Closing Lemmas.
&
Symplectic Geometry.

入江 慶 (RIMS)

Hamiltonian Dynamics

(M, ω) : Symplectic mfd $\overset{\text{def}}{\Longleftrightarrow}$ M: $2n$-dim'l mfd.
$\omega \in \Omega^2(M)$
$d\omega = 0$
$\underline{\omega^n}(p) \neq 0 \quad (\forall p \in M)$
volume form.

$\forall H : M \to \mathbb{R}$ (Hamiltonian)

$\exists ! X_H \in \mathfrak{X}(M)$ s.t. $i_{X_H}\omega = -dH$.
(Hamiltonian vec field)

Darboux's thm

$\exists$ Local chart $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ s.t. $\omega = \sum_{j=1}^{n} dp_j\, dq_j$

Hamilton's eqns: $\dot{q}_j = \dfrac{\partial H}{\partial p_j}(q, p)$, $\dot{p}_j = -\dfrac{\partial H}{\partial q_j}(q, p)$ $(1 \leq j \leq n)$

Assume M: closed (i.e. M: cpt & $\partial M = \phi$)

- $\exists (\varphi_H^t)_{t \in \mathbb{R}}$: isotopy on M. s.t.

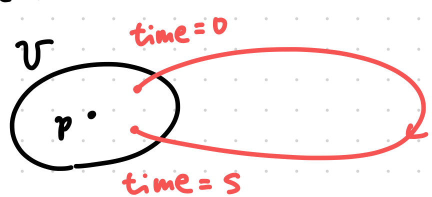
$$\varphi_H^0 = \mathrm{id}_M, \qquad \frac{d}{dt}\varphi_H^t(-) = X_H(\varphi_H^t(-)) \quad (\forall t \in \mathbb{R})$$

- $\forall t \in \mathbb{R}$, φ_H^t preserves ω $(\because L_{X_H}\omega = 0)$.

In particular, 〃 ω^n (volume preserving).

- $\forall p \in M$ is nonwondering (非遊走)

i.e. $p \in {}^{\forall}U \underset{\text{open}}{\subset} M$, ${}^{\forall}t > 0$, ${}^{\exists}s > t$ s.t. $\varphi_H^s(U) \cap U \neq \phi$.



Cf. Poincaré recurrence thm.

$p \in M$: Periodic $\underset{\text{def}}{\iff}$ $\exists t>0,\ \varphi_H^t(p) = p$

Periodic $\Longrightarrow$ Nonwondering

$\cancel{\Longleftarrow}$

C^r-closing Lemma:

"$\Longleftarrow$" is true "modulo C^r-small perturbations".

Hamiltonian C^1-closing Lemma (Pugh-Robinson 1983)

(M, ω): closed. symplectic mfd.

$\emptyset \neq \mathcal{V} \underset{\text{open}}{\subset} M$, $\quad \emptyset \neq \mathcal{U} \underset{\text{open}}{\subseteq} C^2(M)$

$\Rightarrow \exists H \in \mathcal{U}$ s.t. $\mathcal{V} \cap P_\omega(H) \neq \emptyset$.

$$P_\omega(H) = \{ \text{Periodic pts of } X_H \}$$

$\mathcal{V}$

<u>Rmk</u>. $H \underset{C^2}{\doteq} H' \Rightarrow X_H \underset{C^1}{\doteq} X_{H'}$ $\quad$ Recall: $i_{X_H}\omega = -dH$.

<u>Cf</u>. Pugh's C^1-closing Lemma (1967)

Con. (Generic density of periodic orbits)

(M, ω): closed. symplectic mfd.

$\{ H \in C^2(M) \mid P_\omega(H) \text{ is dense in } M \} \ (=: \mathcal{H})$

is residual in $C^2(M)$

i.e. $\exists (\mathcal{O}_i)_{i=1}^{\infty}$: open & dense sets in $C^2(M)$

s.t. $\bigcap_{i=1}^{\infty} \mathcal{O}_i \subset \mathcal{H}$.

<u>Bairé category thm</u> On any completely metrizable top sp.

residual $\Rightarrow$ dense.

- Hamiltonian C^∞-closing Lem fm 2-dim. symp. mfds is (obviously) true.
- Hamiltonian C^r-closing Lem fm 2n-dim symp. mfds ($\forall n \geqq 2$) is NOT true if $r \geqq 2n + \text{const.}$ (in particular, $r = \infty$)

<u>Thm</u> (Herman, 1991)

$$n \geqq 2 \Rightarrow \exists \begin{cases} \bullet\ \omega : \text{symplectic form on } T^{2n} \\ \bullet\ \mathcal{V} : \text{nonempty open set in } T^{2n} \\ \bullet\ \mathcal{U} : \quad \text{"} \quad C^{2n+\text{const}}(T^{2n}) \end{cases}$$

s.t. $H \in \mathcal{U} \Rightarrow \mathcal{V} \cap P_\omega(H) = \emptyset$.

Summary of this talk

Some "nice" Hamiltonian systems (e.g. 3-dim'l Reeb flows)

satisfy "strong closing Lemma"

($\Rightarrow$ C^∞ closing Lemma.)

Proofs are based on min-max theory

in symplectic geometry.

Y : $2n-1$ dim'l mfd.

$\lambda \in \Omega^1(Y)$: contact form (接触形式)

$\underset{\text{defn}}{\Longleftrightarrow}$ $\lambda \wedge d\lambda^{n-1}(p) \neq 0$ for $\forall p \in Y$.

Rmk. λ : contact form

- $\lambda' \in \Omega^1(Y)$, $\lambda \underset{C^1}{\doteqdot} \lambda' \Longrightarrow \lambda'$: contact form
- $\forall h \in C^\infty(Y)$, $e^h \lambda$: contact form

Reeb vector field

λ : Contact form on Y

$\Longrightarrow \exists ! R_\lambda \in \mathfrak{X}(Y)$ s.t.

$$i_{R_\lambda}(d\lambda) \equiv 0, \quad \lambda(R_\lambda) \equiv 1$$

Reeb dynamics are Hamiltonian dynamics

$(Y \times \mathbb{R},\ d(e^r \lambda))$: Symplectic mfd

Coordinate on $\mathbb{R}$ (Symplectization of (Y, λ))

$$X_{e^r} = R_\lambda .$$

Periodic Reeb orbits

Period (or Action) of γ.

$$\mathcal{P}(Y,\lambda) := \left\{ \gamma : \underset{\mathbb{R}/\mathbb{Z}}{\overset{\shortparallel}{S^1}} \to Y \;\middle|\; \begin{array}{l} \exists\, T_\gamma > 0 \text{ s.t.} \\ \dot{\gamma} = T_\gamma \cdot R_\lambda(\gamma) \end{array} \right\}$$

$$A_+(Y,\lambda) := \{0\} \cup \left\{ \sum_{k=1}^{m} T_{\gamma_i} \;\middle|\; \begin{array}{l} m \geqq 1 \\ \gamma_1, \dots, \gamma_m \in \mathcal{P}(Y,\lambda) \end{array} \right\}$$

$$\subset \mathbb{R}_{\geq 0}$$

Lem $\operatorname{meas}(A_+(Y,\lambda)) = 0.$

Ex 1 $\quad 0 < a_1 \leq \cdots \leq a_n.$

$$E_{a_1,\cdots,a_n} := \left\{ \begin{array}{l} (q_1,\cdots,q_n, \\ \quad p_1,\cdots,p_n) \in \mathbb{R}^{2n} \end{array} \;\middle|\; \sum_{j=1}^{n} \frac{\pi (q_j^2 + p_j^2)}{a_j} \leq 1 \right\}$$

$$\lambda_{a_1,\cdots,a_n} := \sum_{j=1}^{n} \frac{p_j\, dq_j - q_j\, dp_j}{2} \bigg|_{\partial E_{a_1,\cdots,a_n}}$$

$$R_{\lambda_{a_1,\cdots,a_n}} = \sum_{j=1}^{n} \frac{2\pi}{a_j} \cdot \left(p_j \frac{\partial}{\partial q_j} - q_j \frac{\partial}{\partial p_j} \right)$$

$a_i / a_j \in \mathbb{Q}$ for any $i \leq j$ $\Rightarrow$ $\forall$orbit is periodic.

$a_i / a_j \notin \mathbb{Q}$ for any $i < j$ $\Rightarrow$ $\# P_{\text{simple}}(Y, \lambda) = n.$

Ex 2. (Geodesic Flow)

N : C^∞-mfd.

Define $\lambda_N \in \Omega^1(T^*N)$ by

$$\lambda_N(v) := p\left(pr_*(v)\right) \qquad \left(\begin{array}{l} q \in N,\ p \in T_q^*N \\ v \in T_{(q,p)}(T^*N) \end{array}\right)$$

g: Riem. met on N

$\Rightarrow$ • $\lambda_N \big|_{\{(q,p) \in T^*N \,\mid\, \|p\|_g = 1\}}$ is a contact form.

unit sphene cot bdl

• Reeb Flow = Geodesic Flow.

• Periodic Reeb orbits. $\underset{1:1}{\longleftrightarrow}$ Nonconst. closed geodesics of (N, g)

<u>Thm</u> Y: Closed. odd-dim'l mfd.

For a C^2-generic contact form λ on Y,

$$\bigcup_{\gamma \in P(Y, \lambda)} \mathrm{Im}(\gamma) \text{ is dense in } Y.$$

∵ Apply Hamiltonian C^1-closing Lemma to $(Y \times \mathbb{R}, d(e^r \lambda))$ with a Hamiltonian e^r

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Thm (I, 2015)

Y: closed. 3-dim'l mfd.

For a C^∞-generic contact form λ on Y,

$$\bigcup_{\gamma \in P(Y,\lambda)} \mathrm{Im}(\gamma) \text{ is dense in } Y.$$

This follows from

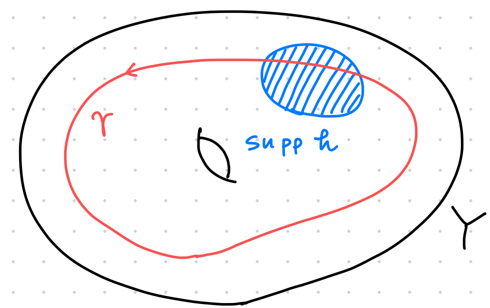
"Strong Closing Lemma" for 3-dim'l Reeb Flows

Def. Y: Closed mfd, λ: Contact form on Y

λ satisfies <u>strong closing property</u> (SCP)

$\overset{\text{def.}}{\Longleftrightarrow}$ $\forall h \in C^\infty(Y, \mathbb{R}_{\geq 0}) \setminus \{0\}$

$\exists \left\{ \begin{array}{l} t \in [0,1] \\ \gamma \in P(e^{th}\lambda) \end{array} \right\}$ s.t. $\mathrm{Im}(\gamma) \cap \mathrm{supp}(h) \neq \phi$.



Rmk λ satisfies SCP $\Rightarrow$ C^∞-CP.

i.e. $\mathcal{V}$: nonempty open set in Y

$\mathcal{U}$: open nbd of $0 \in C^\infty(Y)$

$\Longrightarrow \exists \left\{ \begin{array}{l} h' \in \mathcal{U} \\ \gamma \in P(e^{h'}\lambda) \end{array} \right\}$ s.t. $\mathrm{Im}(\gamma) \cap \mathcal{V} \neq \phi$.

Strong Closing Lemma for 3-dim Reeb Flows (I, 2015)

Y: Closed 3-mfd.

$\Rightarrow$ $^\forall$contact form on Y satisfies SCP

(thus C^∞-CP)

Cor.

For a C^∞-generic contact form λ on Y,

$\bigcup_{\gamma \in \mathcal{P}(Y, \lambda)} \mathrm{Im}(\gamma)$ is dense in Y.

(Y, ξ) : <u>contact mfd</u>

$\overset{\Longleftrightarrow}{\text{def}}$ • Y : $2n-1$ dim'l mfd.

• ξ : $2n-2$ dim'l co-oriented subbdl of TY.

• $\exists \lambda$: contact form on Y. s.t.

$$\boxed{\forall p \in Y,\ \ \xi_p = \ker \lambda_p,\ \ \begin{array}{c} T_pY/\xi_p \simeq \mathbb{R} \\ [v] \mapsto \lambda(v) \end{array} : \text{compatible w/ co-orientation.}} \ \bigstar$$

$$\Lambda(Y, \xi) := \{ \lambda : \text{contact form on } Y \mid \bigstar \}$$

<u>Rmk</u> $\forall \lambda \in \Lambda(Y, \xi)$,

$$\Lambda(Y, \xi) = \{ e^{h} \lambda \mid h \in C^\infty(Y) \}$$

Def (Action selector / Spectral invariant)

(Y, ξ): closed contact mfd.

Action Selector of (Y, ξ) is a map $c : \Delta(Y, \xi) \to \mathbb{R}_{\geq 0}$

such that $\forall \lambda \in \Delta(Y, \xi)$ satisfies:

Spectrality : $c(\lambda) \in A_+(Y.\lambda)$ $:= \{0\} \cup \left\{ \sum_{j=1}^{m} T_{\gamma_j} \,\middle|\, \begin{array}{l} m \geqq 1 \\ \gamma_1, \ldots, \gamma_m \\ \in P(Y.\lambda) \end{array} \right\}$

Conformality : $\forall a > 0, \quad c(a\lambda) = a \cdot c(\lambda)$

Monotonicity : $\forall h \in C^\infty(Y, \mathbb{R}_{\geq 0}), \quad c(e^h \lambda) \geqq c(\lambda)$

C^0-continuity : $\forall \varepsilon > 0, \exists \delta > 0.$ s.t.

$$\|h\|_{C^0} \leq \delta \Rightarrow |c(e^h\lambda) - c(\lambda)| \leq \varepsilon.$$

Typical Construction of Action Selectors

$\mathcal{Z}(Y) := \{ C^\infty\text{-immersions from } S^1 \text{ to } Y \} / \mathrm{Diff}_+(S^1)$

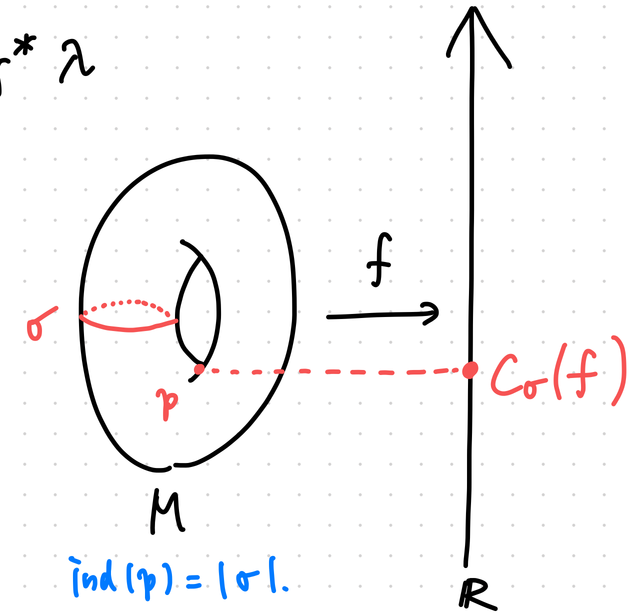
$\mathcal{A}_\lambda : \mathcal{Z}(Y) \to \mathbb{R} : [\gamma] \mapsto \int_{S^1} \gamma^* \lambda$

$\mathrm{Crit}(\mathcal{A}_\lambda) = \mathcal{P}(Y, \lambda) / S^1$

σ: nonzero "homology class" of $\mathcal{Z}(Y)$

⇝ Define AS c_σ by

$$c_\sigma(\lambda) := \inf_{\substack{c:\text{cycle} \\ [c]=\sigma}} \left(\sup \mathcal{A}_\lambda|_c \right)$$



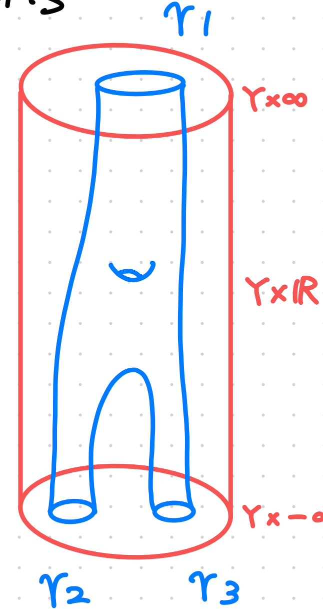
Need ∞-dim'l homology theory. (Floer-type homology)

Construction of Floer-type homology (Eliashberg - Givental - Hofer 2000)

Vec. sp. gen'd by (finite sets of) Periodic Reeb orbits

$\circlearrowright$

∂: counts pseudo-hol curves in the symplectization asymptotic to PROs.



Examples

1. Embedded Contact Homology (Hutchings, H-Taubes)

Def'd only when $\dim Y = 3$. counts "ECH index = 1" curves.

2. Contact Homology. (Pardon, Bao-Honda, Ishikawa)

Def'd for any dim. Counts $\left\{ \begin{array}{l} \cdot \text{ genus} = 0 \\ \cdot \ \# \text{pos. puncture} = 1 \end{array} \right\}$ curves.

Key Lemma

(Y, ξ) : closed, contact mfd.

$\lambda \in \Lambda(Y, \xi)$

$$\boxed{\begin{array}{l} \forall h \in C^\infty(Y, \mathbb{R}_{\geq 0}) \setminus \{0\} \\ \exists c : \text{AS of } (Y, \xi) \text{ s.t. } c(e^h \lambda) > c(\lambda) \end{array}}$$

Local Sensitivity

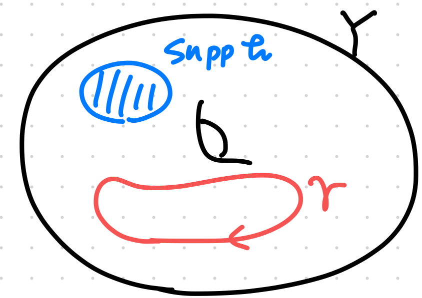
$\Rightarrow \lambda$ satisfies SCP.

Pf of Key Lemma

Assume λ does not satisfy SCP. i.e.

$\exists h \in C^\infty(Y, \mathbb{R}_{\geq 0}) \setminus \{0\}$ s.t.

$\forall t \in [0,1],\ \gamma \in \mathcal{P}(e^{th}\lambda) \Rightarrow \mathrm{Im}(\gamma) \cap \mathrm{supp}(h) = \emptyset$



Then, $\forall t \in [0,1]$,

$\mathcal{P}(e^{th}\lambda) = \mathcal{P}(\lambda),\ A_+(e^{th}\lambda) = A_+(\lambda)$

$\forall c$: AS of (Y, ξ)

$$\underline{c(e^{th}\lambda)} \in A_+(e^{th}\lambda) = \underline{A_+(\lambda)}$$

(∵ spectrality.)

measure zero.

Continuous on t $\Rightarrow$ const 〃

(∵ C^0-continuity) $\therefore\ c(e^h\lambda) = c(\lambda)$: contradicts Local sensitivity. //

Fact. (Y, ξ) : Closed & connected contact 3-mfd

$\Rightarrow \exists$ seq of AS $(C_k^{ECH})_{k \geq 1}$. s.t.

$$\forall \lambda \in \Delta(Y, \xi) \quad \lim_{k \to \infty} \frac{C_k^{ECH.}(\lambda)^2}{2k} = Vol(Y. \lambda) := \int_Y \lambda \wedge d\lambda.$$

(Volume thm / Weyl Law)

Rmk

$(C_k^{ECH})_k$: introduced by Hutchings (2012)

Weyl Law : proved by Cristofaro Gardiner - Hutchings - Ramos (2015)

<u>Cor</u>. (Strong closing Lemma).

(Y, ξ): closed contact 3-mfd.

$\Rightarrow \forall \lambda \in \Lambda(Y, \xi)$ satisfies SCP.

$\odot$ May assume Y: connected.

$\forall h \in C^\infty(Y, \mathbb{R}_{\geq 0}) \setminus \{0\}$

$\mathrm{Vol}(Y, e^h \lambda) > \mathrm{Vol}(Y, \lambda)$

$\therefore k \gg 1 \Rightarrow c_k^{ECH}(e^h \lambda) > c_k^{ECH}(\lambda)$.

$\therefore \lambda$ satisfies Local Sensitivity,

thus SCP. //

Thm (I, 2018)

(Y, ξ): Closed contact 3-mfd.

For a generic $\lambda \in \Lambda(Y, \xi)$, $\exists$ seq of "equidistributed" periodic Reeb orbits.

Cf. Marques-Neves-Song (2017): $\exists$ seq of equidistributed minimal hypersurfaces for generic metrics.

Thm (Cristofaro Gardiner-Prasad-Zhang, Edtmair-Hutchings) both 2021

(S, ω): Closed. symplectic 2-mfd

For a C^∞-generic $\varphi \in \mathrm{Diff}(S, \omega)$, $\mathrm{Per}(\varphi)$ is dense in S.

Rmk • Asaoka-I (2015) proved the parallel statement for $\mathrm{Ham}(S, \omega)$

• CGPZ & EH used Periodic Floer Homology.

Closing Lemmas via Contact Homology?

- (Y,ξ): Closed contact mfd $\rightsquigarrow CH(Y,\xi)$ Contact Homology.
  (of any dim)
- $\forall\sigma\in CH(Y,\xi)\setminus\{0\}\rightsquigarrow C_\sigma$: AS of (Y,ξ)
- when Y: connected, one can define $U \circlearrowright CH(Y,\xi)$ U-map.

$\forall\sigma\in CH(Y,\xi)\setminus\{0\}$, $C_\sigma(\lambda)\geqq C_{U\sigma}(\lambda)$
$\forall\lambda\in\Lambda(Y,\xi)$

Prop. (I, 2022)

$$\inf_{\sigma\in CH(Y,\xi)\setminus\{0\}} C_\sigma(\lambda)-C_{U\sigma}(\lambda)=0 \Rightarrow \lambda \text{ satisfies Local Sensitivity.}$$

(thus SCP).

<u>Recall</u> : For $0 < a_1 \leq \cdots \leq a_n$.

$$E_{a_1,\cdots,a_n} := \left\{ \begin{matrix} (q_1,\cdots,q_n, \\ p_1,\cdots,p_n) \in \mathbb{R}^{2n} \end{matrix} \;\middle|\; \sum_{j=1}^{n} \frac{\pi(q_j^2+p_j^2)}{a_j} \leq 1 \right\}$$

$$\lambda_{a_1,\cdots,a_n} := \sum_{j=1}^{n} \frac{p_j\,dq_j - q_j\,dp_j}{2} \Bigg|_{\partial E_{a_1,\cdots,a_n}}$$

$$S^{2n-1} := \left\{ \begin{matrix} (q_1,\cdots,q_n, \\ p_1,\cdots,p_n) \in \mathbb{R}^{2n} \end{matrix} \;\middle|\; \sum_{j=1}^{n} q_j^2 + p_j^2 = 1 \right\} = \partial E_{\pi,\cdots,\pi}$$

$$\xi_n := \xi_{\lambda_{\pi,\cdots,\pi}}$$

<u>Lem</u> For any $0 < a_1 \leq \cdots \leq a_n$

$$(\partial E_{a_1,\cdots,a_n}, \xi_{\lambda_{a_1,\cdots,a_n}}) \cong (S^{2n-1}, \xi_n)$$

as contact mfds.

Thm (Chaidez-Datta-Prasad-Tanny, 2022)

For any $0 < a_1 \leq \dots \leq a_n$

$$\inf_{\sigma \in CH(S^{2n-1}, \xi_n) \setminus \{0\}} c_\sigma(\lambda a_1 \cdots a_n) - c_{V\sigma}(\lambda a_1 \cdots a_n) = 0.$$

In particular, $\lambda a_1 \cdots a_n$ satisfies SCP.

Idea of pf

Step 1 Rational case ($a_i / a_j \in \mathbb{Q}$ for any $i \leq j$)

Step 2. General case follows from Step 1
by approximation argument (Dirchlet approx thm)

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Cf. Cineli-Seyfaddini

Question

Does SCP (or C^∞-CP) hold

- on a nbd of $\lambda_{a_1 \cdots a_n}$ in $\Lambda(S^{2n-1}, \xi_n)$?

- for any $\lambda \in \Lambda(S^{2n-1}, \xi_n)$?

Cf. Fish-Hofer conj.