Recent progresses on motivic cohomology

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A quest for theory of *motivic cohomology* dates back to work of Grothendieck and early days of algebraic geometry.

Beilinson gave a precise conjectural framework for such hoped-for theory.

It is expected to play fundamental roles in algebraic and arithmetic geometry (e.g. Beilinson's conjecture on special values of L-functions).

The conjecture has been very influential.

Recall Atiyah-Hirzebruch spectral sequence for a CW complex X

$$E_2^{p,q} = H^{p-q}_{sing}(X, \mathbb{Z}) \Rightarrow K^{top}_{-p-q}(X).$$

Beilinson conjectured there should be an algebraic counterpart.

Conjecture 0.1 (Beilinson (1985))

For any reasonable scheme X, there is a natural spectral sequence:

$$E_2^{p,q} = H^{p-q}_{\mathcal{M}}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X)$$

where $K_*(X)$ is the (non-connective) algebraic K-theory of X, and $H^i_{\mathcal{M}}(X, \mathbb{Z}(n))$ is the motivic cohomology of X yet to be defined.

Expected to satisfy universal properties such as the projective bundle formula, the blowup formula, the localization sequence, etc.

Short review on algebraic K-theory

For a comutative ring R, let Proj_R be the groupoid of finitely generated projective R-modules with isomorphisms.

Proj_R is a symmetric monodial category via

$$\mathrm{Proj}_R \times \mathrm{Proj}_R \to \mathrm{Proj}_R \ : \ (M, N) \to M \oplus N.$$

So, $\mathrm{Proj}_R/\sim$ is an abelian monoid with product $\oplus$ and identity 0.

Recall the inclusion of the categories

$$\{\text{commutative groups}\} \hookrightarrow \{\text{commutative monoids}\}$$

admits a left adjoint $M \to M^{gr}$ called the group completion.

Definition $K_0(R) = \left(\mathrm{Proj}_R/\sim\right)^{gr}$.

Recall the inclusions of the (∞-)categories

$$\mathrm{Proj}_R \in \{\text{groupoids}\} \hookrightarrow \mathcal{S}_{\leq 1} \hookrightarrow \mathcal{S},$$

where $\mathcal{S}$ is the ∞-categoy of spaces(=Kan simplicial sets),

$\mathcal{S}_{\leq 1}$ is the full subcategory of objects X s.t. $\pi_i(X) = 0$ for $i > 1$.

The symmetric monoidal structure on Proj_R turns it into an object of $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S})$, the ∞-category of *commutative monoids* in $\mathcal{S}$.

Let $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{S})$ be its full subcategory of *group-like objects*, i.e. those $M \in \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S})$ that $\pi_0(M)$ are groups. The inclusion

$$\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{S}) \hookrightarrow \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S})$$

admits a left adjoint $M \to M^{gr}$.

Definition 0.1

$K(R) = \left(\mathrm{Proj}_R\right)^{gr} \in \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{S})$. *For* $n \geq 0$, $K_n(R) = \pi_n(K(R))$.

$K(R)$ (or any object of $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{S})$) is an *infinite loop space*, i.e. there is a sequence $T_0 = K(R), T_1, T_2, \dots$ in $\mathcal{S}_*$ such that for $n \geq 1$, $T_{n-1} = \Omega T_n$ and $\pi_i(T_n) = 0$ for $i < n$. So, it gives an object

$$\mathbf{K}(R) = (T_0, T_1, T_2, \dots) \in \mathrm{Sp}, \text{ the } \infty\text{-category of spectra.}$$

Note $\mathbf{K}(R)$ is connective i.e. its *stable homotopy group*

$$\pi_i^{\mathrm{st}}(\mathbf{K}(R)) := \varinjlim_n \pi_{i+n}(T_n) \text{ vanishes for } i < 0.$$

By the construction,

$$K_n(R) = \pi_n^{\mathrm{st}}(\mathbf{K}(R)) \text{ for } n \geq 0.$$

Bass construction: The map

$$K_0(R[t]) \oplus K_0(R[t^{-1}]) \xrightarrow{\varphi} K_0(R[t, t^{-1}])$$

is not surjective in general unless R is regular.

H. Bass defined the negative K-groups $K^B_{-1}(R) = \mathrm{Coker}(\varphi)$, and $K^B_{-n}(R)$ for all $n > 0$ inductively as the cokernel of

$$K^B_{-n+1}(R[t]) \oplus K^B_{-n+1}(R[t^{-1}]) \xrightarrow{\varphi} K^B_{-n+1}(R[t, t^{-1}]).$$

The definition can be upgraded to a spectrum version giving the *non-connective K-theory spectrum* $\mathbf{K}^B(R) \in \mathrm{Sp}$.

There is a natural map (connective cover) $\mathbf{K}(R) \to \mathbf{K}^B(R)$ in Sp s.t.

$$\pi_i^{\mathrm{st}}(\mathbf{K}^B(R)) \simeq \begin{cases} \pi_i^{\mathrm{st}}(\mathbf{K}(R)) = K_i(R) & \text{if } i \geq 0, \\ K_i^B(R) \text{ defined above} & \text{if } i < 0, \end{cases}$$

An advantage of $\mathbf{K}^B(R)$ over $\mathbf{K}(R)$ is a theorem of Thomason:

The presheaf $\mathrm{AffSch}^{op} \to \mathrm{Sp}$; $X = \mathrm{Spec}(R) \to \mathbf{K}^B(R)$

is a *Zariski sheaf* so we can extend it to

$$\mathbf{K}^B \in \mathrm{Shv}_{\mathrm{Zar}}(\mathbf{Sch}, \mathrm{Sp})$$

as the *Zariski sheafication* of the presheaf $X \to \mathbf{K}^B(\Gamma(X, \mathcal{O}))$.

In what follows, we write for $\mathbf{K} \to \mathbf{K}^B$

$$K_{\geq 0} \to K.$$

Let $\mathbf{Sch}$ be category of quasi-compact and quasi-separated schemes.

$\mathcal{C} = \mathrm{Sp}$ or $\mathcal{D}(\mathbb{Z})$, (∞-)category of unbounded complex of $\mathbb{Z}$-modules.

Let $\mathsf{PSh}(\mathbf{Sch}, \mathcal{C}) = \mathsf{Fun}(\mathbf{Sch}^{op}, \mathcal{C})$ be the (∞-)category of presheaves on $\mathbf{Sch}$ with values in $\mathcal{C}$.

For a Grothendieck topology τ on $\mathbf{Sch}$, let

$$\mathsf{Shv}_\tau(\mathbf{Sch}, \mathcal{C}) \subset \mathsf{PSh}(\mathbf{Sch}, \mathcal{C})$$

be the full subcategory of τ-sheaves.

The inclusion $\mathsf{Shv}_\tau(\mathbf{Sch}, \mathcal{C}) \to \mathsf{PSh}(\mathbf{Sch}, \mathcal{C})$ admits a left adjoint called the τ-sheafication

$$a_\tau : \mathsf{PSh}(\mathbf{Sch}, \mathcal{C}) \to \mathsf{Shv}_\tau(\mathbf{Sch}, \mathcal{C}).$$

By definition, $F \in \mathsf{PSh}(\mathbf{Sch}, \mathcal{C})$ is a τ-sheaf if for any τ-covering family $\{Y_i \to X\}_{i \in I}$ in $\mathbf{Sch}$, we have an equivalence in $\mathcal{C}$:

$$F(X) \simeq \varprojlim_{[n] \in \Delta} \prod_{(i_0, \dots, i_n) \in I^{n+1}} F(Y_{i_0} \times_X \cdots \times_X Y_{i_n}),$$

$$= \varprojlim \Big(\prod_{i \in I} F(Y_i) \overset{\rightarrow}{\underset{\leftarrow}{\rightarrow}} \prod_{(i,j) \in I^2} F(Y_i \times_X Y_j) \overset{\rightarrow}{\underset{\leftarrow}{\overset{\rightarrow}{\underset{\leftarrow}{\rightarrow}}}} \cdots \Big)$$

where

$$[n] \in \Delta^{op} \to \bigsqcup_{(i_0, \dots, i_n) \in I^{n+1}} Y_{i_0} \times_X \cdots \times_X Y_{i_n}$$

is the Čech nerve of the covering family, which is a simplicial object of schemes.

Motivic complexes of smooth schemes over a field

Conjecture 0.2 (Beilinson (1985))

For any reasonable scheme X, there is a natural spectral sequence:

$$E_2^{p,q} = H_{\mathcal{M}}^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X)$$

where $K_*(X)$ is the non-connective K-theory of X, and

$H^i_{\mathcal{M}}(X, \mathbb{Z}(n))$ is the motivic cohomology of X yet to be defined.

The conjecture was answered in case X is smooth over a field k.

$H^i_{\mathcal{M}}(X, \mathbb{Z}(n))$ can be defined as an algebraic analog of

the singular homology of topological spaces !!

Recall the singular homology $H_q(X, \mathbb{Z})$ of a topological space X is the q-th homology of the chain complex

$$\cdots \to s(X,q) \xrightarrow{\partial} s(X,q-1) \xrightarrow{\partial} \cdots \xrightarrow{\partial} s(X,0),$$

$$s(X,q) = \bigoplus_{\Gamma} \mathbb{Z}[\Gamma] \quad (\Gamma : \Delta^q_{top} \to X \text{ continuous}),$$

$$\Delta^q_{top} = \Big\{(x_0, x_1, \cdots, x_q) \in \mathbb{R}^{q+1} \;\Big|\; \sum_{0 \leqq i \leqq q} x_i = 1,\ x_i \geqq 0\Big\},$$

∂ is the alternating sum of the restriction maps to faces of Δ^q_{top}.

For a scheme X of finite type over a field k and an integer $n \geq 0$, *Bloch's higher Chow groups* $\mathrm{CH}^n(X,q)$ is the q-th homology of

$$\cdots \to z^n(X,q) \xrightarrow{\partial} z^n(X,q-1) \xrightarrow{\partial} \cdots \xrightarrow{\partial} z^n(X,0),$$

$$z^n(X,q) = \bigoplus_{\Gamma \subset X \times_k \Delta^q} \mathbb{Z}[\Gamma],$$

$$\Delta^q = \mathrm{Spec}\left(k[t_0,\cdots,t_q]/(\sum_{i=0}^{q} t_i - 1)\right)$$

where Γ ranges over integral closed subschemes of codimension n in $X \times \Delta^q$ intersecting properly with faces $\{t_{i_1} = \cdots = t_{i_r} = 0\} \subset \Delta^q$, ∂ is the alternating sum of the restriction maps to faces of codimension one.

Example: $\mathrm{CH}^n(X,0)=\mathrm{CH}^n(X)$, $\mathrm{CH}^1(X,1)=\mathcal{O}(X)^\times$.

One can turn the association

$$\mathbb{Z}(n)^{sm}:\mathbf{Sm}_k\to\mathcal{D}(\mathbb{Z})\ ;\ X\to z^n(X,2n-\bullet)$$

into an object $\mathbb{Z}(n)^{sm}\in\mathsf{PSh}(\mathbf{Sm}_k,\mathcal{D}(\mathbb{Z}))$, the category of presheaves on $\mathbf{Sm}_k$ with values in $\mathcal{D}(\mathbb{Z})$, where

$\mathbf{Sm}_k$ is category of smooth schemes separated of finite type over k,

$\mathcal{D}(\mathbb{Z})$ is ∞-category of unbounded complexes of $\mathbb{Z}$-modules.

$\mathbb{Z}(n)^{sm}$ satisfies the projective bundle formula, the smooth blowup formula and the localization sequence.

Theorem 0.1 (Friedlander-Suslin (2002), Levine(2008))

For $X \in \mathbf{Sm}_k$, there exists a complete decreasing $\mathbb{N}$-indexed filtration $\left\{F^n_{\mathrm{mot}} K(X)\right\}_{n\in\mathbb{N}}$ on $K(X)$ and identifications

$$\mathrm{gr}^n_{F_{\mathrm{mot}}} K(X) \simeq \mathbb{Z}(n)^{sm}(X)[2n].$$

Defining motivic cohomology for $X \in \mathbf{Sm}_k$:

$$H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) = H^i(\mathbb{Z}(n)^{sm}(X)) = \mathrm{CH}^n(X, 2n-i),$$

the hoped-for spectral sequence

$$E_2^{p,q} = H^{p-q}_{\mathcal{M}}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X)$$

is formally deduced from the filtration $F^{\bullet}_{\mathrm{mot}} K(X)$.

Remark 0.3

Call a presheaf F on $\mathbf{Sch}_k$ or $\mathbf{Sm}_k$ *$\mathbb{A}^1$-invariant* if

$$F(X) \simeq F(X \times_k \mathbb{A}^1_k) \text{ for all } X.$$

The proof of Theorem 0.1 heavily uses the $\mathbb{A}^1$-invariance of K-theory, which is valid only for regular schemes.

The higher Chow groups of a singular variety over a field are $\mathbb{A}^1$-invariant, while algebraic K-theory does not satisfy the property.

It has been an open problem to give a motivic filtration on K-theory and a motivic cohomology for singular schemes.

Now recall a formal procedure to extend presheaves on $\mathbf{Sm}_k$ to $\mathbf{Sch}_k$.

For $F \in \mathsf{PSh}(\mathbf{Sm}_k, \mathcal{C})$ ($\mathcal{C} = \mathrm{Sp}$ or $\mathcal{D}(\mathbb{Z})$), its left Kan extension $L^{sm}F \in \mathsf{PSh}(\mathbf{Sch}_k, \mathcal{C})$ along $\mathbf{Sm}_k \to \mathbf{Sch}_k$ is defined as follows:
For $X \in \mathbf{Sch}_k$, $L^{sm}F(X)$ is the colimit in $\mathcal{C}$ of the diagram

$$(\mathbf{Sm}_{X/})^{op} \to \mathcal{C};\ (Y, \varphi) \to F(Y),$$

$\mathbf{Sm}_{X/}$ is the category of pairs $(Y, X \xrightarrow{\varphi} Y)$ with $Y \in \mathbf{Sm}_k$, morphisms $(Y, \varphi) \to (Y', \varphi')$ are maps $f : Y \to Y'$ s.t. $\varphi' = f\varphi$.

Remarkable fact(Bhatt-Lurier): $L^{sm}K_{|\mathbf{Sm}_k} = K_{\geq 0}$ on $\mathbf{Sch}_k$.
Warning: Basic properties of $\mathbb{Z}(n)^{sm}$ (projective bundle formula, blowup formula, etc) are not genetic to $L^{sm}\mathbb{Z}(n)^{sm}$.

Voevodsky: Sheafify $L^{sm}\mathbb{Z}(n)^{sm}$ to retrieve them.

Cdh-local motivic complex

A *distinguished Nisnevich square* is a pullback square in $\mathbf{Sch}$:

$$\begin{array}{ccc} W & \xrightarrow{j} & V \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{i} & X \end{array} \tag{1}$$

where i is a quasi-compact open immersion and f is an étale morphism inducing an isomorphism over $X \setminus U$.

The Nisnevich topology on $\mathbf{Sch}$ is the Grothendieck topology generated by coverings families of the form

$$\{U \to X\} \sqcup \{V \to X\},$$

where X, U, V are from (1).

An *abstract blowup square* is a pullback square in $\mathbf{Sch}$:

$$\begin{array}{ccc} E & \xrightarrow{j} & Y \\ \downarrow g & & \downarrow f \\ Z & \xrightarrow{i} & X \end{array} \tag{2}$$

where i is a closed immersion locally of finite presentation and f is a proper morphism inducing an isomorphism over $X \setminus Z$.

The *cdh topology* on $\mathbf{Sch}$ is the Grothendieck topology generated by Nisnevich topology and coverings families of the form

$$\{Z \to X\} \sqcup \{Y \to X\},$$

where X, Z, Y are from (2).

Example 0.4

For a finite ring Λ, the étale cohomology

$$R\Gamma_{\text{ét}}(-, \Lambda) : \mathbf{Sch}^{op} \to \mathcal{D}(\Lambda)$$

is a cdh sheaf by the proper base change theorem.

Remark 0.5

For any scheme X of finite type over a field k of characteristic zero (or assuming resolution of singularities over k), there exist a cdh covering $Y \to X$ with $Y \in \mathbf{Sm}_k$. Thus, many properties of cdh cohomology of X are deduced from smooth case.

For $\mathcal{C} = \mathrm{Sp}$ or $\mathcal{D}(\mathbb{Z})$, let $\mathsf{Shv}_{cdh}(\mathbf{Sch}, \mathcal{C}) \subset \mathsf{PSh}(\mathbf{Sch}, \mathcal{C})$ be the full subcategory of sheaves for cdh topology with the sheafication functor

$$a_{cdh} : \mathsf{PSh}(\mathbf{Sch}, \mathcal{C}) \to \mathsf{Shv}_{cdh}(\mathbf{Sch}, \mathcal{C}).$$

Definition 0.2

For integers $n \geq 0$, we define the cdh-local motivic complex

$$\mathbb{Z}(n)^{cdh} = a_{cdh} L^{sm} \mathbb{Z}(n)^{sm} \in \mathsf{Shv}_{cdh}(\mathbf{Sch}_k, \mathcal{D}(\mathbb{Z})).$$

Basic properties of $\mathbb{Z}(n)^{sm}$ (projective bundle formula, blowup formula, etc) are inherited to $\mathbb{Z}(n)^{cdh}$. Indeed, it gets too much: $\mathbb{Z}(n)^{cdh}$ is $\mathbb{A}^1$-invariant! So, impossible to give rise to the hoped-for spectral sequence for $X \in \mathbf{Sch}_k$

$$E_2^{p,q} = H_{\mathcal{M}}^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X) \quad \text{for } X \in \mathbf{Sch}_k.$$

Elmanto-Morrow's motivic complex

Elmanto-Morrow construct their motivic complex

$$\mathbb{Z}(n)^{EM} \in \mathsf{PSh}(\mathbf{Sch}_{\mathbb{F}}, \mathcal{D}(\mathbb{Z})) \text{ for } \mathbb{F} = \mathbb{Q} \text{ or } \mathbb{F}_p$$

modifying $\mathbb{Z}(n)^{cdh}$ by *Hodge-completed derived de Rham complexes* in case $\mathrm{ch}(\mathbb{F}) = 0$ and *syntomic complexes* in case $\mathrm{ch}(\mathbb{F}) = p > 0$.

For a qcqs $\mathbb{F}$-scheme X ($\mathbb{F} = \mathbb{Q}$ or $\mathbb{F}_p$), $\mathbb{Z}(n)^{EM}(X) \in \mathcal{D}(\mathbb{Z})$ sits in the following pullback squares in $\mathcal{D}(\mathbb{Z})$:

$$\begin{array}{ccc} \mathbb{Z}(n)^{EM}(X) & \longrightarrow & \widehat{L\Omega}^{\geq n}_{X/\mathbb{Q}} \\ \downarrow & & \downarrow \\ \mathbb{Z}(n)^{cdh}(X) & \xrightarrow{c^H} & (a_{cdh}\widehat{L\Omega}^{\geq n}_{-/\mathbb{Q}})(X) \end{array} \qquad \text{if } \mathbb{F} = \mathbb{Q},$$

$$\begin{array}{ccc} \mathbb{Z}(n)^{EM}(X) & \longrightarrow & \mathbb{Z}_p(n)^{syn}(X) \\ \downarrow & & \downarrow \\ \mathbb{Z}(n)^{cdh}(X) & \xrightarrow{c^{syn}} & (a_{cdh}\mathbb{Z}_p(n)^{syn})(X) \end{array} \qquad \text{if } \mathbb{F} = \mathbb{F}_p$$

where c^H and c^{syn} are defined in a non-trivial manner.

Now we recall $\widehat{L\Omega}^{\geq n}_{X/\mathbb{Q}}$ and $\mathbb{Z}_p(n)^{syn}(X)$.

For a field k, let CAlg_k be the category of k-algebras and $\mathsf{CAlg}_k^{poly} \subset \mathsf{CAlg}_k$ be the full subcategory of polynomial k-algebras . Define a functor called *the derived de Rham complex*

$$L\Omega_{-/k} : \mathsf{CAlg}_k \to \mathcal{D}(k)$$

as the left Kan extension along $\mathsf{CAlg}_k^{poly} \to \mathsf{CAlg}_k$ of the functor

$$\mathsf{CAlg}_k^{poly} \to \mathcal{D}(k)\ ;\ R \to \Omega^\bullet_{R/k} = (R \xrightarrow{d} \Omega^1_{R/k} \xrightarrow{d} \Omega^2_{R/k} \xrightarrow{d} \cdots),$$

where $\Omega^\bullet_{R/k}$ is the de Rham complex. Define a decreasing filtration $\left\{L\Omega^{\geq n}_{-/k}\right\}_{n\in\mathbb{N}}$ on $L\Omega_{-/k}$, *the derived Hodge filtration*, by left-Kan extending the Hodge filtration $\left\{\Omega^{\geq n}_{R/k}\right\}_{n\in\mathbb{N}}$ on $\Omega^\bullet_{R/k}$.

Letting $L_{R/k}$ be the *cotangent complex of R/k*, we have

$$L\Omega^{\geq n}_{R/k}/L\Omega^{\geq n+1}_{R/k} \simeq \wedge^n L_{R/k}[-n] \text{ for } \quad R \in \mathsf{CAlg}_k.$$

The Hodge-completed derived de Rham complex $\widehat{L\Omega}_{R/k}$ for $R \in \mathsf{CAlg}_k$ is defined as the limit of the diagram:

$$\mathbb{N} \to \mathcal{D}(k) \ ; \ n \to L\Omega_{R/k}/L\Omega^{\geq n}_{R/k}.$$

We extend $\widehat{L\Omega}_{-/k}$ to an object of $\mathsf{Shv}_{\mathrm{Zar}}(\mathbf{Sch}_k, \mathcal{D}(k))$ as the Zariski sheafication of the presheaf $\mathbf{Sch}_k^{op} \to \mathcal{D}(k) \ ; \ X \to \widehat{L\Omega}_{\Gamma(X,\mathcal{O})/k}$.

The derived Hodge filtration induces a decreasing filtration $\left\{\widehat{L\Omega}^{\geq n}_{-/k}\right\}_{n\in\mathbb{N}}$ on $\widehat{L\Omega}_{-/k}$.

Example 0.6

(i) (Illusie) For $X \in \mathbf{Sm}_k$, $\widehat{L\Omega}_{X/k} = \Omega^\bullet_{X/k}$.

(ii) (Bhatt) If $k = \mathbb{C}$ and X is a scheme of finite type over $\mathbb{C}$, $H^*(\widehat{L\Omega}_{X/\mathbb{C}})$ is canonically isomorphic to the singular cohomology with $\mathbb{C}$-coefficients of the associated $\mathbb{C}$-points of X.

(iii) If k is p-adic, Beilinson used $\widehat{L\Omega}_{X/k}$ to give a new proof of the B_{dR}-conjecture in the p-adic Hodge theory.

For any qcqs $\mathbb{Z}_p$-scheme X, we have the *syntomic complex*

$$\mathbb{Z}_p(n)^{syn}(X) \in \mathcal{D}(\mathbb{Z}).$$

It is a vast generalization of the one by Fontaine-Messing, Kato, Tsuji which treats the case of smooth or semi-stable over DVR.

For a regular $\mathbb{F}_p$-scheme X,

$$\mathbb{Z}_p(n)^{syn}(X) \otimes^{\mathbb{L}} \mathbb{Z}/p\mathbb{Z} = R\Gamma(X_{\text{ét}}, \Omega^n_{X,\log})[-n].$$

In general, $\quad \mathrm{gr}^n_{F_{\mathrm{BMS}}} \mathrm{TC}(X) \simeq \mathbb{Z}_p(n)^{syn}(X)[2n]$

for Bhatt-Morrow-Scholze's filtration $\left\{F^n_{\mathrm{BMS}} \mathrm{TC}(X)\right\}_{n\in\mathbb{N}}$ on the *topological cyclic homology* $\mathrm{TC}(X)$ of X. Bhatt-Scholze gave an alternative definition using their *prismatic cohomology theory.*

For a qcqs $\mathbb{F}$-scheme X ($\mathbb{F} = \mathbb{Q}$ or $\mathbb{F}_p$), Elmanto-Morrow define $\mathbb{Z}(n)^{EM}(X) \in \mathcal{D}(\mathbb{Z})$ which sits in the following pullback squares

$$\begin{array}{ccc} \mathbb{Z}(n)^{EM}(X) & \longrightarrow & \widehat{L\Omega}^{\geq n}_{X/\mathbb{Q}} \\ \downarrow & & \downarrow \\ \mathbb{Z}(n)^{cdh}(X) & \xrightarrow{c^H} & (a_{cdh}\widehat{L\Omega}^{\geq n}_{-/\mathbb{Q}})(X) \end{array} \qquad \text{if } \mathbb{F} = \mathbb{Q},$$

$$\begin{array}{ccc} \mathbb{Z}(n)^{EM}(X) & \longrightarrow & \mathbb{Z}_p(n)^{syn}(X) \\ \downarrow & & \downarrow \\ \mathbb{Z}(n)^{cdh}(X) & \xrightarrow{c^{syn}} & (a_{cdh}\mathbb{Z}_p(n)^{syn})(X) \end{array} \qquad \text{if } \mathbb{F} = \mathbb{F}_p,$$

where c^H and c^{syn} are defined in a non-trivial manner.

Theorem 0.2 (Elmanto-Morrow (2023))

For $X \in \mathbf{Sch}_{\mathbb{F}}$, there exists a complete decreasing $\mathbb{N}$-indexed filtration $\left\{F^n_{\mathrm{EM}} K(X)\right\}_{n\in\mathbb{N}}$ on $K(X)$ and identifications

$$\mathrm{gr}^n_{F_{\mathrm{EM}}} K(X) \simeq \mathbb{Z}(n)^{EM}(X)[2n].$$

This gives an answer to Beilinson's conjecture.

Elmanto-Morrow prove some basic properties of $\mathbb{Z}(n)^{EM}$ such as

- $\mathbb{Z}^{EM}(0) = \mathbb{Z}[0]$ and $\tau^{\leq 2}\mathbb{Z}^{EM}(1) = \mathbb{G}_m[-1]$.
- projective bundle formula and blowup formula.
- $\mathbb{Z}(n)^{EM}(X) = \mathbb{Z}(n)^{sm}(X)$ for $X \in \mathbf{Sm}_k$.

Question 0.7

Possible to define $\mathbb{Z}(n)^{motivic}$ not using other cohomology theories?

Pro-cdh-local motivic complex

Introduce *pro-cdh topology*, a new Grothendieck topology on $\mathbf{Sch}$, and define *pro-cdh-local motivic complex* $\mathbb{Z}(n)^{pcdh}$ as the pro-cdh sheafication of left Kan extension of $\mathbb{Z}(n)^{sm}$ along $\mathbf{Sm} \to \mathbf{Sch}$.

Consider an abstract blowup square in $\mathbf{Sch}$

$$\begin{array}{ccc} E & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array} \tag{3}$$

i.e. i is a closed immersion locally of finite presentation and f is a proper morphism inducing an isomorphism over $X \setminus Z$.

Definition 0.3 (Kelly-S. (2023))

The pro-cdh topology on $\mathbf{Sch}$ is the Grothendieck topology generated by Nisnevich topology and coverings families of the form

$$\{Z_r \to X\}_{r\in\mathbb{N}} \sqcup \{Y \to X\},$$

for all squares (3), where $Z_r = \underline{\mathrm{Spec}}(\mathcal{O}_X/\mathcal{I}_Z^r) \hookrightarrow X$ is the r-th infinitesimal thickening of Z. We have the sheafication functor

$$a_{pcdh} : \mathsf{PSh}(\mathbf{Sch}, \mathcal{C}) \to \mathsf{Shv}_{pcdh}(\mathbf{Sch}, \mathcal{C}),$$

where $\mathsf{Shv}_{pcdh}(\mathbf{Sch}, \mathcal{C})$ is the full subcategory of pro-cdh sheaves.

Recall the cdh topology is generated by Nisnevich topology and coverings families $\{Z \to X\} \sqcup \{Y \to X\}$ for all squares (3).

Cdh sheaves (e.g. $R\Gamma_{\text{ét}}(-,\Lambda)$) are pro-cdh sheaves.

Example 0.8 (Kerz-Strunk-Tamme, Grothendieck, Morrow)

Let $\mathbf{Sch}^{noe}$ be the category of noetherian schemes. Then,

the non-connective K-theory K, $R\Gamma(-,\mathcal{O})$, $L_{R/k}$

are pro-cdh sheaves on $\mathbf{Sch}^{noe}$ but not cdh-sheaves.

Warning: These are not pro-cdh sheaves on $\mathbf{Sch}$.

An application of the pro-cdh topology is a topos-theoretic interpretation of the Bass construction $K_{\geq 0}(X) \to K(X)$.

Theorem 0.3 (Kelly-S.)

For $X \in \mathbf{Sch}^{noe}$ with $\dim(X) < \infty$, there exists a natural equivalence

$$(a_{pcdh} K_{\geq 0})(X) \simeq K(X).$$

Definition 0.4

For integers $n \geq 0$, we define the pro-cdh-local motivic complex

$$\mathbb{Z}(n)^{pcdh} := a_{pcdh} L^{sm} \mathbb{Z}(n)^{sm} \in \mathsf{Shv}_{pcdh}(\mathbf{Sch}_{\mathbb{F}}, \mathcal{D}(\mathbb{Z})),$$

as the pro-cdh-sheafication of the left Kan extension of $\mathbb{Z}(n)^{sm}$ along $\mathbf{Sm}_{\mathbb{F}} \to \mathbf{Sch}_{\mathbb{F}}$, where $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F}_p$.

Theorem 0.4 (Kelly-S.)

For $X \in \mathbf{Sch}_{\mathbb{F}}^{noe}$ with $\dim(X) < \infty$, there exists a complete decreasing $\mathbb{N}$-indexed filtration $\left\{ F^n_{\mathrm{pcdh}} K(X) \right\}_{n \in \mathbb{N}}$ on $K(X)$ and identifications

$$\mathrm{gr}^n_{F_{\mathrm{pcdh}}} K(X) \simeq \mathbb{Z}(n)^{pcdh}(X)[2n].$$

Of course, the above theorem should be compared with the work of Elmanto-Morrow. Thus, the following result is naturally expected.

Theorem 0.5 (Kelly-S.)

There is an equivalence $\mathbb{Z}(n)^{pcdh} \simeq \mathbb{Z}(n)^{EM}$ *in* $\mathrm{PSh}(\mathbf{Sch}_{\mathbb{F}}^{noe}, \mathcal{D}(\mathbb{Z}))$.

We expect that the noetherian hypothesis may be removed working with derived schemes instead of schemes in view of the following.

Theorem 0.6 (Kelly-S.-Tamme)

The non-connective K*-theory is a sheaf for the pro-cdh topology on the category of quasi-compact quasi-separated derived schemes.*

We explain some ingredients in the proofs of the main theorems.

Definition 0.5

A pro-cdh local ring is a henselian local ring R s.t. $R = V \times_\kappa Q$, where Q is a local ring of Krull dimension 0, κ is the residue field of Q and V is a valuation ring of κ.

Theorem 0.7 (Kelly-S.)

A collection of maps $\{Y_i \to X\}_{i \in I}$ in $\mathbf{Sch}$ is a covering for the pro-cdh topology if and only if

$$\coprod_{i \in I} \mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}(R), Y_i) \to \mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}(R), X)$$

is surjective for every pro-cdh local ring R.

Remark 0.9

An important consequence of the above theorem is roughly speaking that the "stalks" of a pro-cdh sheave are given by the sections over the pro-cdh local rings, and the equivalence of a map of pro-cdh sheaves is checked by those stalks.

Recall the stalks of Zariski (resp. étale) sheaves are given by local (resp. henselian local) rings.

An analogue of Theorem 0.7 holds for the cdh topology by replacing pro-cdh local rings by henselian valuation rings (i.e. the case $Q = \kappa$ in Definition 0.5).

A difficulty of Theorem 0.7 compared to the case of the cdh topology is that the pro-cdh toplogy is not *finitary*.

Another key ingredient provides a bound of the cohomological dimension of the pro-cdh cohomology:

Theorem 0.8 (Kelly-S.)

Let F be a pro-cdh sheaf of abelian groups on $\mathbf{Sch}^{noe}$. For $X \in \mathbf{Sch}^{noe}$, we have a vanishing of the pro-cdh cohomology:

$$H^i_{pcdh}(X, F) = 0 \text{ for } \quad i > 2\dim(X).$$

Now we define motivic cohomology of $X \in \mathbf{Sch}^{noe}_{\mathbb{F}}$ as

$$H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) = H^i(\mathbb{Z}(n)^{EM}(X)) = H^i(\mathbb{Z}(n)^{pcdh}(X)).$$

Then, we get the hoped-for spectral sequence

$$E_2^{p,q} = H^{p-q}_{\mathcal{M}}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X)$$

answering Beilinson's conjecture for noetherian schemes X over fields.

Some computations of motivic cohomology

For X be a noetherian qcqs scheme over $\mathbb{F}$, Elmanto-Morrow prove:

- $H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) = \mathrm{CH}^n(X, 2n-i)$ for $X \in \mathbf{Sm}_k$.
- $$H^i_{\mathcal{M}}(X, \mathbb{Z}(1)) = \begin{cases} 0 & \text{if } i = 0, \\ \mathcal{O}(X)^\times & \text{if } i = 1, \\ \mathrm{Pic}(X) & \text{if } i = 2, \\ H^2_{\mathrm{Nis}}(X, \mathbb{G}_m) & \text{if } i = 3. \end{cases}$$
- If X is a reduced quasi-projective surface, then $H^4_{\mathcal{M}}(X, \mathbb{Z}(2)) \simeq \mathrm{CH}_0^{LW}(X)$, the Levin-Weibel Chow group.
- For a local $\mathbb{F}$-algebra A, $H^n_{\mathcal{M}}(\mathrm{Spec}(A), \mathbb{Z}(n)) \simeq \hat{K}^M_n(A)$, the improved Milnor K-group of Gabber-Kerz.

Thank you for your attention!