Geometry of λ-hypersurfaces of the weighted volume-preserving mean curvature flow

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with $X(0) = X$ is called mean curvature flow if they satisfy

$$\frac{\partial X(p,t)}{\partial t} = H(p,t),$$

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This is a smooth flow except at the origin at time $t = T$ when the flow becomes extinct.

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When M_0 is non-convex, the other singularities of the mean curvature flow can occur.

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In fact, the resulting of the singularity converges to a shrinking cylinder.

Let $X : M^n \to \mathbb{R}^{n+1}$ be a hypersurface satisfying

$$H + \langle X, N \rangle = 0,$$

where H denotes the mean curvature of the hypersurface.
One can prove that

$$X(t) = \sqrt{-2t}X : M^n \to \mathbb{R}^{n+1},$$

is a solution of the mean curvature flow equation, which is called a self-similar solution of the mean curvature flow.

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A key starting point for singularity analysis is Huisken's monotonicity formula because the monotonicity implies that the flow is asymptotically self-similar near a given singularity which is modeled by self-shrinking solutions of the flow.

For simple, one calls a hypersurface $X : M^n \to \mathbb{R}^{n+1}$ a self-shrinker if it satisfies

$$H + \langle X, N \rangle = 0.$$

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On the other hand, if we consider weighted area functional

$$\mathcal{F}(s) = \int_M e^{-\frac{|X(s)|^2}{2}} d\mu_s$$

By computing the first variation formula, we know that $X : M^n \to \mathbb{R}^{n+1}$ is a critical point of $\mathcal{F}(s)$ if and only if $X : M^n \to \mathbb{R}^{n+1}$ is a self-shrinker, that is,

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Furthermore, we know that $X : M^n \to \mathbb{R}^{n+1}$ is a minimal hypersurface in $\mathbb{R}^{n+1}$ equipped with the metric $g_{AB} = e^{-\frac{|X|^2}{n}}\delta_{AB}$ if and only if $X : M^n \to \mathbb{R}^{n+1}$ is a self-shrinker, that is,

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1.2. The mean curvature type flow

As one knows, for a family of immersions $X(t) : M \to \mathbb{R}^{n+1}$ with $X(0) = X$, the volume of M is defined by

$$\frac{1}{n+1}\int_M \langle X(t), N(t)\rangle d\mu_t$$

Huisken (J. Reine Angew Math. 1987) studied the mean curvature type flow:

$$\frac{\partial X(t)}{\partial t} = (-h(t)N(t) + \mathbf{H}(t)),$$

where $X(t) = X(\cdot, t)$, $h(t) = \frac{\int_M H(t)d\mu_t}{\int_M d\mu_t}$ and $N(t)$ is the unit normal vector of $X(t) : M \to \mathbb{R}^{n+1}$.

It can be proved the above flow preserves the volume of M. Hence, one calls this flow the volume-preserving mean curvature flow.

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But since this definition of the volume of M is not good enough from the view point of variations for the weighted area functional, we need to find new definitions of the volume and a flow.

In Cheng and Wei (arXiv 2014), we introduce a definition of the weighted volume of M.
For a family of immersions $X(t) : M \to \mathbb{R}^{n+1}$ with $X(0) = X$, we define a weighted volume of M by

$$V(t) = \int_M \langle X(t), N \rangle e^{-\frac{|X|^2}{2}} d\mu.$$

Furthermore, we consider a new type of mean curvature flow:

$$\frac{\partial X(t)}{\partial t} = (-\alpha(t)N(t) + \mathbf{H}(t))$$

with

$$\alpha(t) = \frac{\int_M H(t)\langle N(t), N\rangle e^{-\frac{|X|^2}{2}} d\mu}{\int_M \langle N(t), N\rangle e^{-\frac{|X|^2}{2}} d\mu},$$

where N is the unit normal vector of $X : M \to \mathbb{R}^{n+1}$.
We can prove that the flow:

$$\frac{\partial X(t)}{\partial t} = (-\alpha(t)N(t) + \mathbf{H}(t))$$

preserves the weighted volume $V(t)$. Hence, we call this flow a weighted volume-preserving mean curvature flow.

2.1. The weighted volume-preserving variations

Let $X : M^n \to \mathbb{R}^{n+1}$ be an n-dimensional hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$.

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We denote a variation of X by $X(t) : M \to \mathbb{R}^{n+1}$, $t \in (-\varepsilon, \varepsilon)$ with $X(0) = X$.
We define a weighted area functional $A : (-\varepsilon, \varepsilon) \to \mathbb{R}$ by

$$A(t) = \int_M e^{-\frac{|X(t)|^2}{2}} d\mu_t,$$

where $d\mu_t$ is the area element of M in the metric induced by $X(t)$.

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The weighted volume function $V : (-\varepsilon, \varepsilon) \to \mathbb{R}$ of M is defined by

$$V(t) = \int_M \langle X(t), N\rangle e^{-\frac{|X|^2}{2}} d\mu.$$

We say that a variation $X(t)$ of X is a weighted volume-preserving normal variation if $V(t) = V(0)$ for all t and $\frac{\partial X(t)}{\partial t}|_{t=0} = fN$.

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1. For all weighted volume-preserving variations, $A'(0) = 0$.
2. $\langle X, N\rangle + H = \lambda$, which is constant.

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Remark. When $\lambda = 0$, the λ-hypersurface becomes a self-shrinker of mean curvature flow.

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and compact self-shrinkers with higher genus due to Møller and so on.

Furthermore,
Example 2.1.

$$X : S^n(r) \to \mathbb{R}^{n+1}, \quad r > 0$$

is a compact λ-hypersurface in $\mathbb{R}^{n+1}$ with $\lambda = \frac{n}{r} - r$.

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Example 2.2. For a positive integer k,

$$X : \mathbb{S}^k(r) \times \mathbb{R}^{n-k}$$

is an n-dimensional complete noncompact λ-hypersurface in $\mathbb{R}^{n+1}$ with $\lambda = \frac{k}{r} - r$.

Example 2.3. For $n = 1$, and for some $\lambda < 0$, we can prove that there exist closed embedded λ-curves Γ_λ in $\mathbb{R}^2$, which is not circle.

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Remark.
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Example 2.4. For any positive integer n, there exist complete embedded λ-hypersurfaces, which are given by $\Gamma_\lambda \times \mathbb{R}^{n-1}$ in $\mathbb{R}^{n+1}$.

Theorem 2.2 (Cheng and Wei, 2015). For $n \geq 2$ and $\lambda \geq 0$, there exists embedding revolution λ-hypersurface $X : S^1 \times S^{n-1} \to \mathbb{R}^{n+1}$ in $\mathbb{R}^{n+1}$.

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Proof of theorem 2.2. Let $(x(s), r(s))$, $s \in (a, b)$ be a curve in the xr-plane with $r > 0$ and $S^{n-1}(1)$ denote the standard unit sphere of dimension $n - 1$.

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defined by $X(s, \alpha) = (x(s), r(s)\alpha)$, $s \in (a, b)$, $\alpha \in S^{n-1}(1)$. Namely, X is obtained by rotating the plane curve $(x(s), r(s))$ around x axis.

Thus, $X : (a, b) \times S^{n-1}(1) \to \mathbb{R}^{n+1}$ is a λ-hypersurface if and only if (x, r) satisfies

$$\begin{cases} (x')^2 + (r')^2 = 1 \\ x'' = -r'[xr' + (\frac{n-1}{r} - r)x' + \lambda]. \end{cases}$$

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Let (x_δ, r_δ) be the maximal solution of the above equations with initial value $(x_\delta, r_\delta, x_\delta{}'(0)) = (0, \delta, 1)$. Then for small enough $\delta > 0$, there is a simple closed curve (x_δ, r_δ) in xr-plane.

It can be proved that it is a graph of $x = f_\delta(r)$. Hence, there exists an embedding revolution λ-hypersurface $X : S^1 \times S^{n-1} \to \mathbb{R}^{n+1}$ in $\mathbb{R}^{n+1}$.

3.1. $\mathcal{F}$-functional

We define a $\mathcal{F}$-functional by

$$\begin{aligned}\mathcal{F}(s) &= \mathcal{F}_{X_s,t_s}(X(s)) \\ &= (4\pi t_s)^{-\frac{n}{2}} \int_M e^{-\frac{|X(s)-X_s|^2}{2t_s}} d\mu_s \\ &+ \lambda(4\pi)^{-\frac{n}{2}} \frac{1}{\sqrt{t_s}} \int_M \langle X(s) - X_s, N\rangle e^{-\frac{|X|^2}{2}} d\mu,\end{aligned}$$

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where X_s and t_s denote variations of $X_0 = O$, $t_0 = 1$, respectively and $\frac{\partial X(0)}{\partial s} = fN$.

One calls that $X : M \to \mathbb{R}^{n+1}$ is a critical point of $\mathcal{F}(s)$ if it is critical with respect to all normal variations and all variations X_s and t_s of $X_0 = O$, $t_0 = 1$.

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4. $X : M \to \mathbb{R}^{n+1}$ is a critical point of $\mathcal{F}(s)$.

3.2. Stability of compact λ-hypersurface

Definition 3.1. One calls that a critical point $X : M \to \mathbb{R}^{n+1}$ of the $\mathcal{F}$-functional $\mathcal{F}(s)$ is $\mathcal{F}$-stable

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One calls that a critical point $X : M \to \mathbb{R}^{n+1}$ of the $\mathcal{F}$-functional $\mathcal{F}(s)$ is $\mathcal{F}$-unstable if there exist a normal variation $X(s)$ of X such that for all variations X_s and t_s of $X_0 = O$, $t_0 = 1$,

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holds.

Theorem 3.2 (Cheng and Wei, 2014)

1. If $r \leq \sqrt{n}$ or $r > \sqrt{n+1}$,
   the n-dimensional round sphere

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In order to prove this result, the property that the mean curvature H is an eigenfunction of Jacobi operator plays a very important role.
But for λ-hypersurfaces, the mean curvature H is not an eigenfunction of Jacobi operator in general.

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In fact, for any positive integer n, complete embedded λ-hypersurfaces $\Gamma_\lambda \times \mathbb{R}^{n-1}$ in $\mathbb{R}^{n+1}$ do not satisfy this condition, where Γ_λ is a closed embedded λ-curve in $\mathbb{R}^2$.

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Suppose η is a function with compact support,

$$\begin{aligned}
&\int_M \langle \nabla \eta^2, \nabla \log(H-\lambda)\rangle e^{-\frac{|X|^2}{2}} d\mu \\
= -&\int_M \eta^2 (\mathcal{L} \log(H-\lambda)) e^{-\frac{|X|^2}{2}} d\mu \\
= &\int_M \eta^2 \left(S - 1 - \frac{\lambda}{H-\lambda} + |\nabla \log(H-\lambda)|^2\right) e^{-\frac{|X|^2}{2}} d\mu.
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Combining this with inequality:

$$\langle \nabla \eta^2, \nabla \log(H-\lambda)\rangle \leq \varepsilon |\nabla \eta|^2 + \frac{1}{\varepsilon} \eta^2 |\nabla \log(H-\lambda)|^2,$$

we have

$$\int_M \left(\eta^2 S + \eta^2\left(1 - \frac{1}{\varepsilon}\right)|\nabla \log(H - \lambda)|^2\right)e^{-\frac{|X|^2}{2}} d\mu$$
$$\leq \int_M \left(\varepsilon|\nabla \eta|^2 + \eta^2 + \frac{\lambda}{H - \lambda}\eta^2\right)e^{-\frac{|X|^2}{2}} d\mu,$$

for $\varepsilon > 0$.

we have

$$\int_M (\eta^2 S + \eta^2 (1 - \frac{1}{\varepsilon}) |\nabla \log(H - \lambda)|^2) e^{-\frac{|X|^2}{2}} d\mu$$
$$\leq \int_M (\varepsilon |\nabla \eta|^2 + \eta^2 + \frac{\lambda}{H - \lambda} \eta^2) e^{-\frac{|X|^2}{2}} d\mu,$$

for $\varepsilon > 0$.
Since

$$\frac{\lambda}{H - \lambda} \leq \frac{\lambda f_3}{S} \leq |\lambda| \sqrt{S} \leq |\lambda| (\frac{S}{2\delta} + \frac{\delta}{2})$$

for $\delta > 0$, we have

$$\int_M \left\{ (1 - \frac{|\lambda|}{2\delta})\eta^2 S + \eta^2 (1 - \frac{1}{\varepsilon}) |\nabla \log(H - \lambda)|^2 \right\} e^{-\frac{|X|^2}{2}} d\mu$$
$$\leq \int_M \left(\varepsilon |\nabla \eta|^2 + (1 + \frac{|\lambda|}{2}\delta)\eta^2 \right) e^{-\frac{|X|^2}{2}} d\mu.$$

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By choosing ε, δ and constant $c(n,\lambda)$, we get

$$\int_M \eta^2(S + |\nabla \log(H-\lambda)|^2) e^{-\frac{|X|^2}{2}} d\mu$$
$$\leq c(n,\lambda) \int_M (|\nabla\eta|^2 + \eta^2) e^{-\frac{|X|^2}{2}} d\mu.$$

Since $X : M \to \mathbb{R}^{n+1}$ has polynomial area growth, we can prove, for any $m > 0$,

$$\int_M (1 + |X|^m) e^{-\frac{|X|^2}{2}} d\mu < \infty.$$

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By replacing η with $|X|\eta$, we have

$$\begin{aligned} &\int_M \eta^2 S(1 + |X|^2) e^{-\frac{|X|^2}{2}} d\mu \\ &\leq c(n, \lambda) \int_M (1 + |X|^2) e^{-\frac{|X|^2}{2}} d\mu. \end{aligned}$$

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So, we can apply Stokes formula to our functions.

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$$\begin{aligned}0 \geq &\int_M \left|\nabla \sqrt{S} - \sqrt{S}\nabla \log(H-\lambda)\right|^2 e^{-\frac{|X|^2}{2}} d\mu \\ &+ \int_M \lambda\Big(f_3 - \frac{S}{H-\lambda}\Big)e^{-\frac{|X|^2}{2}} d\mu.\end{aligned}$$

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We obtain that $X:M\to\mathbb{R}^{n+1}$ is isometric to $\mathbf{R}^n$ or $S^k(r)\times\mathbf{R}^{n-k}$ with $\lambda=\frac{k}{r}-r$.

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For n-dimensional complete and non-compact Riemannian manifolds with nonnegative Ricci curvature, geodesic balls have at most polynomial area growth:

$$\mathrm{Area}(B_r(x_0)) \leq Cr^n.$$

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Remark. There exist n-dimensional complete and non-compact gradient shrinking Ricci solitons, which Ricci curvature is not nonnegative.

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For the above problem 5.1, Cheng and Wei (arXiv 2014) have proved the following:

Theorem (Cheng and Wei, arXiv 2014).
Let $X : M \to \mathbb{R}^{n+1}$ be a complete and non-compact proper λ-hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. Then, there is a positive constant C such that for $r \geq 1$,

$$\mathrm{Area}(B_r(0) \cap X(M)) \leq C r^{n+\frac{\lambda^2}{2}-2\beta-\frac{\inf H^2}{2}},$$

where $\beta = \frac{1}{4}\inf(\lambda - H)^2$.

Remark. The estimate in our theorem is best possible because the cylinders $S^k(r_0) \times \mathbb{R}^{n-k}$ satisfy the equality.

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Theorem (Cheng and Wei, arXiv 2014) . A complete and non-compact λ-hypersurface $X : M \to \mathbb{R}^{n+1}$ in the Euclidean space $\mathbb{R}^{n+1}$ has polynomial area growth if and only if $X : M \to \mathbb{R}^{n+1}$ is proper.

5.2. Lower bound growth of area of complete λ-hypersurfaces

Calabi and Yau studied lower bound growth of area for n-dimensional complete and non-compact Riemannian manifolds with nonnegative Ricci curvature. They proved the following:

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Calabi and Yau studied lower bound growth of area for n-dimensional complete and non-compact Riemannian manifolds with nonnegative Ricci curvature. They proved the following:

Theorem (Calabi and Yau).
For n-dimensional complete and non-compact Riemannian manifolds with nonnegative Ricci curvature, geodesic balls have at least linear area growth:

$$\mathrm{Area}(B_r(x_0)) \geq Cr.$$

Cao and Zhou (J. Diff. Geom., 2010) have proved that n-dimensional complete and non-compact gradient shrinking Ricci solitons must have infinite area.

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Furthermore, Munteanu and Wang (Comm. Analy. Geom., 2012) have proved that areas of geodesic balls for n-dimensional complete and non-compact gradient shrinking Ricci solitons have at least linear growth:

$$\mathbf{Area}(B_r(x_0)) \geq Cr.$$

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For complete and noncompact λ-hypersurfaces, Cheng and Wei (arXiv 2014) have proved the following:

Theorem (Cheng and Wei, arXiv 2014).
Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional complete proper λ-hypersurface.
Then, for any $p \in M$, there exists a constant $C > 0$ such that

$$\text{Area}(B_r(0) \cap X(M)) \geq Cr,$$

for all $r > 1$.

Remark. The estimate in our theorem is best possible because the cylinders $S^{n-1}(r_0) \times \mathbb{R}$ satisfy the equality.

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When $\lambda = 0$, that is, for self-shrinkers, Li and Y. Wei (Proc. Amer. math. Soc., 2014) have proved this result.

Thank you!