Random Matrices and Free probability

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The abstract with less typos

Overview

Plan:

1. Random Matrix theory.
2. Free probability theory.
3. Asymptotic freeness and strong asymptotic freeness.
4. Applications and perspectives.

Genesis of RMT 1: John Wishart (1898 - 1956)

First birth of Random Matrix theory: John Wishart, a Scottish mathematician and agricultural statistician.

Genesis of RMT 1: John Wishart

Statistical motivation: one ‘crash’ example:

- Consider N^2 i.i.d. centered real bounded random variables $X_{ij}, 1 \leqslant i, j \leqslant N$. Let $X = (X_{ij})$. X is an $N \times N$ real random variable. It is composed of N iid random vectors $X = (X_1 | \dots | X_N)$.

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- Consider the matrix $B = X^t X$. Up to a multiple, B is the *empirical covariance*, and $\mathbb{E}(B)$ is the *covariance matrix* of X_1. It is symmetric.

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- Consider the matrix $B = X^t X$. Up to a multiple, B is the *empirical covariance*, and $\mathbb{E}(B)$ is the *covariance matrix* of X_1. It is symmetric.
- Since events are assumed to be independent, on average, the matrix should be close to diagonal (at least its expectation is diagonal).

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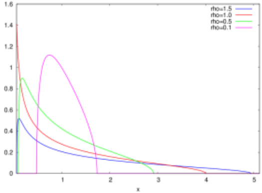
Theorem (Wishart, 1928 / Marchenko Pastur)

The histogram of eigenvalues (properly rescaled) tends to the Wishart distribution as the dimension N grows.

Wishart distribution / Marchenko Pastur



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- He worked with *David Hilbert* at the University of Göttingen. Wigner and *Hermann Weyl* introduced group theory into physics, particularly the theory of symmetry in physics.

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- He worked with *David Hilbert* at the University of Göttingen. Wigner and *Hermann Weyl* introduced group theory into physics, particularly the theory of symmetry in physics.
- In 1930, Princeton University recruited Wigner, along with *John von Neumann*, and he moved to the United States. von Neumann was in the same school as Wigner, a year behind him.

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- In addition, the dimension of H is *big*.

An very powerful old idea from statistical physics:
(1) What would happen if H was random? (symmetries would be a source of randomness)
(2) Would it be a good approximation?
The answer to (2) seems to be YES. The answer to (1) is mathematics and we draw our attention on it.

Wigner's semicircle distribution

- *This idea happened to work*

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- We are interested in $X^{(N)}$ a symmetric $N \times N$ matrix whose (upper triangular) entries are i.i.d. centered L^2 variables.
- Consider, as above, the normalized eigenvalue counting measure $N^{-1} \sum_{i=1}^{N} \delta_{\lambda_i}$.

Theorem (Wigner, 1948)

The histogram of eigenvalues (properly rescaled—to variance $= N^{-1}$) tends to a semi-circle distribution as $N \to \infty$:

$$\mu = \frac{1}{2\pi} \sqrt{4 - x^2} 1_{[-2,2]} dx$$

Wigner's semicircle distribution

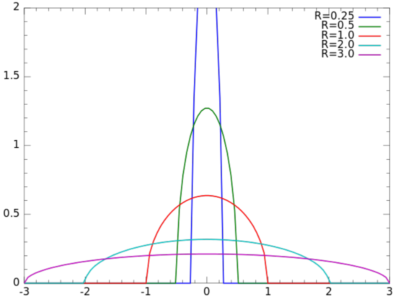

Abbildung: semi-circle distribution

Random Matrix Theory: Subsequent developments

- Mehta, Pastur, Marchenko, Tracy, Widom, etc... : analysis behind single matrix models. Relation to determinantal formulas, etc.
- Dyson and Montgomery (Princeton tea, 1972): relation between spacing of eigenvalues and spacing of zeroes of the Riemann Zeta function.
- Theoretical physics: Matrix Integrals, 2D quantum gravity ('t Hooft, Itzykson, Zuber, Parisi)
- Algebraic geometry, algebraic combinatorics (Harer Zagier) – Representation Theory (Okounkov, Borodin...)

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- ...Until Free probability (see next slides)...
- ...and a wealth of applications: Quantum Information Thoery, wireless transmission, statistics, finance, AI...

Free probability

- Dan V. Voiculescu (1949 –): A mathematician of Romanian origin with very strong early career achievements in operator algebras.
- In the early 80's he was interested in the free group factor isomorphism problem: are $L(F_2)$ and $L(F_3)$ isomorphic?
- Group factors have a canonical (unique) tracial state therefore it is natural to see them as a (non-commutative) *probability space*.
  He decided to try an approach where $L(F_d)$ is a (non-commutative) space of bounded measurable random variables, and the trace is their expectation.

Free probability

- This is a particular case of *non-commutative probability theory*. A non-commutative probability space is (A, τ), where A is a unital algebra and τ a state. It was motivated by quantum mechanics and was exotic in the 80's. The main notion was the notion of tensor independence.
- Voiculescu introduced free independence and it boosted (and temporarily overthrew?) non-commutative probability.
- Lately, NC probability also used for quantum information (quantum games, etc).

Free probability: definition of free independence

- Let $1 \in A_i \subset A$ be a family of unital subalgebras of A. Let τ be a state on A. They are *freely independent* w.r.t τ iff

$$\tau(a_1 \dots a_l) = 0$$

  as soon as $a_1 \in A_{i_1}, \dots, a_l \in A_{i_l}$ with $i_1 \neq i_2, \dots, i_{l-1} \neq i_l$ and $\tau(a_i) = 0$.
- If a group G is a free product $\star_i G_i$, then the group subalgebras are free in the group algebra w.r.t the l.r. state.
- With this property, the value of τ on all A_i determines uniquely the data of τ on the algebra generated by all A_i.

Free probability: the free CLT

- An early discovery of Voiculescu with free probability:
  There exists limit theorems, and the limit of the free central limit theorem is the *semi-circle distribution*.
  Is it a coincidence that semi-circle distribution appears both in the free CLT and Wigner?

Free probability: the free CLT

- An early discovery of Voiculescu with free probability: *There exists limit theorems*, and the limit of the free central limit theorem is the *semi-circle distribution*. Is it a coincidence that semi-circle distribution appears both in the free CLT and Wigner?
- No, because GUE is a stable NC distribution ($l^{-1/2}(X_1 + \ldots + X_l)$ has the same distribution as X) when they are all iid GUE.
- $GUE(N)$ is defined as the probability measure on the $N \times N$ selfadjoint complex matrices whose density is proportional to $\exp[-N/2 Tr(X^2)]dX$

Free probability: convergence in NC distribution

- Step (1): Consider a d-tuple of (random) matrices as non-commutative random variables $X_1^{(N)}, \ldots, X_d^{(N)} \in M_N(\mathbb{C})$. The NC expectation is $N^{-1}Tr = tr$.
- Step (2): If, for any $i_1, \ldots, i_l \in \{1, \ldots, d\}$ we can establish the existence of the limit

$$\lim_N tr(X_{i_1}^{(N)} \ldots X_{i_l}^{(N)})$$

- Step (3): ... and we can identify the limit object, namely (A, τ) and $x_1, \ldots, x_d \in A$ such that the above limit is $\tau(x_{i_1} \ldots x_{i_l})$, then one has convergence in NC distribution (def, Voiculescu). .
- If x_i belong to different algebras that are free in Voiculescu's sense, this is *asymptotic freeness*.

Free probability: asymptotic freeness

The Haar case:

- Consider $U_1^{(N)}, \ldots, U_d^{(N)}$ iid Haar unitaries and w a non-reduced word in formal unitaries (or free group elements) $u_1, \ldots, u_d$ and their inverses.
  Consider $W^{(N)}$ to be the random unitary obtained by replacing u_i by $U_i^{(N)}$ in the word w.

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  Then, $\lim_N tr(W^{(N)}) = 0$ a.s. as $N \to \infty$ (Voiculescu 1992, 1998)
- The limiting object is $L(F_d)$ with its canonical tracial state. It is a free product. Therefore we speak of asymptotic freeness

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- The limiting object is $L(F_d)$ with its canonical tracial state. It is a free product. Therefore we speak of asymptotic freeness
- We can replace one $U_i^{(N)}$ by a *constant* traceless word, and it doesn't change the result. Therefore, we can add any constant matrix after some arithmetic computations.

Free probability: asymptotic freeness

- The calculations behind that are called free probability calculus, it was developed by many people (revolving around Voiculescu and Speicher).
- The GUE case.
  Consider $X_1^{(N)}, \ldots, X_d^{(N)}$ iid GUE
  Then, $\lim_N tr(W^{(N)}) = \#K$ a.s. as $N \to \infty$ where K is the number of admissible non-crossing partitions
- Example:

$$\lim_N tr(X_1^{(N)} X_2^{(N)} X_2^{(N)} X_1^{(N)} X_1^{(N)} X_1^{(N)}) = 2,$$

$$\lim_N tr(X_2^{(N)} X_1^{(N)} X_2^{(N)} X_1^{(N)} X_1^{(N)} X_1^{(N)}) = 0.$$

Knowing the moments imply knowing the distribution of any NC polynomial

A remarkable example: in $\mathbb{M}_{2N}(\mathbb{C})$, take uniformly two independent random selfadjoint projections of rank N, $P^{(N)}$ and $Q^{(N)}$. $(P^{(N)}, Q^{(N)})$ has the same distribution as $(P^{(N)}, UQ^{(N)}U^*)$ for any unitary U.

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Theorem

As the dimension grows, with high probability, the histogram (i.e. the NC distribution) of $P^{(N)}Q^{(N)}P^{(N)}$ and of $P^{(N)} + Q^{(N)}$ has the same shape (up to trivial eigenvalues and with a factor two).

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Proof: For any k, as $N \to \infty$, $2tr[(2PQP)^k] \sim tr[(P+Q)^k]$.

arcsine distribution

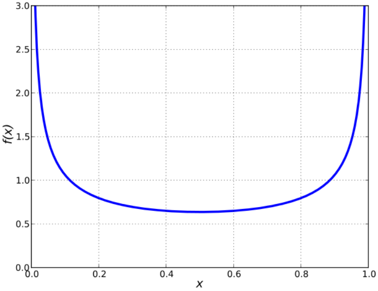

Abbildung: Arcsine distribution

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- When a d-tuple of random matrices converges to a limiting object, then its L^p norms converge to the L^p norm of the limiting object with respect to its trace (at least if p is even).
- Let P be a non-commutative polynomial in d formal variables. Assume that $(X_1^{(N)}, \ldots, X_d^{(N)})$ converge in distribution to $(x_1, \ldots, x_d)$.
  Then, the above observation implies that

$$\liminf_N ||P(X_1^{(N)}, \ldots, X_d^{(N)})|| \geq ||P(x_1, \ldots, x_d)||$$

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  Important question (strong convergence): when is this inequality saturated (in the sense that, for any P, $\limsup_N ||P(X_1^{(N)}, \ldots, X_d^{(N)})|| \leq ||P(x_1, \ldots, x_d)||$?

Strong Asymptotic freeness

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- Both in the GUE and in the unitary case, quantitative estimates were obtained by F. Parraud with free stochastic calculus (KU& Lyon Ph.D.)
  Very strong results by Bandeira, Boedijardjo, van Handel

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  Very strong results by Bandeira, Boedijardjo, van Handel
- Bordenave & C obtained strong convergence for very general models with moment methods [see later]

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- There exists many examples of non-random matrix models (Biane, C, Novak, Śniady, etc) that are asymptotically free.
- However there exists no non-random example that is strongly asymptotically free (specifically, all the above counterexamples fail).
- As far as I can tell, there is only one credible non-random candidate (LPS). Having non-random examples would have really important applications (e.g. in Quantum Information Theory).
- Part of our leitmotiv with Bordenave consisted in reducing the amount of randomness in our strongly convergent models.

The tensor case: setup

We are interested in the following specific problem:

- Consider an $n \times n$ block matrix $Z = (Z_{ij})_{i,j \in \{1,\dots n\}}$
- Each block Z_{ij} is an $N \times N$ matrix.
- We assume given $U_1^{(N)}, \dots, U_d^{(N)}$ unitaries in $N \times N$ (they will be random i.i.d later)
- Each Z_{ij} is a (NC) polynomial in $U_1^{(N)}, \dots, U_d^{(N)}$ and their inverses (we possibly allow constant matrices in addition).

The tensor case: the problem

- Example, $n = 3$ (let's focus on selfadjoint)
  $Z_N =$
  $\Big(U_1^{(N)2} + (U_1^{(N)})^{-2} \quad U_2^{(N)} \quad 0 \; (U_2^{(N)})^{-1} \quad 3I_N \quad U_1^{(N)} + (U_3^{(N)})^{-1} \; 0$
- What is the operator norm of such a matrix?
- We are particularly interested in the case $U_i^{(N)}$ random and N large.

Reduction: the linearization trick

- We may assume that the entries are affine functions in U_i, U_i^*
- Example:

$$||3 \cdot 1_N + U^2 + U^{-2}|| = ||U + U^{-1}||^2 + 1$$

$$||2 \cdot 1_N + U_2 U_1^* + U_1 U_2^*|| = || \begin{pmatrix} 0 & U_1 & U_2 & U_1^* & 0 & 0 & U_2^* & 0 & 0 \end{pmatrix} ||^2$$

- Similar recipes work for multiple matrices and matrix coefficients.

Reduction: the linearization trick

- This is called the *unitary linearization trick*.
- It was discovered by Gilles Pisier in the 90's for unitaries. His statement was a theoretical *global* equivalence in the context of operator spaces.
  "*Understand the norm for all Z with affine matrix coefficients (for all n)*" is equivalent to
  "*Understand the norm for all Z in general (for all degree, for all n)*"

Reduction: the linearization trick

- An explicit version of the trick (given a Z, find a linearized Z' whose norm allows us to deduce the norm of Z) was found by Lehner in the free case.
- Bordenave, C, 2023: we did the general case (without freeness assumption, as in Pisier's original result).
- It relies heavily on the fact that $UU^* = U^*U = 1$ (but how to use this trick for RMT questions if we don't know how to make analysis on unitaries?)

Reduction: the linearization trick (The selfadjoint case and RMT)

- A version of the linearization problem was also found in Haagerup-Thorbjørnsen in 2005 for the *selfadjoint case*.
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  They needed it to understand the norm of polynomials in *GUE*.

Reduction: the linearization trick (The selfadjoint case and RMT)

- A version of the linearization problem was also found in Haagerup-Thorbjørnsen in 2005 for the *selfadjoint case*. (10 years after Pisier)
  They needed it to understand the norm of polynomials in GUE.
- The first linearization trick that was useful for RMT was historically the one discovered later. The Pisier trick started becoming useful for Random Matrix Theory only with Bordenave & C (after understanding how to do analysis on random unitaries)

Why does the tensor setup matter? One concrete example with graphs and permutations.

- What is the operator norm of $U_1^{(N)} + U_1^{(N)*} + \ldots + U_d^{(N)} + U_d^{(N)*}$?
- If $U_i^{(N)}$ of permutations: This is the adjacency matrix of a $2d$-regular (random) graph. The norm is $2d$ (Perron Frobenius).

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- On the orthogonal of the PF eigenvector The norm is at least

$$2\sqrt{2d-1} - f(N)$$

with $f(N) = o_N(1)$ (Alon-Boppana).

One concrete example: more comments

- Note: $Tr(w)$ is a number of fixed points if $U_i^{(N)}$ are permutations. Asymptotic freeness is easy to derive (Nica) and it can be used to rederive the Alon-Boppana bound (the Kesten McKay distribution can be rederived from free probability).

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- Friedman, Bordenave: What is the second largest eigenvalue (norm on the orthogonal of the PF space) is at most $2\sqrt{2d-1}+f(N)$ (strong convergence on the orthogonal).
- Important for mixing times of RW's (in random environment). If the second largest e.v. is $2\sqrt{2d-1}$ then this is the adjacency matrix of a Ramanujan graph.
- Marcus Spielman Srivastava: such graphs exist with probability > 0 (relation to the paving problem / Kadison-Singer problem).

One concrete example: the relation with tensors

- Similar lower and upper bound hold for other random unitaries (e.g. the Haar measure on the unitary group, orthogonal group). The lower bounds are achieved with Weingarten calculus.

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- Similar lower and upper bound hold for other random unitaries (e.g. the Haar measure on the unitary group, orthogonal group). The lower bounds are achieved with Weingarten calculus.
- Remark: if we replaced $U_i^{(N)}$ by $v_i \otimes U_i \in \mathbb{U}_{nN}$ with v_i completely arbitrary unitaries in $\mathbb{U}_n$, the free probability result would give the same lower bound for n very large.
- There seems to be no constraint on n for a lower bound on the operator norm (we show that this is the case later). Is it the case for the upper bound?

One concrete example: predictable constraints on n for the upper bound

- Is it the case for the upper bound? YES
- Exercise:

$$\sum_{i=1}^{d} \overline{U_i} \otimes U_i$$

has operator norm d

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- Pick an epsilon net of $\mathbb{U}_N^d$, of cardinal L. It yields $v_i \in M_N(\mathbb{C})^L$ such that

$$||\sum_{i=1}^{d} v_i \otimes U_i|| \geq d - \varepsilon$$

- The upper bound can be much larger than Alon Boppana (in this example, this can happen if we have $n >> \exp O(N^2)$

Tensor setup: General result

- We aim at comparing the operator norm of $Z_{N,n}$ with the operator norm of $\tilde{Z}_n$, where $\tilde{Z}_n$ is an operator of $B(l^2(F_d)^n)$, and the matrix unitaries U_i are replaced by abstract unitaries $\tilde{U}_i$ with
- (1) The $\tilde{U}_i$'s act by left multiplication on $l^2(F_d)$.
  (2) $B(l^2(F_d)^n) = M_n(l^2(F_d))$.

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  (2) $B(l^2(F_d)^n) = M_n(l^2(F_d))$.
- $Z_{N,n}$ is in general, a random matrix.
  $\tilde{Z}_n$ is a non-random concrete operator (albeit of infinite dimension). It is the candidate for the limit given by all previous developments in free probability.

Tensor setup: General result

Theorem (Bordenave, C)

With high probability (as N grows), the operator norm of $Z_{N,n}$ and $\tilde{Z}_n$ are close as long as $n << \exp O(N^\alpha)$.
[$\alpha > 0$, explicit, depends on d]

Theorem (Bordenave, C)

With high probability (as N grows), the operator norm of $Z_{N,n}$ is bigger than $||\tilde{Z}_n|| - \varepsilon$ INDEPENDENTLY on n. This is true with probability one if $U_i^{(N)}$ are permutations and the coefficients are positive (generalized Alon-Boppana)

Elements of proof: the Moment method

- Our proof relies on the moment method.
- For N dimensional matrices, the L^p norm and the L^∞ (operator) norm are close as soon as $p >> \log N$ (the multiplicative error term is $N^{1/p}$).
- We need to compute moments of order at least $\log(\mathit{dimension})$ (here, dimension = nN)

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- We need to compute moments of order at least $\log(\textit{dimension})$ (here, dimension = nN)
- We don't know how to do it directly. Previous proofs for norm of random matrix (in the multimatrix case) all involved complex analysis. We need to transform the problem first.

Elements of proof: Operator valued non-backtracking theory

- We consider $(b_1, \ldots, b_l)$ elements in $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space. We assume that the index set is endowed with an involution $i \mapsto i^*$ (and $i^{**} = i$ for all i).
- Typically: $l = 2d + 1$ with the notation $i* = -i$ and $U_{-i}^{(N)} = U_i^{(N)*}$

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- Typically: $l = 2d + 1$ with the notation $i* = -i$ and $U_{-i}^{(N)} = U_i^{(N)*}$
- The *non-backtracking operator* associated to the ℓ-tuple of matrices $(b_1, \ldots, b_l)$ is the operator on $\mathcal{B}(\mathcal{H} \otimes \mathbb{C}^l)$ defined by

$$B = \sum_{j \neq i^*} b_j \otimes E_{ij}, \tag{1}$$

Elements of proof: Operator valued non-backtracking theory

Theorem (Bordenave, C)

Let $\lambda \in \mathbb{C}$ satisfy $\lambda^2 \notin \{spec(b_i b_{i^*}) : i \in \{1, \dots, l\}\}$. Define the operator A_λ on $\mathcal{H}$ through

$$A_\lambda = b_0(\lambda) + \sum_{i=1}^{\ell} b_i(\lambda), \qquad b_i(\lambda) = \lambda b_i(\lambda^2 - b_{i^*} b_i)^{-1}$$

and

$$b_0(\lambda) = -1 - \sum_{i=1}^{\ell} b_i(\lambda^2 - b_{i^*} b_i)^{-1} b_{i^*}.$$

Then $\lambda \in \sigma(B)$ if and only if $0 \in \sigma(A_\lambda)$.

Putting the proof together

- In practice we have to understand the spectral radius of the operator and therefore, evaluate $Tr(B^T B^{*T})$ with T growing with the matrix dimension.
- The non backtracking structure makes calculations tractable... through Weingarten calculus.
- *Weingarten calculus* is a systematic method relying on representation theory and algebraic combinatorics to compute integrals of type

$$\int_{U \in G} u_{i_1 j_1} \dots u_{i_k j_k} \overline{u_{i'_1 j'_1} \dots u_{i'_k j'_k}} dU,$$

Where $U = (u_{ij})$ is an element of a matrix compact group G, and dU is the Haar measure.

Concluding remarks: more motivations and perspectives

- Letting both n and N grow to infinity was not natural originally in free probability theory (n fix was natural).
- There was a very strong sense that $n \geq N$ would be very hard.

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- Letting both n and N grow to infinity was not natural originally in free probability theory (n fix was natural).
- There was a very strong sense that $n \geq N$ would be very hard.
- Ben Hayes proved that $n = N$ is enough to solve the Peterson-Thom conjecture. It says that "any diffuse, amenable subalgebra of the free group factor $L(F_2)$ is contained in a unique maximal amenable subalgebra".
  In particular, our result implies the Peterson Thom conjecture (a different, more tailored proof was proposed a bit earlier by Belinschi and Capitaine)

Concluding remarks: more motivations and perspectives

- The proof techniques we developed have important applications in Random geometry (Hide, Magee – maximal spectral gap for a Laplacian on random surfaces of high genus)...

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- The proof techniques we developed have important applications in Random geometry (Hide, Magee – maximal spectral gap for a Laplacian on random surfaces of high genus)...
- ...and for other representations of random groups (Magee, Thomas – mapping class groups, right-angled Artin groups)

Thank you!

- arXiv:1801.00876 Eigenvalues of random lifts and polynomials of random permutation matrices C Bordenave, B Collins Journal-ref: Annals of Mathematics 190 (2019), no. 3, 811-75
- arXiv:2012.08759 Strong asymptotic freeness for independent uniform variables on compact groups associated to non-trivial representations C Bordenave, B Collins
- arXiv:2304.05714 Norm of matrix-valued polynomials in random unitaries and permutations C Bordenave, B Collins

The abstract with less typos