathsf{exp}\left\{C(\|b\|_n^n + 1)\right\}$$

$$\kappa := \|f^+\|^n_{L^n(\Gamma_r[u])}$$

３．ABP最大値原理（放物型）

Krylov (1976)

Tso (1985)

Parabolic upper contact set (放物型上接集合)

$$u : Q_T := \Omega \times (0,T) \to \mathbf{R}$$

$$\Pi_r[u] := \left\{ (x,t) \in Q_T \;\middle|\; \begin{array}{l} \exists p \in B_r \text{ such that} \\ u(y,s) \le u(x,t) + \langle p, y-x \rangle \\ \qquad \forall (y,s) \in \Omega \times [0,t] \end{array} \right\}$$

$$\Pi[u] := \bigcup_{r>0} \Pi_r[u]$$

ABP最大値原理

$$\exists C_2 = C_2(n, \|b\|_{n+1}, \mathrm{diam}(\Omega), T) > 0$$

such that

$$\frac{\partial u}{\partial t} - \triangle u + \langle b, Du \rangle \leq f \quad \text{in } \Omega \times (0, T]$$

$$\Downarrow$$

$$\max_{\overline{Q}_T} u \leq \max_{\partial_p Q_T} u + C_2 \mathrm{diam}(\Omega) \|f^+\|_{L^{n+1}(\Pi[u])}$$

$$\partial_p Q_T := (\partial\Omega \times [0, T]) \cup (\Omega \times \{0\})$$

$$D := \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right), \qquad \nabla := \left(D, \frac{\partial}{\partial t}\right)^T$$

Fix $a \in \mathbf{R}^n$. Define $\Phi : \mathbf{R}^{n+1} \longrightarrow \mathbf{R}^{n+1}$ by

$$\Phi := (Du, u - \langle Du, x - a\rangle).$$

$$\begin{aligned} \nabla\Phi(x,t) &= \begin{pmatrix} D^2u & D^T(u - \langle Du, x-a\rangle) \\ Du_t & \frac{\partial}{\partial t}(u - \langle Du, x-a\rangle) \end{pmatrix} \\ &= \begin{pmatrix} D^2u & -D^2u(x-a)^T \\ Du_t & u_t - \langle Du_t, x-a\rangle \end{pmatrix} \end{aligned}$$

Lemma

$$|\det \nabla\Phi| = u_t \times \det D^2 u$$

$$\because$$

$$\nabla\Phi := \begin{pmatrix} D^2u & -D^2u(x-a)^T \\ Du_t & u_t - \langle Du_t, x-a\rangle \end{pmatrix}$$

jth column $\times(x_j - a_j)$ be added to $n+1$th column

$$\Downarrow$$

$$\begin{pmatrix} D^2u & \vec{0} \\ *** & u_t \end{pmatrix}$$

Assume $\operatorname{diam}(\Omega) = 1$.

$$m_0 := \max_{\partial_p Q_T} u, \qquad r_0 := \max_{\overline{Q}_T} u - m_0 > 0$$

$$D := \{(\xi, h) \in \mathbf{R}^n \times \mathbf{R} \mid |\xi| < r_0, |\xi| < h - m_0 < r_0\}$$

$(a, t_0) \in \Omega \times (0, T]$ such that $r_0 = u(a, t_0) - m_0$

Claim : $D \subset \Phi(\Pi_{r_0}[u])$

Fix $\forall(\xi, h) \in D$

$P(y) := h + \langle \xi, y - a \rangle$ affine function

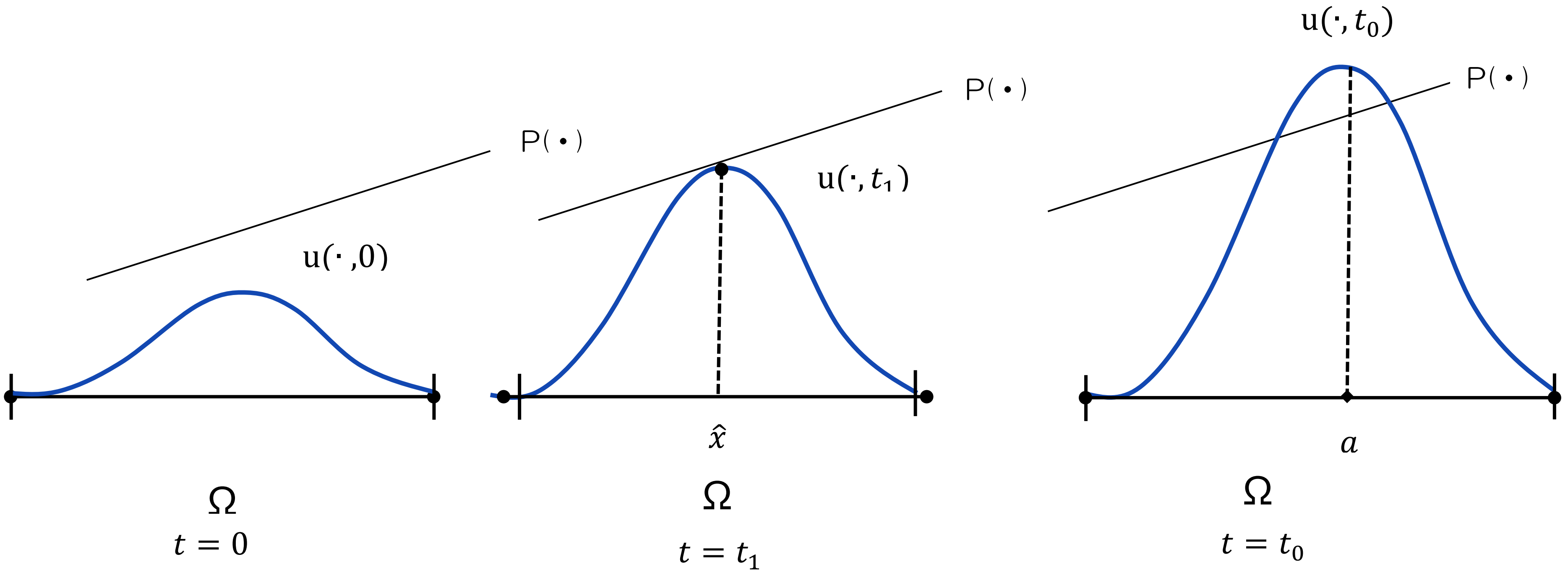
$P(y) \geq h - |\xi| > m_0$ i.e. $P \geq u$ on $\partial_p Q_T$

$\Leftarrow \ |\xi| + m_0 < h < r_0 + m_0$

$P(a) = h < r_0 + m_0 = u(a, t_0)$

$\exists(\hat{x}, t_1) \in \Omega \times (0, t_0)$ such that

$$u(\hat{x}, t_1) = P(\hat{x}), \quad u(y, s) \leq P(y) \ (y \in \Omega, \ 0 \leq s \leq t_1)$$



P(•)
u(· ,0)
Ω
t = 0
P(•)
u(·, t₁)
x̂
Ω
t = t₁
u(·, t₀)
P(•)
a
Ω
t = t₀

$$D := \{(\xi, h) \mid |\xi| \le r_0, |\xi| < h - m_0 < r_0\}$$

$$\kappa := \|f^+\|^{n+1}_{L^{n+1}(\Pi[u])}.$$

$$\int_D \frac{1}{\kappa + |\xi|^{n+1}} dh d\xi = C_0 \int_0^{r_0} \frac{r^{n-1}}{\kappa + r^{n+1}} \left(\int_{r+m_0}^{r_0+m_0} dh \right) dr$$

$$= C_0 \int_0^{r_0} \frac{r_0 r^{n-1} - r^n}{\kappa + r^{n+1}} dr$$

$$\int_D \frac{1}{\kappa + |\xi|^{n+1}} dh d\xi \leq \int_{\Pi_{r_0}[u]} \frac{|\mathrm{det} \nabla \Phi|}{\kappa + |Du|^{n+1}} dx dt$$

$$\leq C_1 \int_{\Pi_{r_0}[u]} \frac{(u_t - \triangle u)^{n+1}}{\kappa + |Du|^{n+1}} dx dt$$

$$\leq C_2 \int_{\Pi_{r_0}[u]} \frac{(|b||Du|)^{n+1} + (f^+)^{n+1}}{\kappa + |Du|^{n+1}} dx dt$$

$$\text{RHS} \le C_2 \int_{\Pi_{r_0}[u]} |b|^{n+1} dx dt + C_2$$

$$\text{LHD} = C_0 \int_0^{r_0} \frac{r_0 r^{n-1} - r^n}{\kappa + r^{n+1}} dr$$

$$\ge C_3 \int_0^{r_0} \frac{r_0 r^{n-1} - r^n}{\kappa + \frac{r_0}{n} r^n - \frac{1}{n+1} r^{n+1}} dr$$

$$= C_3 \log \left\{ \left(\kappa + \frac{r_0}{n} r_0^n - \frac{1}{n+1} r_0^{n+1} \right) / \kappa \right\}$$

$$= C_3 \log \left(1 + \frac{r_0^{n+1}}{n(n+1)\kappa} \right)$$

$$\log\left(1+\frac{r_0^{n+1}}{n(n+1)\kappa}\right) \leq C_4$$

$$\sup_{Q_T} u - \sup_{\partial_p Q_T} u = r_0 \leq C_5 \kappa^{\frac{1}{n+1}} = C_5 \|f^+\|_{L^{n+1}(\Pi[u])}$$

4．応用

4.1 Weak Harnack inequality

i.e.

$u \geq 0$: supersolution in Ω

$$-\triangle u + \langle b, Du \rangle \geq f \quad \text{in } \Omega$$

$$B_2 \Subset \Omega$$

$$\|u\|_{L^{p_0}(B_1)} \leq C_0 \left(\inf_{B_1} u + \|f^-\|_{L^n(\Omega)} \right)$$

for $\exists p_0, C_0 > 0$ constants.

Decay estimate of distribution function

$$v := \frac{u}{\inf_{B_1} u + \varepsilon_0^{-1} \|f^-\|_n}, \ \varepsilon_0 > 0 \text{ fixed later}$$

Note $\inf_{B_1} v \leq 1$, $\|f^-\|_n \leq \varepsilon_0$

$$\exists C_0 > 0 \text{ such that } \|v\|_{L^{p_0}(B_1)} \leq C_0$$

$$\Downarrow$$

$$\|u\|_{L^{p_0}(B_1)} \leq C_0 \left(\inf_{B_1} u + \|f\|_{L^n(\Omega)} \right)$$

$$\|u\|_{L^{p_0}(B_1)} \leq C_0$$
$$\Uparrow$$
$\exists \alpha > 0$ such that $|\{x \in B_1 \mid u(x) > t\}| \leq C t^{-\alpha}$

$\Leftrightarrow \quad \exists M > 1$ and $\exists \theta \in (0, 1)$ such that

$$|\{x \in B_1 \mid u(x) > M^m\}| \leq \theta^m$$

Let us show the case when $m = 1$.

$$-\triangle u \geq f \text{ in } B_2 \qquad (u|_{\partial B_2} = 0 \text{ for simplicity !})$$

$$\phi(x) := -A(|x|^{2-n} - 2^{2-n}): \text{ “fundamental solution” } (A > 0)$$

$$\phi|_{\partial B_2} = 0,\ \phi \leq -2 \text{ in } B_1$$

$$\hat{\phi} \in C^2 \text{ such that } \phi = \hat{\phi} \text{ except near } 0$$

$$-\triangle \hat{\phi} = 0 \text{ except near } 0, \text{ i.e. supp} \triangle \hat{\phi} \Subset B_1$$

$$w := -u - \hat{\phi}, \text{ Recall } \inf_{B_1} u \leq 1$$

$$- \triangle w \leq f^- + \triangle \phi \text{ in } B_2$$

$$\Downarrow$$

$$\sup_{B_2} w \leq \sup_{\partial B_2} w + C \| f^- + \triangle \hat{\phi} \|_{L^n(\Gamma[w;B_2])}$$

$$\text{LHS} \geq \sup_{B_1}(-u + 2) \geq - \inf_{B_1} u + 2 \geq 1$$

$$\text{RHS} \leq C\varepsilon_0 + C \left(\int_{B_1 \cap \{w \geq 0\}} 1 dx \right)^{\frac{1}{n}}$$

$\exists\boldsymbol{\sigma} > \mathbf{0}$ such that

$$\boldsymbol{\sigma} \leq |\{\boldsymbol{x} \in \boldsymbol{B_1} \mid \boldsymbol{w(x)} \geq \mathbf{0}\}| \leq |\{\boldsymbol{x} \in \boldsymbol{B_1} \mid \boldsymbol{u(x)} \leq \boldsymbol{M} := \sup_{\boldsymbol{B_2}}(-\hat{\boldsymbol{\phi}})\}|$$

$$\Downarrow$$

$\exists\boldsymbol{\theta} > \mathbf{0}$ such that $|\{\boldsymbol{x} \in \boldsymbol{B_1} \mid \boldsymbol{u(x)} > \boldsymbol{M}\}| \leq \boldsymbol{\theta}$

$\cdots$ we had to work with a unit cube instaed of $\boldsymbol{B_1}$ to have $\boldsymbol{\theta} < \mathbf{1}$ $\cdots$

4.2 Local maximum principle

Unbounded coefficient case

is excluded by a classical argument.

Use weak Harnack inequality (by Caffarelli)

4.3 Calderón-Zygmund estimate   Caffarelli 1989

$$D^2 u \approx \Theta := \max\{\Theta^+, \Theta^-\}$$

$$\Theta^\pm(x) := \inf \left\{ M > 0 \;\middle|\; \begin{array}{c} u(y) \le \pm M|y|^2 + \text{affine} \\ = \text{ holds at } x \end{array} \right\}$$

$$\left| \int u \phi_{x_i x_j} dx \right| \le C \|\Theta\|_{L^p} \|\phi\|_{L^{p'}} \ (\forall \phi \in C_0^\infty)$$

and

$$\|D^2 u\|_{L^p} \le C \|\Theta\|_{L^p} \Leftarrow \text{Decay estimate on } \Theta$$

Enough to show the case of $i = j$

$$\delta_h f(x) := \frac{1}{h^2}\{f(x + he_i) + f(x - he_i) - 2f(x)\}$$

$$\int u(x)\phi_{x_i x_i} dx \approx \int u \delta_h \phi(x) dx$$

$$= \int \phi(x)\delta_h u(x) dx \leq \int \phi(x)\Theta(x) dx$$

$$\Uparrow$$

$$|u(x + y) - u(x)| \leq M|y|^2 + \langle \xi, y \rangle \quad (\exists \xi \in \mathbf{R}^n)$$

5．一般化

Asuume $\mathrm{diam}(\boldsymbol{\Omega}) = \mathbf{1}$

$\boldsymbol{u} \geq \mathbf{0}$ in $\boldsymbol{\Omega}$

$\boldsymbol{u} = \mathbf{0}$ on $\boldsymbol{\partial\Omega}$

5.1 p-Laplacian cf. Imbert 2011

$$-\triangle_p u \le f \quad \text{in } \Omega$$

$$-\triangle_p u = -\sum_{k=1}^{n} \frac{\partial}{\partial x_k}\left(|Du|^{p-2}\frac{\partial u}{\partial x_k}\right)$$

$$= -|Du|^{p-2}\left\{\triangle u + (p-2)|Du|^{-2}\sum_{k,j=1}^{n} u_{x_k}u_{x_j}u_{x_k x_j}\right\}$$

$\{\cdots\}$ is uniformly elliptic if $p > 1$ and $Du \neq 0$

$$\max_{\overline{\Omega}} u \le \max_{\partial\Omega} u + C\|f^+\|_{L^n(\Gamma[u])}^{\frac{1}{p-1}}$$

$r_0 := \max_{\overline{\Omega}} u - \max_{\partial\Omega} u > 0$ として導けばよい.

$$\int_{B_{r_0}} g(|\xi|)d\xi \le \int_{\Gamma_{r_0}[u]} g(|Du|)|\mathrm{det}D^2u|dx$$

$$\le C_p \int_{\Gamma[u]} g(|Du|)\left(\frac{-\triangle u}{|Du|^{p-2}}\right)^n dx$$

$$\le C_p' \int_{\Gamma[u]} g(|Du|)\frac{(f^+)^n}{|Du|^{n(p-2)}}dx$$

$$g(r) := r^{n(p-2)}$$

$$\text{左辺} = n\omega_n \left[\frac{r^{n(p-1)}}{n(p-1)} \right]_0^{r_0} = \frac{\omega_n}{p-1} r_0^{n(p-1)}$$

$$r_0^{n(p-1)} \leq C_p'' \|f^+\|_{L^n}^n$$

$$\therefore$$

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u + C\|f^+\|_{L^n}^{\frac{1}{p-1}}$$

5.2 ∞-Laplacian

$$-\triangle_p u = -\sum_{k=1}^{n} \frac{\partial}{\partial x_k}\left(|Du|^{p-2}\frac{\partial u}{\partial x_k}\right)$$

$$= -|Du|^{p-2} \triangle u - (p-2)|Du|^{p-4} \sum_{k,j=1}^{n} u_{x_k} u_{x_j} u_{x_k x_j}$$

$\times \frac{1}{p-2}|Du|^{p-4}$ $p \to \infty$ with $|Du| \neq 0$

$$\Downarrow$$

$$-\triangle_\infty u := -\sum_{k,j=1}^{n} u_{x_k} u_{x_j} u_{x_k x_j}$$

Charro-De Phillips-Di Castro-Máximo 2013

Consider normalized ∞-Laplacian:

$$-\triangle_p u \times \frac{1}{p-2}|Du|^{p-2} \quad (p \to \infty)$$

$$\Downarrow$$

$$-\triangle_\infty^N u := -\sum_{k,j=1}^{n} \frac{u_{x_k} u_{x_j}}{|Du|^2} u_{x_k x_j}$$

$$-\triangle_\infty^N u \le f \text{ in } \Omega$$

$$\left(\sup_{\Omega} u - \sup_{\partial\Omega} u\right)^2 \le C \int_{\sup_{\partial\Omega} u}^{\sup_{\Omega} u} \|f^+ 1_{\Gamma[u]}\|_{L^\infty(\{u=r\})} dr$$

5.3 Fractional Laplacian $\quad \sigma \in (0,2)$

$$-L[u] \leq f \quad \text{in } \Omega$$

$$L[u](x) := \int_{\mathbf{R}^n} \frac{u(x+y) - u(x) - \langle Du(x), y\rangle \chi_{B_1}(y)}{|y|^{n+\sigma}} dy$$

$$= \frac{1}{2} \int_{\mathbf{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+\sigma}} dy$$

$$L[u] \leq f(x) \text{ in } \Omega$$

$$u \geq 0 \text{ in } \mathbf{R}^n \setminus \Omega$$

$$\Downarrow$$

$$\max_{\overline{\Omega}} u \leq C \|f^+\|_{L^\infty(K_u)}^{\frac{2-\sigma}{2}} \|f^+\|_{L^n(K_u)}^{\frac{\sigma}{2}}$$

N. Guillen and R. W. Schwab, 2012

5.4 Monge-Ampère equation

$$-\mathrm{det} D^2 u = f \quad \text{in } \Omega$$

$F(X) := -\mathrm{det} X$ should be elliptic;

Restrict solutions to be convex.

If u is convex in Ω (Ω must be convex), then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u, \text{ and}$$

$$\min_{\overline{\Omega}} u \geq \min_{\partial\Omega} u - C\|f\|_{L^1(\Omega)}$$

ご清聴ありがとうございました。

まだ時間があるようなので …

5.5 k-Hessian

$$1 < k < n$$

$$\sigma_k(D^2u) := \sum_{1\le j_1<\cdots<j_k\le n} \lambda_{j_\ell}(D^2u)$$

$\lambda_j(D^2u(x))$: eigen-values of $D^2u(x)$

Restrict solutions to be convex again !

Assume $r := \min_{\partial\Omega} u - \min_{\overline{\Omega}} u > 0$.

$$\prod_{i=1}^{n} \lambda_i = \prod_{i_1<\cdots<i_k} (\lambda_{i_1}\cdots\lambda_{i_k})^{\frac{1}{{}_{n-1}C_{k-1}}}$$

$$\leq \left(\sum_{i_1<\cdots<i_k} \frac{(\lambda_{i_1}\cdots\lambda_{i_k})^{\frac{1}{{}_{n-1}C_{k-1}}}}{{}_nC_k} \right)^{{}_nC_k} \qquad \Leftarrow \text{相加相乗}$$

$$\leq \frac{1}{{}_nC_k}\sigma_k(D^2u)^{\frac{n}{k}}$$

$$\Uparrow$$

$$\left(\sum_{i=1}^{N} a_i\right)^{\alpha} \leq N^{\alpha-1}\sum_{i=1}^{N} a_i^{\alpha} \qquad (\alpha > 1, a_i \geq 0)$$

u: convex solution of

$$-\sigma_k(D^2u) \geq f \text{ in } \Omega$$

$$\int_{B_r} d\xi \leq \frac{1}{{}_nC_k} \|f^-\|^{\frac{n}{k}}_{L^{\frac{n}{k}}(\Omega)}$$

$$\min_{\overline{\Omega}} u \geq \min_{\partial\Omega} u - C\|f^-\|^{\frac{1}{k}}_{L^{\frac{n}{k}}(\Omega)}$$

ご清聴ありがとうございました！