ON THE DEFORMATION OF CYCLIC QUOTIENT SINGULARITIES

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1. Preliminaries

1.1. **The link.** One of the objects which connects local complex analytic singularities with low-dimensional topology is the link of isolated singularities. Namely, if $(\mathcal{X}, o)$ is an isolated complex analytic germ of complex dimension n, then for any real analytic map (so-called *rug-function*) $\rho : (\mathcal{X}, o) \to [0, \infty)$ with $\rho^{-1}(0) = \{o\}$, the diffeomorphism type of $M_\epsilon := \rho^{-1}(\epsilon)$ $(0 < \epsilon \ll 1)$ is independent of the choice of ρ and ϵ. This oriented real $(2n-1)$-manifold M is called the *link* of $(\mathcal{X}, o)$, it is connected whenever $n \geq 2$. It determines the topology of $(\mathcal{X}, o)$ completely. One of the big challenges of singularity theory is to relate invariants, properties of the analytic structure with the topology of M.

If $n = 1$ then M determines only the number of irreducible components of $(\mathcal{X}, o)$, i.e. it does not even differentiate smooth and non-smooth germs, but for $n = 2$ the information in M is extremely rich (codified in its fundamental group). Notice that although the classification of 3-manifolds is an open problem, singularity links are classified: they are the negative definite plumbings.

The list of relations connecting analytic and topological properties started with the following result of Mumford [27]: M is homeomorphic to S^3 (or, equivalently, $\pi_1(M)$ is trivial) if and only if $(\mathcal{X}, o)$ is smooth. This is just the top of the iceberg: e.g., M characterizes rational and elliptic singularities providing even their Hilbert functions (cf. [1, 2, 21, 28]), or, if M is rational homology sphere and $(\mathcal{X}, o)$ too has some special analytic structure (e.g. weighted homogeneous, or splice-quotient, or $\mathbb{Q}$-Gorenstein) then sheaf-cohomology invariants (e.g. the geometric genus) can also be recovered from M (see e.g. [43, 29, 30, 31, 35, 36, 37, 32, 33, 8, 9]). We emphasize that most of these results (e.g. Mumford's theorem) cannot be generalized to higher dimensions. This imposes the necessity to enrich M with some additional structure induced from the analytic structure of $(\mathcal{X}, o)$. One of the candidates is the contact structure on M. In fact, this has a crucial role even for surface singularities creating the bridge between contact 3-manifolds and symplectic 4-manifolds/complex surface theory. In the present talk, we will restrict ourselves to situation $n = 2$.

1.2. **Contact structures.** The definition of the contact structure is the following: we take a rug function of type $\rho := \sum_{k=1}^{N} |\phi_k|^2$, where $\phi_1, \cdots, \phi_N$ are germs from the maximal ideal of $(\mathcal{X}, o)$ defining an immersion of $\mathcal{X} \setminus \{o\}$ into $\mathbb{C}^N$.

In this case ρ is strict pseudosubharmonic, and $\xi := TM \cap J(TM)$, the J-invariant subspace of TM, defines a contact structure on M. Its isotopy class is called the *canonical contact structure* of M induced by the analytic structure of $(\mathcal{X}, o)$. (Here J is the almost complex structure of $T\mathcal{X}$.)

This definition imposes several questions/problems for any fixed M:

1. *Classify all the possible contact structures induces by different analytic structures supported by the topological type determined by M.*
2. *How is the subclass from (1) related with the class of all contact structures of M?*

A (partial) answer to (1) was given in [10, 11]: all the possible analytic structures induces by different analytic structures are contactomorphic (and we conjecture that they, in fact, are even isotopic). Notice again that this fact is not valid in higher dimensions, as was shown by Ustilovsky.

Although there is an intense activity in classification of contact structures, and a considerably impressive list of positive results finishing the classification for lens spaces, torus bundles over circles, circle bundles over surfaces, some Seifert manifolds (thanks to the work of Giroux, Etnyre, Honda, Lisca, Stipsicz and others, see e.g. [14, 15, 17, 18, 25] and the references listed therein), Part (2) is mainly open. Recall that a contact structure is either *overtwisted* or *tight*, and all overtwisted structure are characterized by the homotopy of their underlying oriented plane field (by a result of Eliashberg). Hence, the difficulty appears in the classification of tight structures. The canonical structure is one of them. Surprisingly, even the finiteness of the possible tight structures on M is not guaranteed in general (as was shown by Colin, Giroux and Honda): although the number of corresponding homotopy classes of plane fields is finite, the number of isotopy classes of tight contact structures in finite if and only if M is non-toroidal (i.e. the resolution graph is star-shaped). It is not clear at all how one can identify in this multitude of structures the canonical one.

In the sequel the canonical contact structure will be denoted by (M, ξ_{st}).

1.3. **Fillings of** (M, ξ_{st})**.** Although in the literature there are many different versions of fillability (holomorphic, Stein, strong/weak symplectic), here we will deal only with Stein one: any Stein manifold whose contact boundary is contactomorphic to (M, ξ_{st}) is a Stein filling of (M, ξ_{st}). For a singularity link, the following questions/problems are natural:

1. *Is any (M, ξ_{st}) Stein fillable?*
2. *Classify all the Stein fillings of (M, ξ_{st}).*
3. *Determine all the Stein fillings 'coming from singularity theory'.*

The answer to (1) is yes: Consider the minimal resolution of $(\mathcal{X}, o)$. It is a holomorphic, non-Stein filling of (M, ξ_{st}), but by a theorem of Bogomolov and de Oliveira [7] this holomorphic structure can be deformed into a Stein one. This construction already provides an example for (3); another one is given by the Milnor fibers of the smoothings of different analytic realizations $(\mathcal{X}, o)$ of the topological type fixed by M (if there are any). More precisely, the existence (and uniqness) of the miniversal deformation of isolated singularities is guaranteed by results of Schlessinger [45] and Grauert [16]. In general, its base space has many irreducible components. A component is called smoothing if the generic fiber over it (the so-called Milnor fiber) is smooth. In general, different analytic structures might have different smoothings; or for a fixed $(\mathcal{X}, o)$, different smoothing components might produce diffeomorphic Milnor fibers. It might also happen that smoothing does not exist at all (for the last two situation see e.g. the case of some simple elliptic singularities [22, 38]).

In fact, in the literature basically only these two constructions are present regarding (3); it is high desire to find some other general constructions too. We notice that rational singularities are always smoothable, moreover, the Milnor fiber of one of the smoothing component (the Artin component) is diffeomorphic with the space of the minimal resolution. The list regarding part (2) starts with a result of Eliashberg [13] showing that $(\mathbb{S}^3, \xi_{st})$ has only one filling, namely the ball. For links of simple and simple elliptic singularities the classification was finished by Ohta and Ono [38, 39]. Moreover, in all these cases all possible Stein fillings are provided by the minimal resolution or Milnor fibers. This parallelism sometimes is really striking. E.g. for simple elliptic singularities with degree $k > 0$, Ohta and Ono proved that the existence of a Stein filling with vanishing first Chern class imposes $k \leqq 9$. This can be compared with the fact that in the case of a Milnor fiber the Chern class is vanishing (by [46]) and the smoothability condition is the same $k \leqq 9$ (cf. [42]).

Fillings of links of quotient surface singularities were classified by Bhupal and Ono [6], and of lens spaces $L(p, q)$ by Lisca [23, 24] (as a generalization a result of McDuff [26] valid for the spaces $L(p, 1)$, for all $p \geq 2$). We will return to Lisca's result in the next sections showing that Lisca's list agrees perfectly with the list of Milnor fibers (or, with the smoothing/all deformation components).

We would like to notice that this phenomenon, namely that all the Stein fillings are obtained either by minimal resolution or Milnor fibers, at some point of the singularity complexity, might stop. (This emphasizes the importance of the research in direction (3) even more.) Also, in general, the finiteness of the Stein fillings might fail too: Ohta and Ono produced on some singularity links infinitely many symplectic fillings [40], also recently Ozbagci and Stipsicz (and independently Smith) have shown that certain contact structures have infinitely many Stein fillings (although they are not singularity links, one expects that at some moment similar fact will be established for some singularity links as well).

2. The main result of [34]

2.1. We recall briefly the classification of fillings of lens spaces established by Lisca. Let p, q be coprime integers such that $p > q > 0$. Let $L(p, q)$ be an oriented lens space. Lisca provides by surgery diagrams a list of compact oriented 4-manifolds $W_{p,q}(\underline{k})$ with boundary $L(p, q)$. They are parametrized by a set $K_r(\frac{p}{p-q})$ of sequences of integers $\underline{k} \in \mathbb{N}^r$ (cf. (2.3)). He showed that each manifold $W_{p,q}(\underline{k})$ admits a structure of Stein surface, filling $(L(p, q), \xi_{st})$, and that any symplectic filling of $(L(p, q), \xi_{st})$ is orientation-preserving diffeomorphic to a manifold obtained from one of the $W_{p,q}(\underline{k})$ by a composition of blow-ups. In general, the oriented diffeomorphism type of the boundary and the parameter $\underline{k}$ does not determine uniquely the (orientation-preserving) diffeomorphism type of the fillings: for some pairs the corresponding types might coincide.

Lisca also noted that, following the works of Christophersen [12] and Stevens [47], $K_r(\frac{p}{p-q})$ parametrizes also the irreducible components of the reduced miniversal base space of deformations of the *cyclic quotient* (or *Hirzebruch-Jung*) singularity $\mathcal{X}_{p,q}$. By definition, $(\mathcal{X}_{p,q}, o)$ is the germ of the quotient $\mathcal{X}_{p,q}$ of $\mathbb{C}^2$ by the action $(x,y) \to (\xi x, \xi^q y)$ of the cyclic group $\{\xi \in \mathbb{C}\,,\ \xi^p = 1\} \simeq \mathbb{Z}/p\mathbb{Z}$.

Its oriented link is the oriented lens space $L(p,q)$.

This singularity is 'taut', i.e. the topological type supports only one analytic structure. Since each component of the miniversal space is in this case a smoothing component, Lisca conjectured in [24, page 768] that *the Milnor fiber of the irreducible component of the reduced miniversal base space of the cyclic quotient singularity $\mathcal{X}_{p,q}$, parametrized in* [47] *by $\underline{k} \in K_r(\frac{p}{p-q})$ is diffeomorphic to $W_{p,q}(\underline{k})$.*

On the other hand, in [19], de Jong and van Straten studied by an approach completely different from Christophersen and Stevens the deformation theory of cyclic quotient singularities (as a particular case of sandwiched singularities). They also parametrized the Milnor fibers of $\mathcal{X}_{p,q}$ using the elements of the set $K_r(\frac{p}{p-q})$. Therefore, one can formulate the previous conjecture for their parametrization as well.

2.2. [34] answers positively these questions. Its main results are the following:

1. *One defines an additional structure associated with any (non-necessarily oriented) lens space: the 'order'. Its meaning is the following: geometrically it is a (total) order of the two solid tori separated by the (unique) splitting torus of the lens space; in plumbing language, it is an order of the two ends of the plumbing graph (provided that this graph has at least two vertices). Then one shows that the oriented diffeomorphism type and the order of the boundary, together with the parameter $\underline{k}$ determines uniquely the filling.*
2. *One endows in a natural way all the boundaries of the spaces involved (Lisca's fillings $W_{p,q}(\underline{k})$, Christophersen-Stevens' Milnor fibers $F_{p,q}(\underline{k})$, and de Jong-van Straten's Milnor fibers $F'_{p,q}(\underline{k})$) with orders (this extra structure is denoted by $*$). Then one proves that all these spaces are connected by orientation diffeomorphisms which preserve the order of their boundaries: $W_{p,q}(\underline{k})^* \simeq F_{p,q}(\underline{k})^* \simeq F'_{p,q}(\underline{k})^*$. This is an even stronger statement than the result expected by Lisca's conjecture since it eliminates the ambiguities present in Lisca's classification.*
3. *In fact,* [34] *even provides a fourth description of the Milnor fibers constructed by a minimal sequence of blow ups of the projective plane which eliminates the indeterminacies of a rational function which depends on $\underline{k}$. This is in the spirit of Balke's work* [4].
4. *As a byproduct it follows that both Christophersen-Stevens and de Jong-van Straten parametrized the components of the miniversal base space in the same way (a fact not known before, as far as we know).*

5. *Moreover, one obtains that the Milnor fibers corresponding to the various irreducible components of the miniversal space of deformations of $\mathcal{X}_{p,q}$ are pairwise non-diffeomorphic by orientation-preserving diffeomorphisms whose restrictions to the boundaries preserve the order.*

2.3. **Notations.** If $\underline{x} = (x_1, \dots, x_n)$ are variables, the *Hirzebruch-Jung continued fraction* $[x_1, \dots, x_n]$ can be defined by induction on n through the formulae: $[x_1] = x_1$ and $[x_1, \dots, x_n] = x_1 - 1/[x_2, \dots, x_n]$ for $n \geq 2$. One shows that:

$$[x_1, \dots, x_n] = \frac{Z_n(x_1, \dots, x_n)}{Z_{n-1}(x_2, \dots, x_n)}$$

where the polynomials $Z_n \in \mathbb{Z}[x_1, \dots, x_n]$ satisfy the inductive formulae:

(2.3.2.1)
$$Z_n(x_1, \dots, x_n) = x_1 \cdot Z_{n-1}(x_2, \dots, x_n) - Z_{n-2}(x_3, \dots, x_n) \text{ for all } n \geq 1,$$

with $Z_{-1} \equiv 0$, $Z_0 \equiv 1$ and $Z_1(x) = x$. In fact, $Z_n(\underline{x})$ equals the determinant of the matrix $M(\underline{x}) \in \mathrm{Mat}_{n,n}(\mathbb{Z})$, whose entries are $M_{i,i} = x_i$, $M_{i,j} = -1$ if $|i - j| = 1$, $M_{ij} = 0$ otherwise. Hence, besides (2.3.2.1), they satisfy many 'determinantal relations'. E.g.: $Z_n(x_1, \dots, x_n) = Z_n(x_n, \dots, x_1)$. Following [41], $\underline{x} \in \mathbb{N}^n$ is *admissible* if the matrix $M(\underline{x})$ is positive semi-definite of rank $\geq n - 1$.

Denote by $\mathrm{adm}(\mathbb{N}^n)$ the set of admissible n-tuples.

If $\underline{x}$ is admissible and $n > 1$, then each $x_i > 0$. Moreover, if $[x_1, \dots, x_n]$ is admissible then $[x_n, \dots, x_1]$ is admissible too. For any $r \geq 1$, denote:

$$K_r := \{\underline{k} = (k_1, \dots, k_r) \in \mathrm{adm}(\mathbb{N}^r) | \ \ [k_1, \dots, k_r] = 0\}.$$

For $\underline{k} = (k_1, \dots, k_r) \in K_r$ set $\underline{k}' := (k_r, \dots, k_1) \in K_r$.

For p, q as above and HJ-expansion $\frac{p}{p-q} = [a_1, \dots, a_r]$, set:

$$K_r(\tfrac{p}{p-q}) = K_r(\underline{a}) := \{\underline{k} \in K_r \mid \underline{k} \leqq \underline{a}\} \subset K_r.$$

Here, $\underline{k} \leqq \underline{a}$ means that $k_i \leqq a_i$ for all i.

3. The description of the Milnor fibers

In the sequel we will not say more about Lisca's construction, instead, we will describe briefly the construction (2.1)(3) of the Milnor fibers. One starts with the description of Christophersen and Stevens on the structure of the reduced miniversal base space of cyclic quotients [5, 12, 48] (cf. also with [3]).

3.1. **The space $\mathcal{X}_{p,q}$.** First we concentrate on $\mathcal{X}_{p,q}$. It can be embedded into $\mathbb{C}^{r+2}$ by some regular functions $z_0, \dots, z_{r+1}$. Some of the equations of the embedding are

(3.1.3.1)
$$z_{i-1}z_{i+1} - z_i^{a_i} = 0 \ \text{ for all } i \in \{1, \dots, r\}.$$

Using equations (3.1.3.1) and induction, one shows that the restriction of each z_i to $\mathcal{X}_{p,q}$ is a rational function in (z_0, z_1) of the form

$$z_i = z_1^{Z_{i-1}(a_1,\dots,a_{i-1})} \cdot z_0^{-Z_{i-2}(a_2,\dots,a_{i-1})} \text{ for } i \in \{1, \dots, r+1\}.$$

The equations are weighted homogeneous, however the weights $w_i := w(z_i)$ are not unique. With the choice $w_0 = w_1 = 1$ one has $w_i = Z_{i-1}(a_1, \dots, a_{i-1}) - Z_{i-2}(a_2, \dots, a_{i-1})$ for all $i \geq 1$, and $1 = w_0 = w_1 \leqq w_2 \leqq \cdots \leqq w_{r+1} = q$.

3.2. **The deformation.** Next, we fix $\underline{k} \in K_r(\underline{a})$, and we denote by $S^{CS}_{\underline{k}}$ the corresponding deformation component (as it is described by Christophersen and Stevens). Then, we consider a special 1-parameter deformation with equations $\mathcal{E}^t_{\underline{k}}$ of $\mathcal{X}_{p,q}$. This deformation is determined by the deformed equations of (3.1.3.1) (cf. [3], [47, (2.2)]). These are:

$$z_{i-1}z_{i+1} = z_i^{a_i} + t \cdot z_i^{k_i} \text{ for all } i \in \{1, \dots, r\}, \tag{3.2.3.1}$$

where $t \in \mathbb{C}$. Let $\mathcal{X}^t_{\underline{k}}$ be the affine space determined by the equations $\mathcal{E}^t_{\underline{k}}$ in $\mathbb{C}^{r+2}$.

Lemma 3.2.3.2. *The deformation $t \mapsto \mathcal{X}^t_{\underline{k}}$ has negative weight and is a smoothing belonging to the component $S^{CS}_{\underline{k}}$. Hence, $\mathcal{X}^t_{\underline{k}}$ is a smooth affine space for $t \neq 0$.*

In particular, by [49, (2.2)] $\mathcal{X}^t_{\underline{k}}$ is diffeomorphic to the Milnor fiber of $S^{CS}_{\underline{k}}$.

In the sequel we will denote by $\widehat{\mathcal{X}^t_{\underline{k}}}$ the closure of $\mathcal{X}^t_{\underline{k}}$ in $\mathbb{P}^{r+2}$.

surface.]$\mathcal{X}^t_{\underline{k}}$ as a rational
surface.
Similarly as for $\mathcal{X}_{p,q}$, using (3.2.3.1), on
$\mathcal{X}^t_{\underline{k}}$ all the coordinates z_i can be
expressed as rational functions in (z_0, z_1):

Lemma 3.2.3.3. *For each $i \in \{1, \dots, r+1\}$, on $\mathcal{X}^t_{\underline{k}}$ one has:*

$$z_i = z_0^{-Z_{i-2}(a_2,\dots,a_{i-1})} P_i$$

for some $P_i \in \mathbb{Z}[t, z_0, z_1]$. The polynomials P_i satisfy the inductive relations:

$$P_{i-1} \cdot P_{i+1} = P_i^{a_i} + tP_i^{k_i} \cdot z_0^{(a_i-k_i)\cdot Z_{i-2}(a_2,\dots,a_{i-1})} \tag{3.2.3.4}$$

with $P_1 = z_1$ and with the convention $P_0 = 1$.

Consider the application $\pi : \mathbb{C}^2 \setminus \{z_0 = 0\} \longrightarrow \mathcal{X}^t_{\underline{k}}$ given by

$$(z_0, z_1) \mapsto (z_0, z_1, P_2, \dots, z_0^{-Z_{i-2}(a_2,\dots,a_{i-1})} P_i, \dots, z_0^{-(p-q)} P_{r+1}) \in \mathbb{C}^{r+2}.$$

We are interested in the birational map $\mathbb{C}^2 \dashrightarrow \mathcal{X}^t_{\underline{k}}$, still denoted by π, and its extension $\widehat{\pi} : \mathbb{P}^2 \dashrightarrow \widehat{\mathcal{X}^t_{\underline{k}}}$. Let $\rho'_{\underline{k}} : B'\mathbb{P}^2 \to \mathbb{P}^2$ be the minimal sequence of blow ups such that $\widehat{\pi} \circ \rho'_{\underline{k}}$ extends to a regular map $B'\mathbb{P}^2 \to \widehat{\mathcal{X}^t_{\underline{k}}}$. Let $L_\infty \subset \mathbb{P}^2$ be the line at infinity and by L_0 the closure in $\mathbb{P}^2$ of $\{z_0 = 0\}$. We use the same notations for their strict transforms via blow ups of $\mathbb{P}^2$. Since the projection $pr : \widehat{\mathcal{X}^t_{\underline{k}}} \to \mathbb{C}^2$ is regular and $pr \circ \pi$ is the identity, one gets that $\widehat{\pi} \circ \rho'_{\underline{k}}$ sends L_0 and the total transform of L_∞ in the curve at infinity $\widehat{\mathcal{X}^t_{\underline{k}}} \setminus \mathcal{X}^t_{\underline{k}}$.

Therefore, let us modify $\rho'_{\underline{k}}$ into $\rho_{\underline{k}} : B\mathbb{P}^2 \to \mathbb{P}^2$, the *minimal* sequence of blow ups which resolve the indeterminacies of $\widehat{\pi}$ *sitting in* $\mathbb{C}^2$ (hence $\rho'_{\underline{k}}$ and $\rho_{\underline{k}}$ over $\mathbb{C}^2$ coincide). Denote by E_π its exceptional curve and by C_π the union of those irreducible components of E_π which are sent to $C^\infty_{\underline{k}}$. Set $B\mathbb{C}^2 := B\mathbb{P}^2 \setminus L_\infty$. Summing up all the above discussions, one obtains:

Theorem 3.2.3.5. *The restriction of $\widehat{\pi} \circ \rho_{\underline{k}}$ induces an isomorphism $B\mathbb{C}^2 \setminus (L_0 \cup C_\pi) \to \mathcal{X}^t_{\underline{k}}$. In particular, the Milnor fiber can be realized as the complement of the projective curve $L_\infty \cup L_0 \cup C_\pi$ in $B\mathbb{P}^2$.*

The point is that the indeterminacies of $\widehat{\pi}$ above $\mathbb{C}^2$, hence the modification $\rho_{\underline{k}}$ too, can be described precisely. This leads to the following description of the Milnor fiber.

Corollary 3.2.3.6. *Consider the lines L_∞ and L_0 on $\mathbb{P}^2$ as above. Blow up $r - 1 + \sum_{i=1}^r (a_i - k_i)$ infinitely closed points of L_0 in order to get the dual graph in Figure 1 of the configuration of the total transform of $L_\infty \cup L_0$ (this procedure topologically is unique, and its existence is guaranteed by the fact that $\underline{k} \in K_r(\underline{a})$). Denote the space obtained by this modification by $B\mathbb{P}^2$. Then the Milnor fiber $\mathcal{X}^t_{\underline{k}}$ of $S^{CS}_{\underline{k}}$ is diffeomorphic to $B\mathbb{P}^2 \setminus (\cup_{j=0}^r V_j)$.*

Moreover, let T be a small open tubular neighbourhood of $\cup_{j=0}^r V_j$, and set $F_{p,q}(\underline{k}) = B\mathbb{P}^2 \setminus T$. Then $F_{p,q}(\underline{k})$ is a representative of the Milnor fiber of $S^{CS}_{\underline{k}}$ as a manifold with boundary whose boundary is $L(p, q)$.

Furthermore, the marking $\{V_i\}_i$ as in the Figure 1, defines on the boundary of $F_{p,q}(\underline{k})$ an order; denote this supplemented space by $F_{p,q}(\underline{k})^*$. Then its ordered boundary is $L(p, q)^*$ endowed with the preferred order.

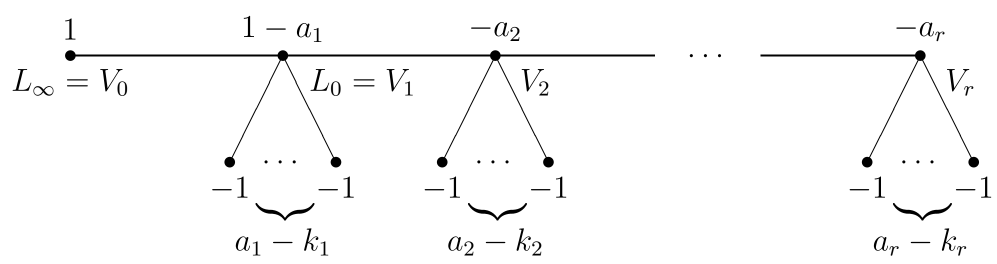

Figure 1

Remark 3.2.3.7. In fact, $\rho_{\underline{k}}$ serves also as the minimal modification which eliminates the indeterminacy of the last component of π from (**??**), namely of the rational function $z_{r+1} = P_{r+1}/z_0^{p-q}$. In particular, we find the following alternative description of the Milnor fiber $F_{p,q}(\underline{k})$:

For each $\underline{k} \in K_r(\underline{a})$, define the polynomial P_{r+1} via the inductive system (3.2.3.4). Let $\rho_{\underline{k}} : B\mathbb{P}^2 \to \mathbb{P}^2$ be the minimal modification of $\mathbb{P}^2$ which eliminates the indeterminacy points of P_{r+1}/z_0^{p-q} sitting in $\mathbb{C}^2$. Then the dual graph of the total transform of $L_\infty \cup L_0$ has the form indicated in Figure 1, and $F_{p,q}(\underline{k})$ is orientation-preserving diffeomorphic to $B\mathbb{P}^2 \setminus (\cup_{j=0}^r V_j)$.

3.3. **An identification criterion of the Milnor fibers.** The next criterion generalizes Lisca's criterion [24, §7] to recognize the fillings of $L(p,q)$, it is valid for the spaces with *ordered* boundaries. It also implies immediately part (2) of (2.1), namely that $W_{p,q}(\underline{k})^* \simeq F_{p,q}(\underline{k})^*$.

Associate to the sequence $\underline{a}$ the string $G(\underline{a})$ decorated with the entries $a_1, \dots, a_r$ (this is also the minimal resolution graph of $\mathcal{X}_{p,p-q}$). Regarded $G(\underline{a})$ as a plumbed graph, it determines the plumbed 4-manifold $\Pi(\underline{a})$ (which is diffeomorphic to the minimal resolution space of $\mathcal{X}_{p,p-q}$), whose oriented boundary is $\overline{L(p,q)}$.

Let F be a Stein filling of $(L(p,q), \xi_{st})$, e.g. one of the Milnor fibers considered above. Set V for the closed 4-manifold obtained by gluing F and $\Pi(\underline{a})$ via an orientation preserving diffeomorphism $\phi : \partial F \to \partial\overline{\Pi(\underline{a})}$ of their boundaries. Denote by $\{s_i\}_{1\leqq i\leqq r}$ the classes of 2-spheres $\{S_i\}_{1\leqq i\leqq r}$ in $H_2(\Pi(\underline{a}))$ (listed in the same order as $\{a_i\}_{1\leqq i\leqq r}$), and also their images via the monomorphism $H_2(\Pi(\underline{a})) \to H_2(V)$ induced by the inclusion.

Lisca's criterion (implied also by the results of (3.2)) reads as follows:

Proposition 3.3.3.1. *For all $i \in \{1, \dots, r\}$ one has*

$$\#\{e \in H_2(V) \mid \ e^2 = -1,\ s_i \cdot e \neq 0,\ s_j \cdot e = 0 \text{ for all } j \neq i\} = 2(a_i - k_i)$$

for some $\underline{k} \in K_r(\underline{a})$. In this way one gets the pair $(\underline{a}, \underline{k})$ and F is orientation-preserving diffeomorphic to $F_{p,q}(\underline{k})$ ($\simeq F_{p,q'}(\underline{k}')$).

One verifies that the above criterion is independent of the choice of the diffeomorphism ϕ. In fact, even the diffeomorphism type of the manifold V is independent of the choice of ϕ.

Notice that $\{S_i\}_{1\leqq i\leqq r}$ and $\{S_{r-i}\}_{1\leqq i\leqq r}$ cannot be distinguished, hence the above algorithm does not differentiate $(\underline{a}, \underline{k})$ from $(\underline{a}', \underline{k}')$, or $F_{p,q}(\underline{k})$ from $F_{p,q'}(\underline{k}')$. On the other hand, these are the only ambiguities. (In fact, if $r = 1$, or even of $r > 1$ but $\underline{a}$ and $\underline{k}$ are symmetric, then there is no ambiguity, since $(p,q,\underline{k}) = (p,q',\underline{k}')$.)

Using the notion of 'order' of the boundaries, one can eliminate the above ambiguity. Notice that any diffeomorphism $F_{p,q}(\underline{k}) \to F_{p,q'}(\underline{k}')$ (whenever $(p,q,\underline{k}) \neq (p,q',\underline{k}')$) does not preserve any fixed order of the boundary. One has:

Theorem 3.3.3.2. *All the spaces $F_{p,q}(\underline{k})^*$ are different, hence their boundaries $L(p,q)^*$ and $\underline{k} \in K_r(\underline{a})$ determines uniquely all the Milnor fibers up to orientation-preserving diffeomorphisms which preserve the order of the boundary.*

In order to prove this, the criterion (3.3.3.1) is modified as follows. Let F^* be a Stein filling of $(L(p,q), \xi_{st})$ with an order on its boundary. Consider $\Pi(\underline{a})^*$ with its preferred order (provided a well-determined order of the s_i's, cf. [34]).

Construct V as in (3.3.3.1), and consider the two pairs $(q, \underline{k})$ and $(q', \underline{k}')$ provided (but undecided) by (3.3.3.1).

Proposition 3.3.3.3. *If ϕ preserves (resp. reverses) the orders of the boundary then F^* is orientation and order preserving diffeomorphic to $F_{p,q}(\underline{k})^*$ (resp. to $F_{p,q'}(\underline{k}')^*$).*

References

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