Diophantine approximations, substituions, and fractals

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1 Substitution

Let W^* be the set of words with the alphabet $\{1,2,3\}$, that is,

$$W^* := \bigcup_{n=0}^{\infty} \{1,2,3\}^n$$

For any words $W = w_1 \cdots w_k$, $V = v_1 \cdots v_l \in W^*$, let us define the product $W \cdot V$ by the concatination:

$$W \cdot V = w_1 \cdots w_k v_1 \cdots v_l.$$

Then W^* is a semi-group. Let $\sigma : W^* \to W^*$ be an endomorphism, that is, let us assume the following property:

$$\sigma(W) = \sigma(w_1)\sigma(w_2)\cdots\sigma(w_k) \quad \text{for} \quad W = w_1 \cdots w_k \in W^*.$$

We call σ satisfying above property by a substitution.

Let $f : W^* \to \boldsymbol{Z}^3$ be the canonical homomorphism, that is,

$$\begin{aligned} f(i) &:= \mathbf{e}_i, \quad i = 1,2,3, \\ f(\phi) &:= \mathbf{o}, \\ f(W) &:= \sum_{i=1}^{k} f(w_i) \quad \text{for} \quad W = w_1 \cdots w_k \in W^* \end{aligned}$$

where $\{\mathbf{e}_i \mid i = 1,2,3\}$ the canonical basis of $\boldsymbol{Z}^3 \subset \boldsymbol{R}^3$. For each substitution $\sigma : W^* \to W^*$ let us define the 3×3 matrix L_σ by

$$L_\sigma = [f(\sigma(1)), f(\sigma(2)), f(\sigma(3))],$$

then the following lemma holds.

Lemma 1.1 *The commutative relation holds:*

$$\begin{array}{ccccc} & W^* & \xrightarrow{\sigma} & W^* & \\ f & \downarrow & & \downarrow & f \\ & \boldsymbol{Z}^3 & \xrightarrow{L_\sigma} & \boldsymbol{Z}^3 & \end{array}$$

The matrix L_σ is called the matrix of σ or abeliarization of σ.

Example 1 (Rauzy substitution [R]) *Let us define the substitution σ as follows:*

$$\sigma : \begin{array}{ccc} 1 & \longrightarrow & 12 \\ 2 & \longrightarrow & 13 \\ 3 & \longrightarrow & 1 \end{array},$$

then the matrix of σ is given by

$$L_\sigma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The substitution σ is called Rauzy substitution.

Example 2 (Modified Jacobi-Perron substitution [I-O 93]) *For each $a \in \boldsymbol{N}$ and $\varepsilon \in \{0,1\}$ let us define the substitution $\sigma_{\binom{a}{\varepsilon}}$ as follows:*

$$\sigma_{\binom{a}{0}} : \begin{array}{ccc} 1 & \longrightarrow & \overbrace{11\cdots1}^{a\ times}2 \\ 2 & \longrightarrow & 3 \\ 3 & \longrightarrow & 1 \end{array}, \quad \sigma_{\binom{a}{1}} : \begin{array}{ccc} 1 & \longrightarrow & \overbrace{11\cdots1}^{a\ times}3 \\ 2 & \longrightarrow & 1 \\ 3 & \longrightarrow & 2 \end{array},$$

then the matrix of $\sigma_{\binom{a}{0}}$ and $\sigma_{\binom{a}{1}}$ are given by

$$L_{\sigma_{\binom{a}{0}}} = \begin{bmatrix} a & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad and \quad L_{\sigma_{\binom{a}{1}}} = \begin{bmatrix} a & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

These substitutions $\sigma_{\binom{a}{\varepsilon}}$, $a \in \boldsymbol{N}$, $\varepsilon \in \{0,1\}$ are called Modified Jacobi-Perron substituions (The definition of Modified Jacobi-Perron algorithm is given in the section 6).

Definition 1.2 *The substitution $\sigma : W^* \to W^*$ is said to be Pisot substitution if*

(1) *det $L_\sigma = \pm 1$,*

(2) *there exist N such that*

$$L_\sigma^N > O \quad (Aperiodic\ condition),$$

(3) *the eigenvalues λ, λ' and λ'' of L_σ satisfy*

$$\lambda > 1 > |\lambda'|, |\lambda''| \quad \textit{(Pisot condition)} .$$

In this lecture, we discuss only on Pisot substitutions.

Let us denote the column and row eigenvectors of L_σ associated with the maximum eigenvalue λ by

$$L_\sigma \begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix}, \quad {}^tL_\sigma \begin{bmatrix} 1 \\ \gamma \\ \delta \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \gamma \\ \delta \end{bmatrix}$$

Let $\mathcal{P}$ be the plane given by

$$\mathcal{P} := \left\{ \boldsymbol{x} \in \boldsymbol{R}^3 \;\middle|\; \left\langle \boldsymbol{x}, \begin{bmatrix} 1 \\ \gamma \\ \delta \end{bmatrix} \right\rangle = 0 \right\},$$

then it is easy to see the following lemma.

Lemma 1.3 *The plane $\mathcal{P}$ is the* <u>*contracting invariant plane*</u> *with respect to L_σ, that is,*

$$L_\sigma \mathcal{P} = \mathcal{P}.$$

More precisely,

(1) *In the case of $\lambda', \lambda'' \in \boldsymbol{R}$, put the eigenvectors of λ', λ'' by*

$$\boldsymbol{v}' := \begin{bmatrix} 1 \\ \alpha' \\ \beta' \end{bmatrix}, \quad \boldsymbol{v}'' := \begin{bmatrix} 1 \\ \alpha'' \\ \beta'' \end{bmatrix} \in \boldsymbol{R}^3,$$

then

$$\boldsymbol{v}', \boldsymbol{v}'' \in \mathcal{P}$$

and

$$[L_\sigma \boldsymbol{v}', L_\sigma \boldsymbol{v}''] = [\boldsymbol{v}', \boldsymbol{v}''] \begin{bmatrix} \lambda' & 0 \\ 0 & \lambda'' \end{bmatrix}$$

(2) *In the case of $\overline{\lambda'} = \lambda'' \in \boldsymbol{C}$, put the eigenvectors of λ', λ'' by*

$$\boldsymbol{v}' := \begin{bmatrix} 1 \\ \alpha' \\ \beta' \end{bmatrix}, \quad \boldsymbol{v}'' := \begin{bmatrix} 1 \\ \alpha'' \\ \beta'' \end{bmatrix} \in \boldsymbol{C}^3,$$

and
$$\boldsymbol{u}' := \frac{1}{2}(\boldsymbol{v}' + \boldsymbol{v}''), \quad \boldsymbol{u}'' := \frac{1}{2i}(\boldsymbol{v}' - \boldsymbol{v}''),$$
then
$$\boldsymbol{u}', \boldsymbol{u}'' \in \mathcal{P}$$
and there exists θ such that
$$[L_\sigma \boldsymbol{u}', L_\sigma \boldsymbol{u}''] = \frac{1}{\sqrt{\lambda}}[\boldsymbol{u}', \boldsymbol{u}''] \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Let us assume $\sigma(1) = 1 \cdot W_1$, that is, the first alphabet of $\sigma(1)$ coinsides with 1. Then there exists the sequence u of $\{1, 2, 3\}$ given by $u = \lim_{n\to\infty} \sigma^n(1)$ and satisfies
$$\sigma(u) = u,$$
that is, u is the fixed point of σ.

For each Pisot substitution, let $\pi : \boldsymbol{R}^3 \to \mathcal{P}$ be the projection along ${}^t[1, \alpha, \beta]$, and let us define the set
$$\begin{aligned}
Y_N &:= \left\{ \pi \sum_{j=1}^{k} \boldsymbol{e}_{s_j} \;\middle|\; k = 1, 2, \cdots, N \right\}, \\
Y_{N,i} &:= \left\{ \pi \sum_{j=1}^{k} \boldsymbol{e}_{s_j} \;\middle|\; s_j = i,\ k = 1, 2, \cdots, N,\ i = 1, 2, 3 \right\}, \\
Y &:= \bigcup_{N=1}^{\infty} Y_N, \\
Y_i &:= \bigcup_{N=1}^{\infty} Y_{N,i}, \\
X &:= \text{the closure of } Y, \\
X_i &:= \text{the closure of } Y_i,
\end{aligned}$$
where s_j, $j \in \boldsymbol{N}$ are given by
$$u = \lim_{n\to\infty} \sigma^n(1) = s_1 s_2 \cdots s_k \cdots\cdots.$$
Then we can find the domain X and X_i, $i = 1, 2, 3$ with fractal boundaries.

On Example 1 [Rauzy substitution]:

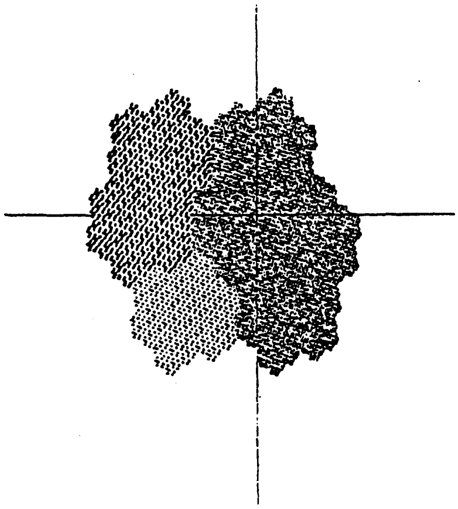

Figure 1: The figure of the domain X with fractal boundaries on Rauzy substitution

On Example 2 [Modified Jacobi-Perron substitution]: in the case of $a = 1$ and $\varepsilon = 0$.

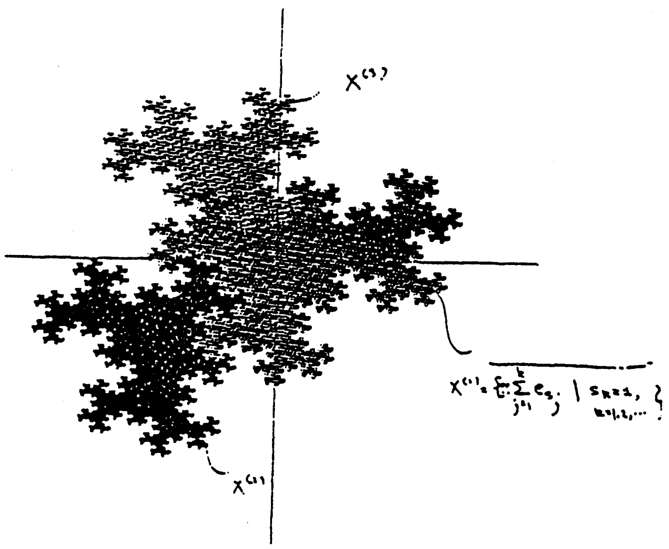

Figure 2: The figure of the domain X with fractal boundaries on Modified Jacobi-Perron substitutions

Note. We introduce the substitution on $W^* = \bigcup_{n=1}^{\infty}\{1,2,3\}^n$ in this lecture, but it is easy to extend the definition of substitutions on the alphabet $\{1,2\cdots,N\}$. In particular, the simplest case of $N = 2$ is fundamental. In this note, we will give a kind of survey at the end of each sections in the case of $N = 2$.

On the alphabet$\{1,2\}$, we can define Pisot substitution σ and its matrix L_σ. Using the column and row eigenvectors with respect to the maximum eignvalue λ:

$$L_\sigma \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \alpha \end{bmatrix}, \quad {}^tL_\sigma \begin{bmatrix} 1 \\ \gamma \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \gamma \end{bmatrix},$$

we can define the contracting invariant line l_σ with respect to L_σ by

$$l_\sigma = \left\{ \boldsymbol{x} \;\middle|\; \left\langle \boldsymbol{x}, \begin{bmatrix} 1 \\ \gamma \end{bmatrix} \right\rangle = 0 \right\}$$

and the projection $\pi : \boldsymbol{R}^2 \to l_\sigma$ along ${}^t[1,\alpha]$ analogously. And for the substituion satisfying

$$\sigma(1) = 1 \cdot W_1$$

let Y, Y_i, $i = 1,2$, and X, X_i, $i = 1,2$ be projective sets as in Page 4. Then we find the set X, X_i, $i = 1,2$ which are intervals usually on the following example.

Example 3 *In the case of $N = 2$, for each $a \in \boldsymbol{N}$ let us define the substitutions σ_a by*

$$\sigma_a : \begin{array}{ccc} 1 & \longrightarrow & \overbrace{11\cdots1}^{a\ times}2 \, . \\ 2 & \longrightarrow & 1 \end{array}$$

The matrix of σ_a are given by

$$L_{\sigma_a} = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}$$

The substituion σ_a, $a \in \boldsymbol{N}$ are called continued fraction substituions (The definition of the continued fraction algorithm can be found in the section 6).

2 Endomorphism

For $\boldsymbol{x} \in \boldsymbol{Z}^3$ and $i \in \{1,2,3\}$ let us consider the pair $(\boldsymbol{x}, i^*)$ by

$$(\boldsymbol{x}, i^*) := \{\boldsymbol{x} + \lambda \boldsymbol{e}_j + \mu \boldsymbol{e}_k \mid 0 \le \lambda \le 1, \quad 0 \le \mu \le 1\}$$

where (j,k) is taken as $(i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}$.

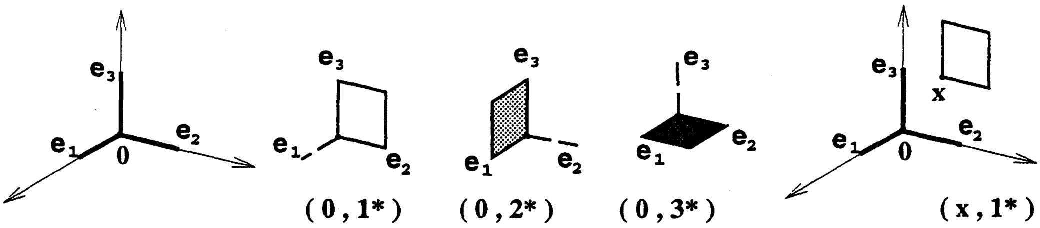

Figure 3: The figure of $(0, i^*)$, $i = 1,2,3$ and $(x, 1^*)$

All of the pair $(\boldsymbol{x}, i^*)$, $\boldsymbol{x} \in \boldsymbol{Z}^3$ and $i \in \{1,2,3\}$ is denoted by Λ, that is,

$$\Lambda = \left\{(\boldsymbol{x}, i^*) \;\middle|\; \boldsymbol{x} \in \boldsymbol{Z}^3, i = \{1,2,3\}\right\}.$$

Let $\mathcal{G}$ be the $\boldsymbol{Z}$-module generated by Λ as follows:

$$\mathcal{G} = \left\{ \sum_{\lambda \in \Lambda} m_\lambda \lambda \;\middle|\; m_\lambda \in \boldsymbol{Z}, \; \#\{\lambda \mid m_\lambda \neq 0\} < +\infty \right\}.$$

Let us consider the following <u>endomorphism Θ of $\mathcal{G}$ associated with σ</u>:

$$\begin{aligned} \Theta(\mathbf{o}, i^*) &:= \sum_{j=1,2,3} \sum_{\substack{W: \\ \sigma(j)=Y \cdot i \cdot W}} \left(L_\sigma^{-1}(f(W)), j^*\right), \\ \Theta(\boldsymbol{x}, i^*) &:= L_\sigma^{-1}\boldsymbol{x} + \Theta(\mathbf{o}, i^*) \end{aligned}$$

where $\boldsymbol{y} + (\boldsymbol{x}, i^*) := (\boldsymbol{y} + \boldsymbol{x}, i^*)$, and for $\sum_{\lambda \in \Lambda} m_\lambda \lambda \in \mathcal{G}$

$$\Theta\left(\sum_{\lambda \in \Lambda} m_\lambda \lambda\right) := \sum_{\lambda \in \Lambda} m_\lambda \Theta(\lambda).$$

To discuss the geometrical property of Θ, we introduce the stepped surface of a plane.

For any $0 < \gamma, \delta < 1$ let us consider the plane $\mathcal{P}_{\gamma,\delta}$, that is,

$$\mathcal{P}_{\gamma,\delta} := \left\{ \boldsymbol{x} \;\middle|\; \left\langle \boldsymbol{x}, \begin{bmatrix} 1 \\ \gamma \\ \delta \end{bmatrix} \right\rangle = 0 \right\},$$

and for each plane $\mathcal{P}_{\gamma,\delta}$ let us conider the stepped surface $\mathcal{S}^{+}_{\gamma,\delta}$ $\left(\mathcal{S}^{-}_{\gamma,\delta}\right)$ as follows:

$$\mathbf{S}^{+}_{\gamma,\delta} := \left\{ (\boldsymbol{x}, i^{*}) \;\middle|\; \left\langle \boldsymbol{x}, \begin{bmatrix} 1 \\ \gamma \\ \delta \end{bmatrix} \right\rangle > 0,\ \left\langle (\boldsymbol{x} - e_i), \begin{bmatrix} 1 \\ \gamma \\ \delta \end{bmatrix} \right\rangle \leq 0 \right\}$$

$$\left(\mathbf{S}^{-}_{\gamma,\delta} := \left\{ (\boldsymbol{x}, i^{*}) \;\middle|\; \left\langle \boldsymbol{x}, \begin{bmatrix} 1 \\ \gamma \\ \delta \end{bmatrix} \right\rangle \geq 0, \left\langle (\boldsymbol{x} - e_i), \begin{bmatrix} 1 \\ \gamma \\ \delta \end{bmatrix} \right\rangle < 0 \right\} \right)$$

and

$$\mathcal{S}^{+}_{\gamma,\delta} := \bigcup_{(\boldsymbol{x}, i^{*}) \in \mathbf{S}^{+}_{\gamma,\delta}} (\boldsymbol{x}, i^{*})$$

$$\left(\mathcal{S}^{-}_{\gamma,\delta} := \bigcup_{(\boldsymbol{x}, i^{*}) \in \mathbf{S}^{-}_{\gamma,\delta}} (\boldsymbol{x}, i^{*}) \right).$$

Let $\Gamma^{+}_{\gamma,\delta}$ $\left(\Gamma^{-}_{\gamma,\delta}\right)$ be

$$\Gamma^{+}_{\gamma,\delta} := \left\{ \bigcup_{\lambda \in \mathbf{S}^{+}_{\gamma,\delta}} n_\lambda \lambda \;\middle|\; n_\lambda \in \{0,1\},\ \#\{\lambda \mid n_\lambda = 1\} < +\infty \right\}$$

$$\left(\Gamma^{-}_{\gamma,\delta} := \left\{ \bigcup_{\lambda \in \mathbf{S}^{-}_{\gamma,\delta}} n_\lambda \lambda \;\middle|\; n_\lambda \in \{0,1\},\ \#\{\lambda \mid n_\lambda = 1\} < +\infty \right\} \right).$$

The set $\Gamma^{+}_{\gamma,\delta}$ $\left(\Gamma^{-}_{\gamma,\delta}\right) \subset \mathcal{G}$ is a family of $\mathbf{S}^{+}_{\gamma,\delta}$ $\left(\mathbf{S}^{+}_{\gamma,\delta}\right)$ on $\mathcal{S}^{+}_{\gamma,\delta}$ $\left(\mathcal{S}^{-}_{\gamma,\delta}\right)$. Then we find the following theorem.

Theorem 2.1 ([A-I]) *For each Pisot substitution let us consider the endomorphism Θ of $\mathcal{G}$ associated with σ and $\Gamma^{+}_{\gamma,\delta}$ with respect to the contracting* *invariant plane $\mathcal{P}_{\gamma,\delta}$, then $\Gamma^+_{\gamma,\delta}$ is invariant with respect to the endomorphism Θ, that is,*

$$\bigcup_{\lambda\in \mathbf{S}^+_{\gamma,\delta}} n_\lambda\lambda \in \Gamma^+_{\gamma,\delta} \quad \textit{implies} \quad \Theta\left(\bigcup_{\lambda\in \mathbf{S}^+_{\gamma,\delta}} n_\lambda\lambda\right) \in \Gamma^+_{\gamma,\delta}$$

$$\left(\bigcup_{\lambda\in \mathbf{S}^-_{\gamma,\delta}} n_\lambda\lambda \in \Gamma^-_{\gamma,\delta} \quad \textit{implies} \quad \Theta\left(\bigcup_{\lambda\in \mathbf{S}^-_{\gamma,\delta}} n_\lambda\lambda\right) \in \Gamma^-_{\gamma,\delta}\right).$$

On Example 1, for the substitution $\sigma : \begin{array}{ccc} 1 & \rightarrow & 12 \\ 2 & \rightarrow & 13 \\ 3 & \rightarrow & 1 \end{array}$ the endomorphism Θ is given by

$$\Theta : \begin{array}{ccl} (\mathbf{o}, 1^*) & \longrightarrow & (\mathbf{o}, 3^*) + (\boldsymbol{f}_2, 1^*) + (\boldsymbol{f}_3, 2^*) \\ (\mathbf{o}, 2^*) & \longrightarrow & (\mathbf{o}, 1^*) \\ (\mathbf{o}, 3^*) & \longrightarrow & (\mathbf{o}, 2^*) \end{array}$$

where $L_\sigma^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} = [\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3]$. The figure of Θ is given as follows:

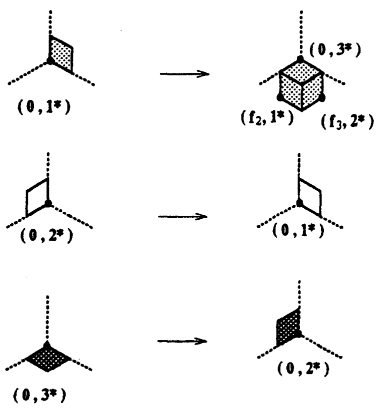

Figure 4: The figure of Θ

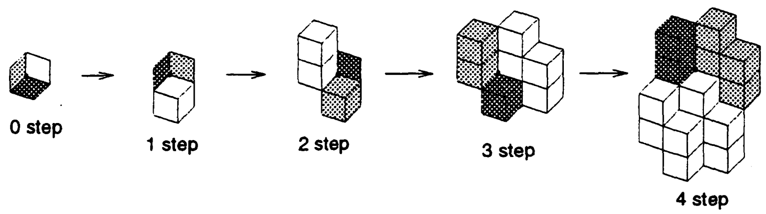

Figure 5: The figure of $\bigcup_{i=1,2,3} \Theta^n(0, i^*)$, $n = 0, 1, 2, 3, 4$

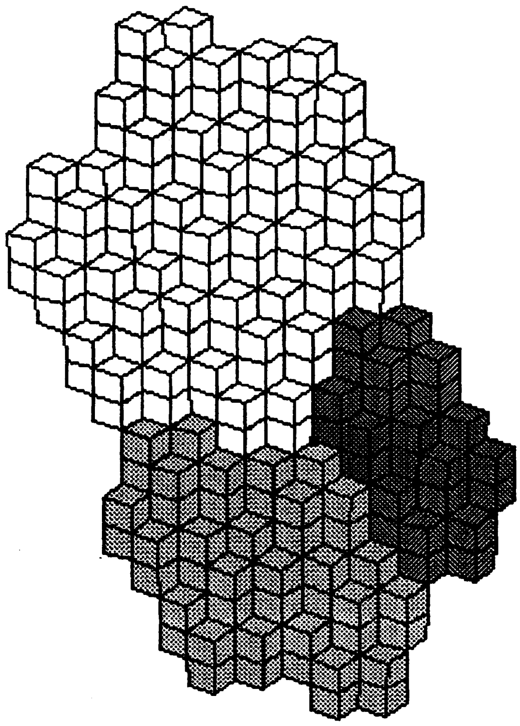

Figure 6: The figure of $\bigcup_{i=1,2,3} \Theta^8(0, i^*)$

On Example 2, for the substitution $\sigma_{\binom{a}{0}}: \begin{array}{lcl} 1 & \longrightarrow & \overbrace{11\cdots1}^{a\ times}2 \\ 2 & \longrightarrow & 3 \\ 3 & \longrightarrow & 1 \end{array}$ and

$\sigma_{\binom{a}{1}}: \begin{array}{lcl} 1 & \longrightarrow & \overbrace{11\cdots1}^{a\ times}3 \\ 2 & \longrightarrow & 1 \\ 3 & \longrightarrow & 2 \end{array}$ the endomorphisms $\Theta_{\binom{a}{0}}$ and $\Theta_{\binom{a}{1}}$ are given by

$$\Theta_{\binom{a}{0}}: \begin{array}{lcl} (\mathbf{o},1^*) & \longrightarrow & (\mathbf{o},3^*) + \sum_{1\leq k\leq a} ((e_1 - ke_3),1^*) \\ (\mathbf{o},2^*) & \longrightarrow & (\mathbf{o},1^*) \\ (\mathbf{o},3^*) & \longrightarrow & (\mathbf{o},2^*) \end{array}$$

and

$$\Theta_{\binom{a}{1}}: \begin{array}{lcl} (\mathbf{o},1^*) & \longrightarrow & (\mathbf{o},2^*) + \sum_{1\leq k\leq a} ((e_1 - ke_2),1^*) \\ (\mathbf{o},2^*) & \longrightarrow & (\mathbf{o},3^*) \\ (\mathbf{o},3^*) & \longrightarrow & (\mathbf{o},1^*) \end{array}$$

The figures of $\Theta_{\binom{a}{\varepsilon}}$, $\varepsilon = 0, 1$ are as follows:

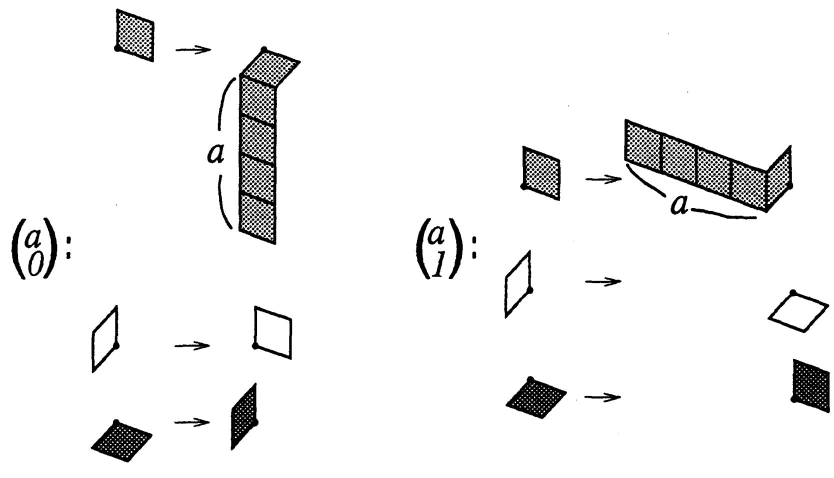

Figure 7: The figure of Θ

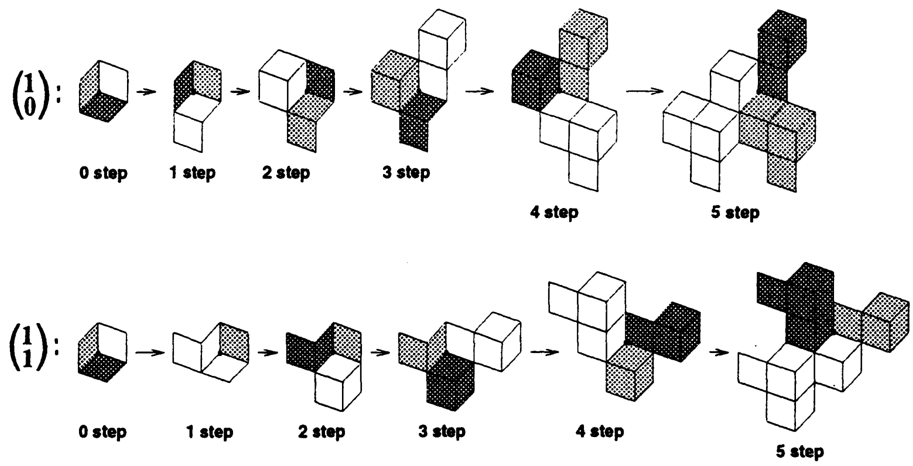

Figure 8: The figure of $\bigcup_{i=1,2,3} \Theta^n(o, i^*), n = 0, 1, \cdots, 5$

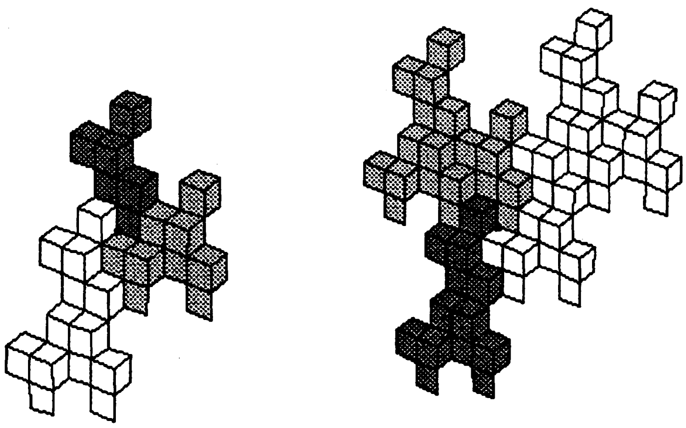

Figure 9: The figure of $\bigcup_{i=1,2,3} \Theta^8(0, i^*)$ and $\bigcup_{i=1,2,3} \Theta^{10}(0, i^*)$ in the case of $a = 1$, $\varepsilon = 0$

On Example 1 and Example 2, we can see that $\Theta^n(\mathbf{o}, i^*)$, $i = 1, 2, 3$ are simply connected. See [I-O] and [F-I] if you are interested in the geometrical properties of $\Theta^n(\mathbf{o}, i^*)$, $i = 1, 2, 3$. In general, $\Theta^n(\mathbf{o}, i^*)$, $i = 1, 2, 3$ are not simply connected.

Example 4 *This is an example that* $\Theta^n(\mathbf{o}, i^*)$, $i = 1, 2, 3$ *are not simply connected. For the substitution* $\sigma : \begin{array}{ccc} 1 & \rightarrow & 21 \\ 2 & \rightarrow & 13 \\ 3 & \rightarrow & 1 \end{array}$, *the endomorphism* Θ *is given by*

$$\Theta : \begin{array}{ccl} (\mathbf{o}, 1^*) & \longrightarrow & (\mathbf{o}, 1^*) + (\mathbf{o}, 3^*) + (\boldsymbol{f}_3, 2^*) \\ (\mathbf{o}, 2^*) & \longrightarrow & (\boldsymbol{f}_1, 1^*) \\ (\mathbf{o}, 3^*) & \longrightarrow & (\mathbf{o}, 2^*) \end{array}$$

where $L_\sigma^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} = [\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3]$. *The figure of* Θ *is given as follows:*

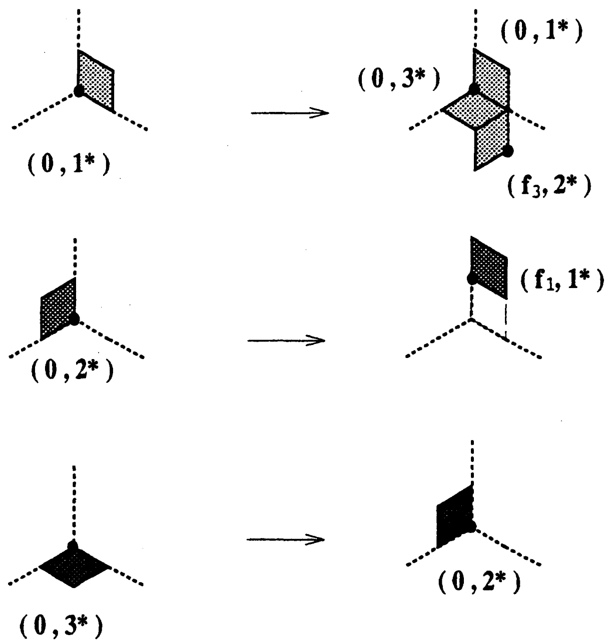

Figure 10: The figure of Θ

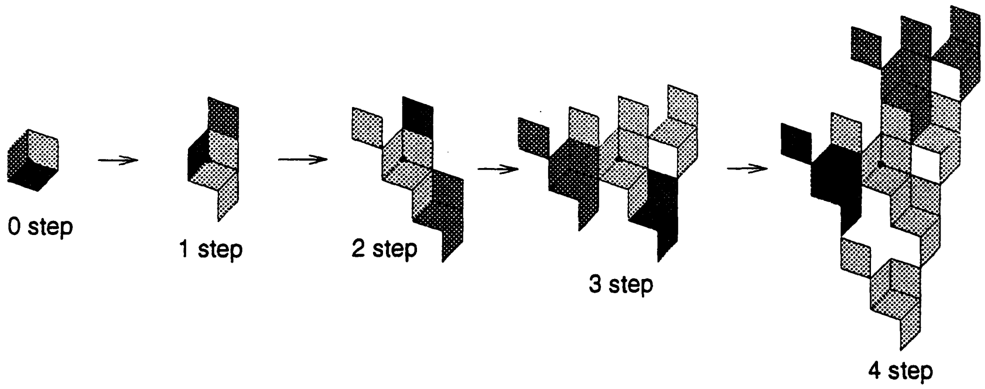

Figure 11: The figure of $\bigcup_{i=1,2,3} \Theta^n(0, i^*), n = 0, 1, 2, 3, 4$

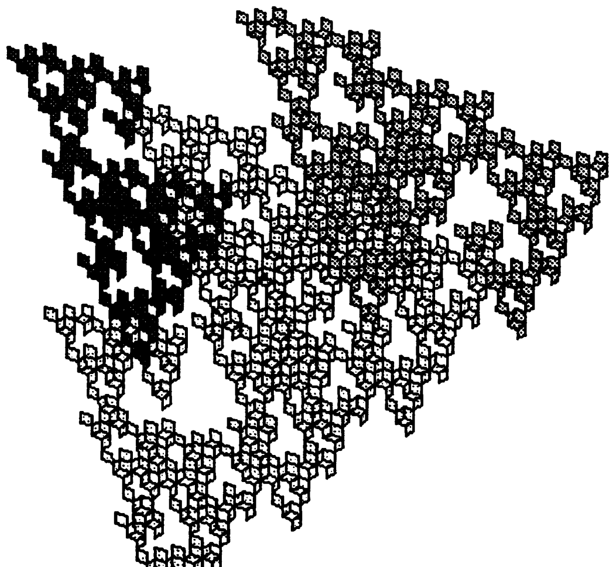

Figure 12: The figure of $\bigcup_{i=1,2,3} \Theta^{10}(0, i^*)$

Notation. For $\gamma = \sum_{\lambda \in \mathbf{S}^+_{\gamma,\delta}} n_\lambda \lambda$, $\delta = \sum_{\lambda \in \mathbf{S}^+_{\gamma,\delta}} m_\lambda \lambda \in \Gamma^+_{\gamma,\delta}$, $\gamma \succ \delta$ means that $n_\lambda \neq 0$ if $m_\lambda \neq 0$, that is, the patch δ is the subpatch of γ (See Figure 13).

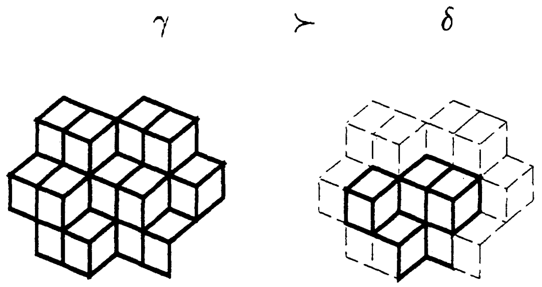

Figure 13: The figure of $\gamma \succ \delta$

On the above notation we have the following lemma.

Lemma 2.2 *Let*

$$\mathcal{U} := \bigcup_{i=1,2,3} (e_i, i^*), \quad \mathcal{U}' := \bigcup_{i=1,2,3} (\mathbf{o}, i^*),$$

then

(i) $\Theta(\mathcal{U}) \succ \mathcal{U}$

(ii) $\Theta(\mathcal{U}') \succ \mathcal{U}'$.

Note. In the case of alphabet $\{1,2\}$, the element $(\boldsymbol{x}, i^*) \in \boldsymbol{Z}^2 \times \{1^*, 2^*\}$ of the set Λ is given by

$$(\boldsymbol{x}, i^*) = \{\boldsymbol{x} + \lambda \boldsymbol{e}_j \mid 0 \leq \lambda \leq 1\}$$

where $(i,j) \in \{(1,2),(2,1)\}$, and $\mathcal{G}$ is given by

$$\mathcal{G} = \left\{ \sum_{\lambda \in \Lambda} m_\lambda \lambda \;\middle|\; m_\lambda \in \boldsymbol{Z},\ \#\{\lambda \mid m_\lambda \neq 0\} < +\infty \right\}.$$

The endomorphism Θ associated with σ of $\mathcal{G}$ is given analogously by

$$\begin{aligned} \Theta(\mathbf{o}, i^*) &:= \sum_{j=1,2} \sum_{\substack{W: \\ \sigma(j)=Y \cdot i \cdot W}} \left(L_\sigma^{-1}(f(W)), j^*\right), \\ \Theta(\boldsymbol{x}, i^*) &:= L_\sigma^{-1}\boldsymbol{x} + \Theta(\mathbf{o}, i^*), \end{aligned}$$

and for $\sum_{\lambda\in\Lambda} m_\lambda\lambda \in \mathcal{G}$

$$\Theta\left(\sum_{\lambda\in\Lambda} m_\lambda\lambda\right) := \sum_{\lambda\in\Lambda} m_\lambda\Theta(\lambda).$$

For any $0 < \gamma < 1$, let us consider the line l_γ, that is,

$$l_\gamma = \left\{\boldsymbol{x} \,\middle|\, \left\langle \boldsymbol{x}, \begin{bmatrix} 1 \\ \gamma \end{bmatrix} \right\rangle = 0 \right\},$$

and for each line l_γ let us consider the <u>stepped curve</u> $\mathcal{S}^+$ ($\mathcal{S}^-$) as follows:

$$\mathcal{S}^+ := \left\{(\boldsymbol{x}, i^*) \,\middle|\, \left\langle \boldsymbol{x}, \begin{bmatrix} 1 \\ \gamma \end{bmatrix} \right\rangle > 0,\ \left\langle (\boldsymbol{x} - \boldsymbol{e}_i), \begin{bmatrix} 1 \\ \gamma \end{bmatrix} \right\rangle \le 0 \right\}$$

$$\left(\mathcal{S}^- := \left\{(\boldsymbol{x}, i^*) \,\middle|\, \left\langle \boldsymbol{x}, \begin{bmatrix} 1 \\ \gamma \end{bmatrix} \right\rangle \ge 0, \left\langle (\boldsymbol{x} - \boldsymbol{e}_i), \begin{bmatrix} 1 \\ \gamma \end{bmatrix} \right\rangle < 0 \right\}\right)$$

and

$$\mathcal{S}^+ := \bigcup_{(\boldsymbol{x},i^*)\in \mathbf{S}^+} (\boldsymbol{x}, i^*)$$

$$\left(\mathcal{S}^- := \bigcup_{(\boldsymbol{x},i^*)\in \mathbf{S}^-} (\boldsymbol{x}, i^*)\right).$$

Let Γ^+ (Γ^-) be

$$\Gamma^+ := \left\{ \bigcup_{\lambda\in \mathbf{S}^+} n_\lambda\lambda \,\middle|\, n_\lambda \in \{0,1\},\ \#\{\lambda \mid n_\lambda = 1\} < +\infty \right\}$$

$$\left(\Gamma^- := \left\{ \bigcup_{\lambda\in \mathbf{S}^-} n_\lambda\lambda \,\middle|\, n_\lambda \in \{0,1\},\ \#\{\lambda \mid n_\lambda = 1\} < +\infty \right\}\right),$$

then the same statement of Theorem 2.1 holds on this frame work.

On Example 3, for the substitution

$$\begin{array}{rccl} \sigma: & 1 & \longrightarrow & \overbrace{11\cdots1}^{a\ times}2 \\ & 2 & \longrightarrow & 1 \end{array}$$

the endomorphism Θ_a is given by

$$\Theta_a : \begin{array}{lcl} (\mathrm{o},1^*) & \longrightarrow & (\mathrm{o},2^*) + \sum_{k=1}^{a}((e_1 - ke_2),1^*) \\ (\mathrm{o},2^*) & \longrightarrow & (\mathrm{o},1^*) \end{array}$$

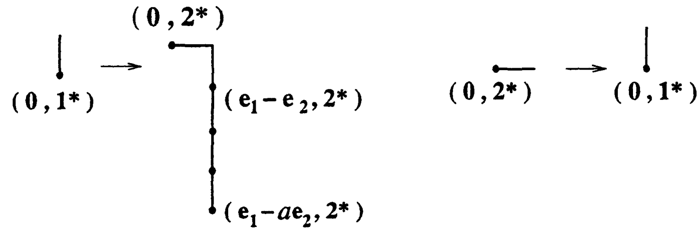

Figure 14: The figure of Θ

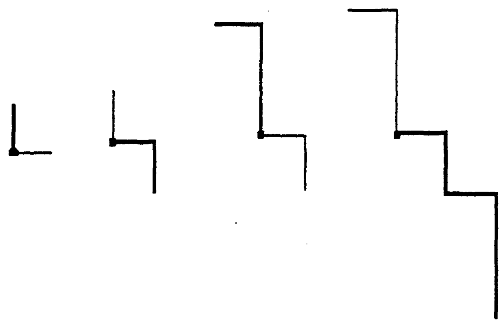

Figure 15: The figure of $\bigcup_{i=1,2}\Theta^n(0,i^*)$, $n = 0,1,2,3$, in the case of $a = 1$

Example 5 *For the substitution* $\sigma : \begin{array}{lcl} 1 & \longrightarrow & 121 \\ 2 & \longrightarrow & 12 \end{array}$ *the endomorphism* Θ *is given by*

$$\Theta : \begin{array}{lcl} (\mathbf{o},1^*) & \longrightarrow & (\mathbf{o},1^*) + (\boldsymbol{f}_1 + \boldsymbol{f}_2,1^*) + (\boldsymbol{f}_2,2^*) \\ (\mathbf{o},2^*) & \longrightarrow & (\boldsymbol{f}_1,1^*) + (\mathbf{o},2^*) \end{array}$$

where $L_\sigma^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = [\boldsymbol{f}_1, \boldsymbol{f}_2]$. *The figure of* Θ *is given as follows:*

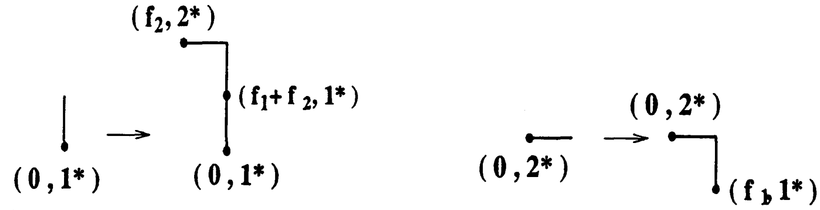

Figure 16: The figure of Θ

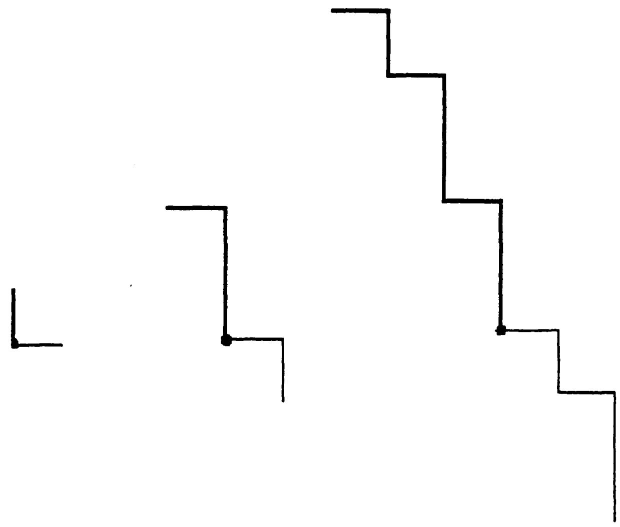

Figure 17: The figure of $\bigcup_{i=1,2} \Theta^n(0, i^*), n = 0, 1, 2$

On Example 3 and Example 5, we see that $\Theta^n(\mathbf{o}, i^*)$, $i = 1, 2$ are simply connected. But in general it is not true.

Example 6 *This is an example that* $\Theta^n(\mathbf{o}, i^*)$, $i = 1, 2$ *are not simply con-*

nected. For the substitution

$$\sigma : \begin{array}{ccc} 1 & \longrightarrow & 112 \\ 2 & \longrightarrow & 21 \end{array}$$

the endomorphism Θ_a *is given by*

$$\Theta : \begin{array}{ccl} (\mathbf{o}, 1^*) & \longrightarrow & (\mathbf{o}, 2^*) + (\boldsymbol{f}_2, 1^*) + (\boldsymbol{f}_1 + \boldsymbol{f}_2, 1^*) \\ (\mathbf{o}, 2^*) & \longrightarrow & (\mathbf{o}, 1^*) + (\boldsymbol{f}_1, 2^*) \end{array}$$

where $L_\sigma^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = [\boldsymbol{f}_1, \boldsymbol{f}_2]$.

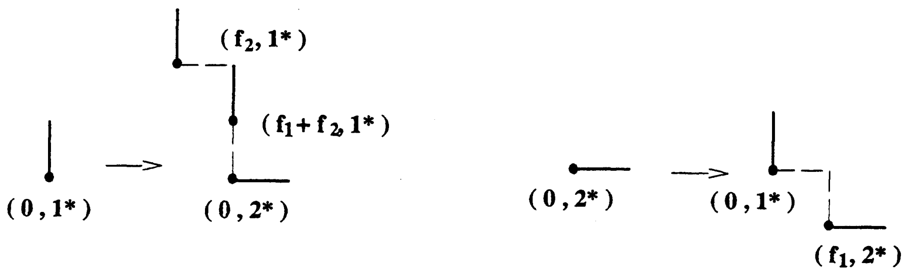

Figure 18: The figure of Θ

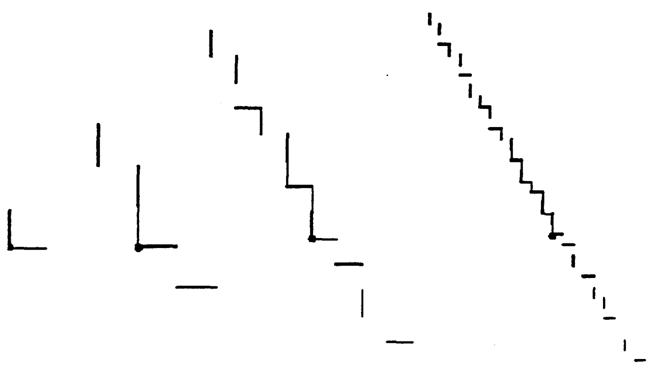

Figure 19: The figure of $\bigcup_{i=1,2} \Theta^n(0, i^*), n = 0, 1, 2, 3$

We have the following interested theorem.

Theorem 2.3 (**[W]** and **[E-I]**) $\Theta^n(\mathbf{o}, i^*)$, $i = 1, 2$ *are simply connected for all* n *iff the substitution* σ *is invertible substitution, that is, there exists the automorphism* $\theta : G\{1, 2\} \to G\{1, 2\}$ *such that*

$$\sigma \circ \theta = \theta \circ \sigma$$

where $G\{1, 2\}$ *be a free group of rank 2 generated by* $\{1, 2\}$.

3 Dynamical system

For each Pisot substitution σ and its endomorphism Θ let us consider the domains on the contracting invariant plane $\mathcal{P}_{\gamma,\delta}$ as follows:

$$\begin{aligned} D_n^{(i)} &:= \pi\left(\Theta^n(e_i, i^*)\right), \quad i = 1,2,3, \\ D_n^{(i)\prime} &:= \pi\left(\Theta^n(\mathbf{o}, i^*)\right), \quad i = 1,2,3, \\ D_n &:= \pi\left(\Theta^n(\mathcal{U})\right), \\ D'_n &:= \pi\left(\Theta^n(\mathcal{U}')\right) \end{aligned}$$

where $\mathcal{U} = \bigcup_{i=1,2,3}(e_i, i^*)$, $\mathcal{U}' = \bigcup_{i=1,2,3}(\mathbf{o}, i^*)$.
Notice that $\mathcal{U} \in \Gamma^+_{\alpha,\beta}$, $\mathcal{U}' \in \Gamma^-_{\alpha,\beta}$ and by Theorem 2.1 and Lemma 2.2 we know that

$$\begin{aligned} \Theta^n(\mathcal{U}) &\succ \Theta^{n-1}(\mathcal{U}) \\ \Theta^n(\mathcal{U}') &\succ \Theta^{n-1}(\mathcal{U}'). \end{aligned}$$

Therefore, the sets $D_n^{(i)}$, $D_n^{(i)\prime}$ are well-defined and $D_n = D'_n$ (See [A-I] in detail).

We have the following theorem.

Theorem 3.1 *Let L_σ^{-n} be denoted by*

$$L_\sigma^{-n} = \left[\boldsymbol{f}_1^{(n)}, \boldsymbol{f}_2^{(n)}, \boldsymbol{f}_3^{(n)}\right],$$

and the following dynamical system W_n on D_n is well-defined:

$$\begin{array}{ccll} D_n & \xrightarrow{W_n} & D_n & \\ \boldsymbol{x} & \mapsto & \boldsymbol{x} - \pi \boldsymbol{f}_i^{(n)} & \textit{if} \quad \boldsymbol{x} \in D_n^{(i)}, \ i = 1,2,3 \end{array}$$

and the dynamical systems (D_n, W_n) are isomorphic each other.

We will show the figures of W_n on the examples.

On Example 1, for the substitution $\sigma : \begin{array}{ccc} 1 & \longrightarrow & 12 \\ 2 & \longrightarrow & 13 \\ 3 & \longrightarrow & 1 \end{array}$, the figure of W_n is as follows:

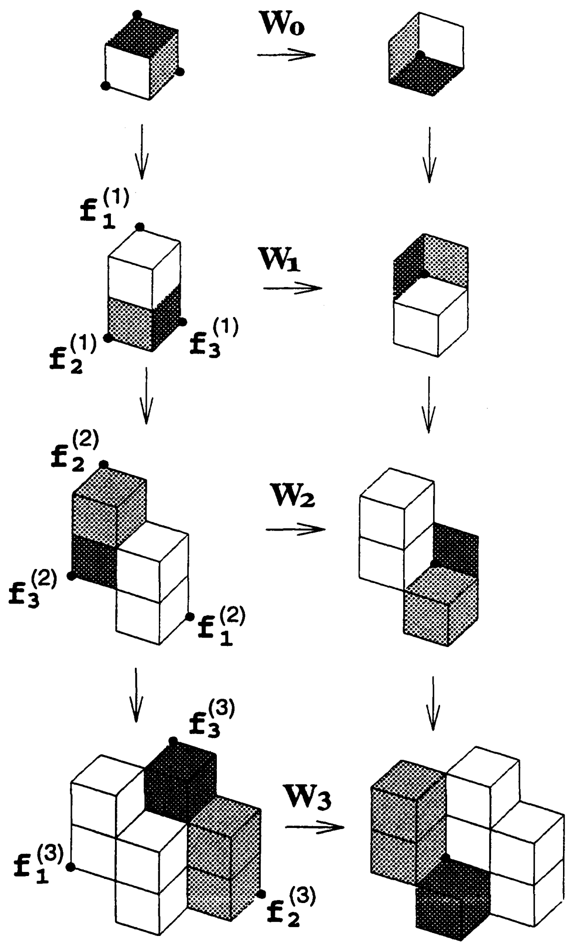

Figure 20: The figures of W_n

On Example 2, for the substituion $\sigma_{\binom{1}{0}}$: $\begin{array}{ccc} 1 & \longrightarrow & 12 \\ 2 & \longrightarrow & 3 \\ 3 & \longrightarrow & 1 \end{array}$, the figure of W_n is as follows:

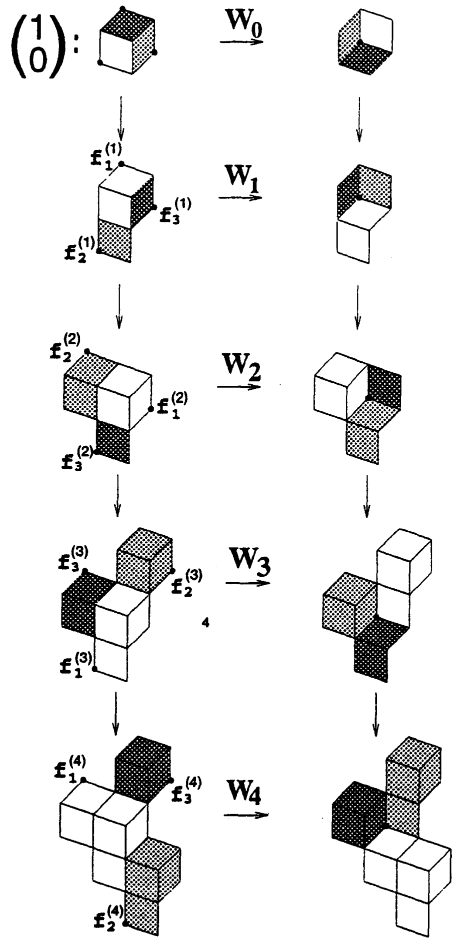

Figure 21: The figure of W_n

Theorem 3.2 *The induced transformation $W_n|_{D_{n-1}}$ of W_n into D_{n-1} coincides with W_{n-1}. More precisely, let us denote the substitution σ by*

$$\sigma : \begin{array}{ccc} 1 & \longrightarrow & s_1^{(1)} s_2^{(1)} \cdots s_{l(1)}^{(1)} \\ 2 & \longrightarrow & s_1^{(2)} s_2^{(2)} \cdots s_{l(2)}^{(2)} \\ 3 & \longrightarrow & s_1^{(3)} s_2^{(3)} \cdots s_{l(3)}^{(3)} \end{array},$$

then the following relation holds:

$$\textbf{(1)}\ W_n^{k-1}\left(D_{n-1}^{(i)}\right) \subset D_n^{(s_k^{(i)})}, \quad k = 1, 2, \cdots, l(i)$$

$$\textbf{(2)}\ W_n^{l(i)}\left(D_{n-1}^{(i)}\right) = D_{n-1}^{(i)'}.$$

On Example 1, in the case of $n = 3$:

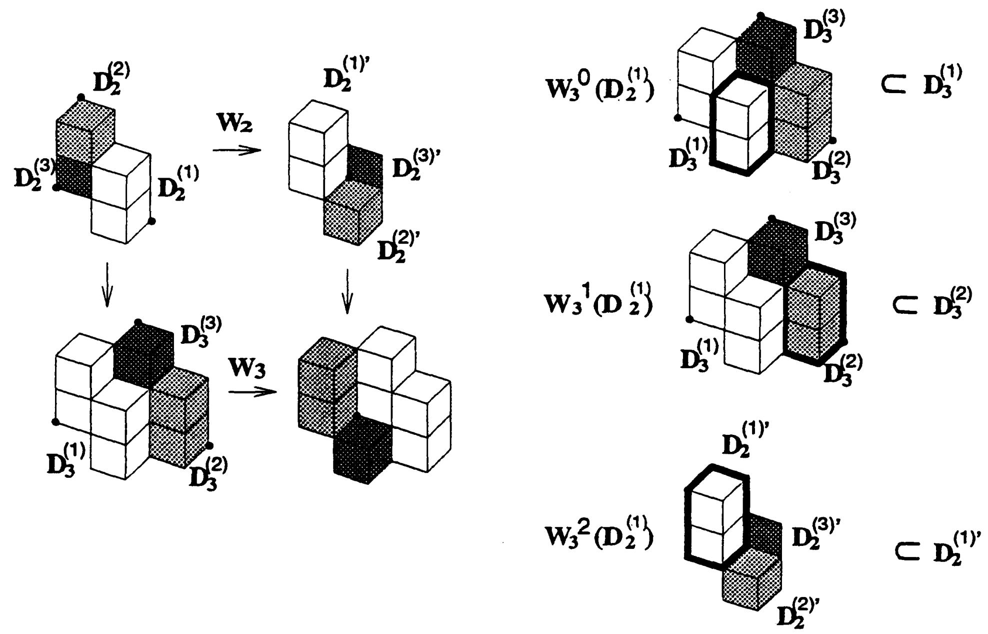

Figure 22:

Corollary 3.3 *Let us denote σ^n by*

$$\sigma^n(i) = s_1^{(n,i)} s_2^{(n,i)} \cdots s_{l(n,i)}^{(n,i)}, \quad i = 1, 2, 3,$$

then

(1)

$$\begin{aligned} W_n^{k-1}\left(D_0^{(i)}\right) &\subset D_n^{\left(s_k^{(n,i)}\right)}, \; k = 1, 2, \cdots, l(n,i) \\ W_n^{l(n,i)}\left(D_0^{(i)}\right) &= D_0^{(i)\prime}, \end{aligned}$$

(See Figure 23), and in particular, we have

(2)

$$\left\{ W_n^k(\mathbf{o}) \;\middle|\; k = 1, 2, \cdots, l(n,1) \right\} = \left\{ -\pi \sum_{j=1}^{k} \boldsymbol{f}^{(n)}_{s_j^{(n,1)}} \;\middle|\; k = 1, 2, \cdots, l(n,1) \right\}$$

where $L_\sigma^{-n} = \left[\boldsymbol{f}_1^{(n)}, \boldsymbol{f}_2^{(n)}, \boldsymbol{f}_3^{(n)} \right]$ *(See Figure 24).*

On Example 1, in the case of n=3:

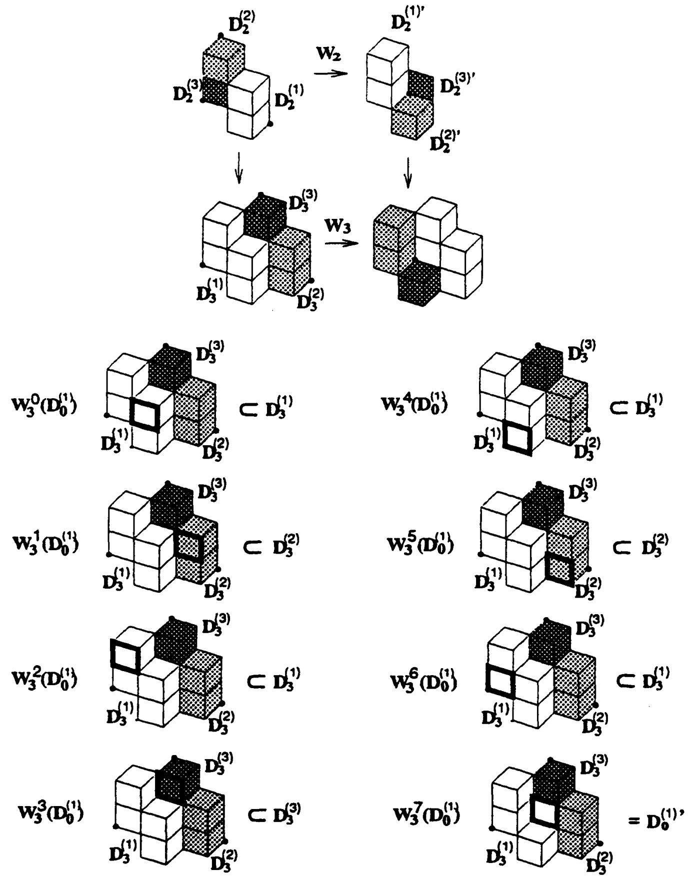

Figure 23:

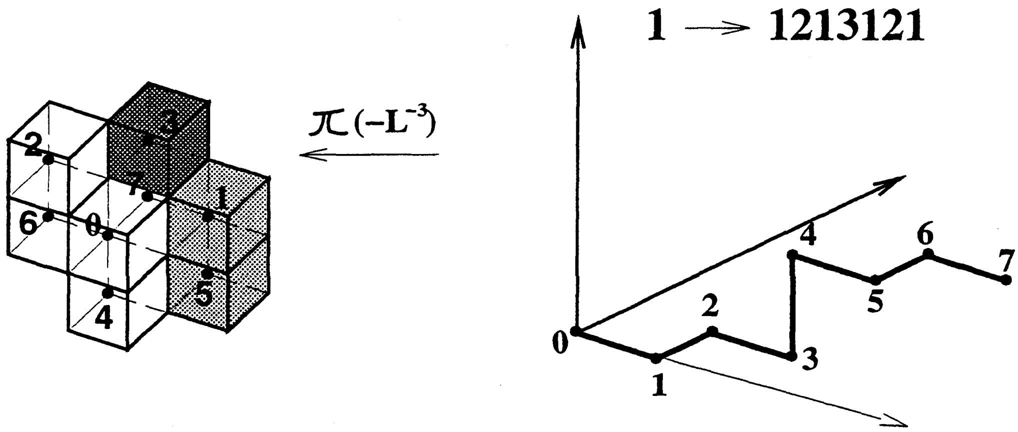

Figure 24:

where $\sigma^3(1) = 1213121$. and $\left[\boldsymbol{f}_1^{(3)}, \boldsymbol{f}_2^{(3)}, \boldsymbol{f}_3^{(3)}\right] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix}$.

Remark 1 *The domain exchange transformaiton $W_0 : D_0 \to D_0$ coincides with the quasi-periodic motion, that is,*

$$W_0 : \begin{array}{ccc} D_0 & \longrightarrow & D_0 \\ \boldsymbol{x} & \mapsto & \boldsymbol{x} - \pi \boldsymbol{e}_1 \end{array} \quad (mod\ \boldsymbol{L}_0)$$

where $\boldsymbol{L}_0 = \{n\pi(\boldsymbol{e}_2 - \boldsymbol{e}_1) + m\pi(\boldsymbol{e}_3 - \boldsymbol{e}_1) \mid m, n \in \boldsymbol{Z}\}$.

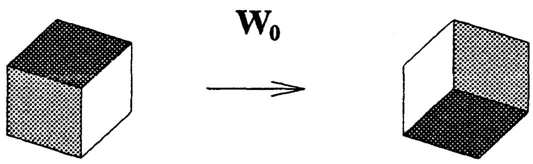

Figure 25: The figure of W_0

$$x \quad \mapsto \quad x - \pi e_1$$

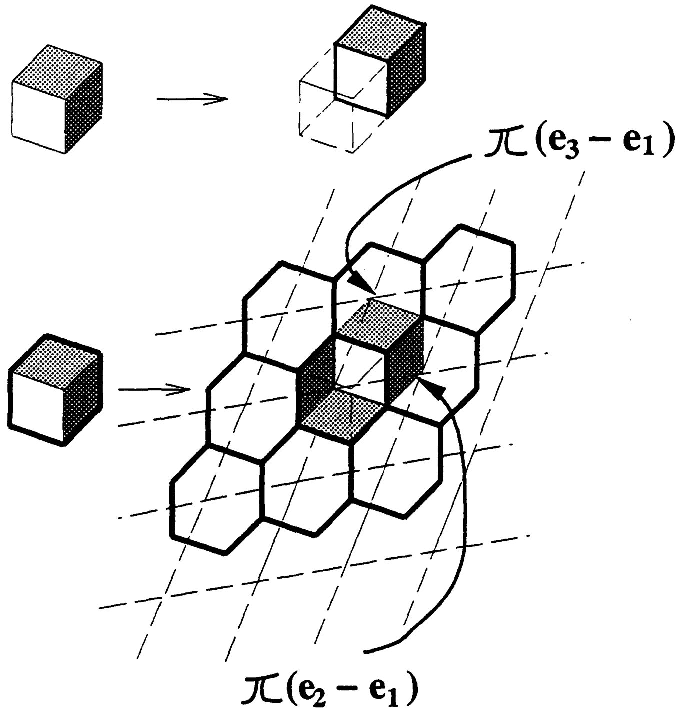

Figure 26: The figure of the quasi-periodic motion

Note. In the case of two alphabets, let us introduce the union of intervals on the line l_γ analogously:

$$\begin{aligned}
D_n^{(i)} &:= \pi\left(\Theta^n(e_i, i^*)\right), \quad i = 1, 2, \\
D_n^{(i)\prime} &:= \pi\left(\Theta^n(\mathbf{o}, i^*)\right), \quad i = 1, 2, \\
D_n &:= \pi\left(\Theta^n\left(\bigcup_{i=1,2}(e_i, i^*)\right)\right), \\
D_n' &:= \pi\left(\Theta^n\left(\bigcup_{i=1,2}(\mathbf{o}, i^*)\right)\right),
\end{aligned}$$

then the following "union of intervals" exchange on D_n is well defined:

$$\begin{array}{ccc} D_n & \xrightarrow{W_n} & D_n \\ \boldsymbol{x} & \mapsto & \boldsymbol{x} - \pi \boldsymbol{f}_i^{(n)} \quad \text{if } \boldsymbol{x} \in D_n^{(i)}, \end{array}$$

and it is isomorphic to the interval exchange

$$\begin{array}{ccc} D_0 & \xrightarrow{W_0} & D_0 \\ \boldsymbol{x} & \mapsto & \boldsymbol{x} - \pi \boldsymbol{e}_i \quad \text{if } \boldsymbol{x} \in D_0^{(i)} \end{array}$$

where $L_\sigma^{-n} = \left[\boldsymbol{f}_1^{(n)}, \boldsymbol{f}_2^{(n)}\right]$.

On Example 5, for the substitution $\sigma : \begin{array}{ccc} 1 & \longrightarrow & 121 \\ 2 & \longrightarrow & 12 \end{array}$, the figure of W_n is as follows:

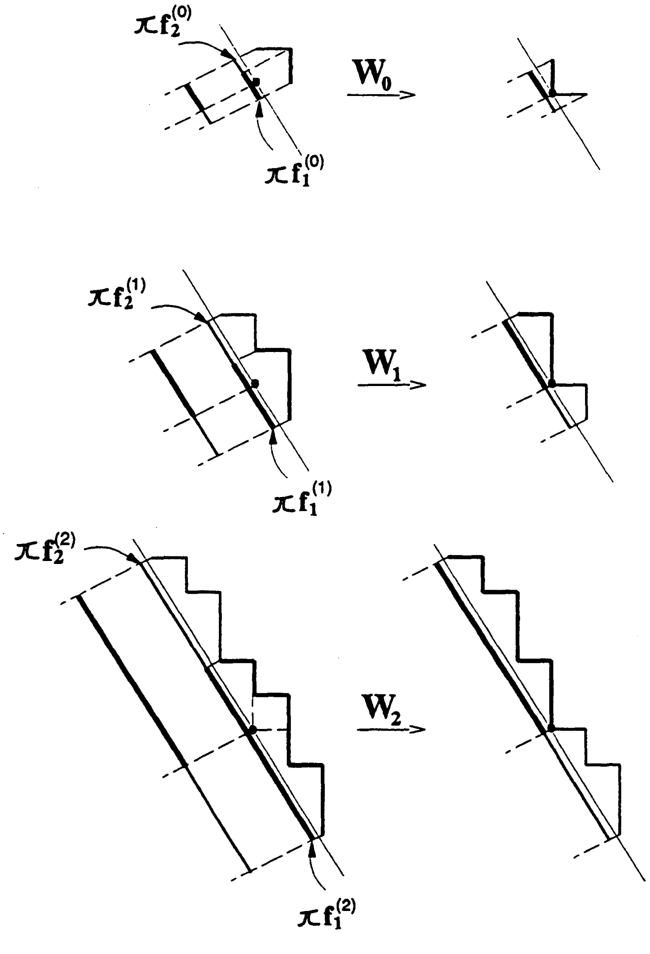

Figure 27: The figures of W_n

On Example 6, for the substitution $\sigma : \begin{array}{ccc} 1 & \longrightarrow & 112 \\ 2 & \longrightarrow & 21 \end{array}$, the figure of W_n is as follows:

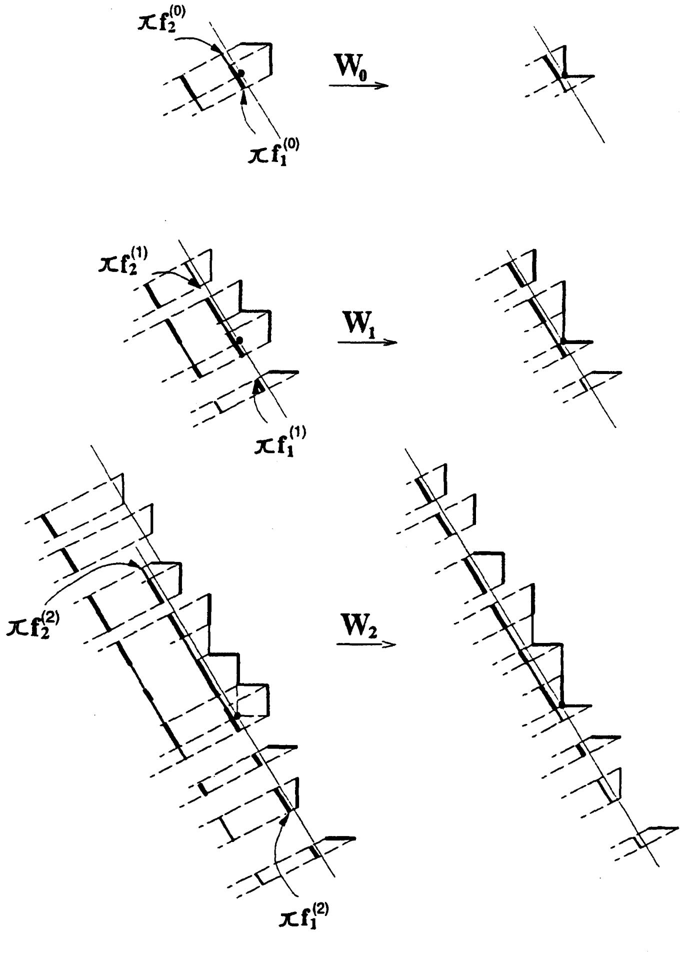

Figure 28: The figures of W_n

The same statements given by Theorem 3.2, Corollary 3.3 hold in the case of alphabets $\{1, 2\}$.

4 Renormalization

Let us consider the following limit set in the sense of Hausdorff metric of the family of compact subsets of $\mathcal{P}_{\gamma,\delta}$:

$$\begin{aligned} X &:= \lim_{n\to\infty} L_\sigma^n\left(\pi\left(\Theta^n(\mathcal{U})\right)\right) \left(= \lim_{n\to\infty} L_\sigma^n\left(\pi\left(\Theta^n(\mathcal{U}')\right)\right)\right), \\ X_i &:= \lim_{n\to\infty} L_\sigma^n\left(\pi\left(\Theta^n(e_i, i^*)\right)\right), \\ X_i' &:= \lim_{n\to\infty} L_\sigma^n\left(\pi\left(\Theta^n(\mathbf{o}, i^*)\right)\right). \end{aligned}$$

Then the following theorem holds.

Theorem 4.1 *Let σ be Pisot substitution and let us assume that there exists $N > 0$, $N \in \boldsymbol{N}$ and i such that*

$$\Theta^N(e_i, i^*) \succ \mathcal{U}.$$

Then the limit set satisfies the following properties:

(1)

$$X = \bigcup_{i=1,2,3} X_i \quad (disjoint)\ ,$$

that is, $|\ X_i \bigcap X_j\ | = 0 \quad (i \neq j)$ *in the sense of Lebesgue measure, and*

$$\mathcal{P}_{\gamma,\delta} = \bigcup_{\boldsymbol{z}\in\boldsymbol{L}_0} (X + \boldsymbol{z}) \quad (disjoint)\ ,$$

that is, $|\ (X+\boldsymbol{z}) \bigcap (X+\boldsymbol{z}')\ | = 0 \quad (\boldsymbol{z} \neq \boldsymbol{z}')$ *in the sense of Lebesgue measure, where* $\boldsymbol{L}_0 = \{n\pi(e_2 - e_1) + m\pi(e_3 - e_1)\ \mid m, n \in \boldsymbol{Z}\}$.

(2) *The transformation* $W : X \to X$

$$W\boldsymbol{x} = \boldsymbol{x} - \pi e_i \quad \textit{if} \quad \boldsymbol{x} \in X_i$$

is well-difined and isomorphic to $W_0 : D_0 \to D_0$.

(3) *The induced transformation* $W|_{L_\sigma X}$ *is isomorphic to* W *and satisfies*

$$\begin{aligned} W^{k-1}(X_i^{(1)}) &\subset X_{s_k^{(i)}}, \quad k = 1, 2, \cdots, l(i) \\ W^{l(i)}(X_i^{(1)}) &= X_i^{(1)'} \end{aligned}$$

where $\sigma(i) = s_1^{(i)} \cdots s_{l(i)}^{(i)}$ *and* $X_i^{(1)} := L_\sigma X_i$ $(X_i^{(1)'} := L_\sigma X_i')$.

(4) *The transformation* $T : X \longrightarrow X$

$$Tx = L_\sigma^{-1}x - L_\sigma^{-1}(f(W)) + e_j \quad \textit{if} \quad L_\sigma^{-1}x \in L_\sigma^{-1}(f(W)) + X_j$$

where the word W *is given by the following formula:*

$$\Theta(\mathbf{o}, i^*) := \bigcup_{j=1,2,3} \bigcup_{\substack{W: \\ \sigma(j)=Y\cdot i\cdot W}} \left(L_\sigma^{-1}(f(W)), j^*\right)$$

is well-defined and the transformation T *is Markov endomorphism whose structure matrix is* L_σ.

Remark. In Theorem 4.1, we find the assumption that there exists N and i such that

$$\Theta^N(e_i, i^*) \succ \mathcal{U}.$$

We don't have the example which does not hold above. I believe that the assumption holds for any Pisot substitution. But it is still open problem even in the case of two alphabets. If it is O.K., then we can say the dynamical system associated with the substitution has the discrete spectrum, moreover it is isomorphic to the quasi-periodic motions.

On Example 1, the transformation W is given as follows:

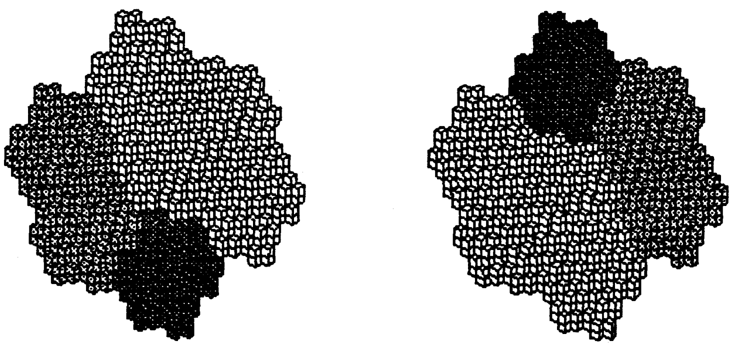

Figure 29: The figure of W

On Example 2 (in the case of $a = 1$, $\varepsilon = 0$), the transformation W is given as follows:

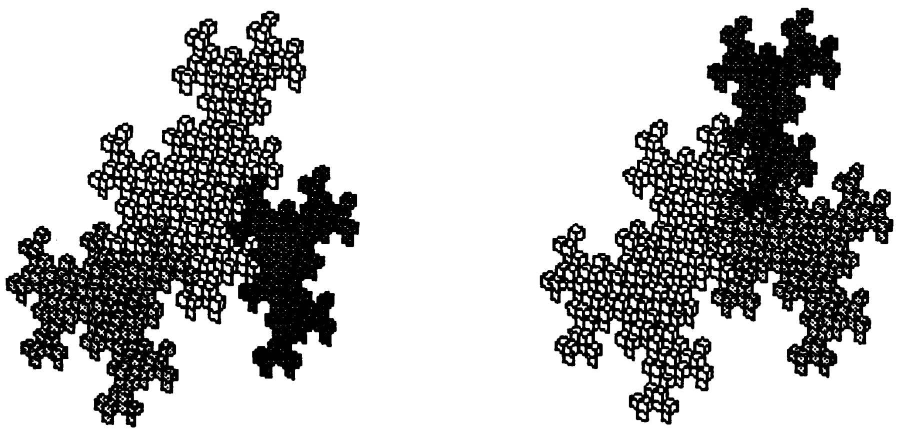

Figure 30: The figure of W

On Example 4, the transformation W is given as follows:

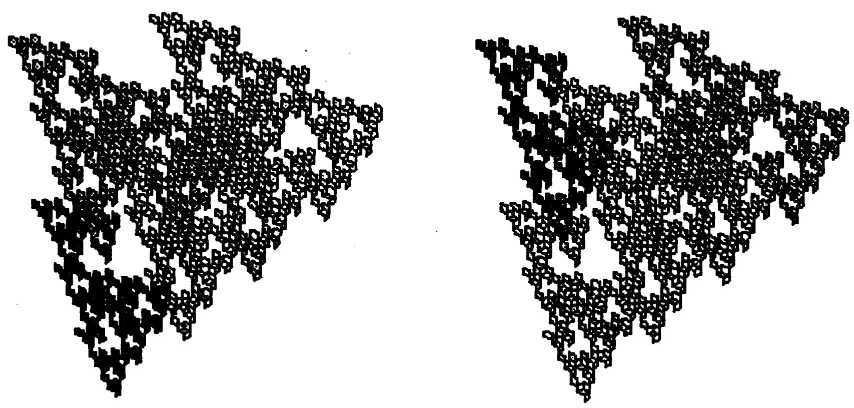

Figure 31: The figure of W

5 Fractal boundaries

The boundary of the domain X seems to be fractal. To observe the property of boundaries, let us introduce some notations.

Let $(\boldsymbol{x}, i) \in \boldsymbol{Z}^3 \times \{1, 2, 3\}$ be

$$(\boldsymbol{x}, i) = \{\boldsymbol{x} + \lambda \boldsymbol{e}_i \mid 0 \leq \lambda < 1\}, \quad i = 1, 2, 3,$$

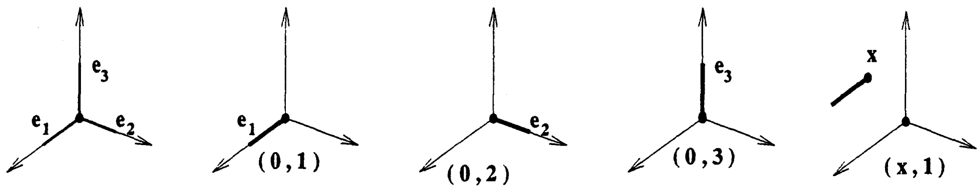

Figure 32: The figure of $(0, i^*)$, $i = 1, 2, 3$

and let us define the boundary map ∂ by

$$\partial(\boldsymbol{x}, i^*) \ := \ (\boldsymbol{x}, j) + (\boldsymbol{x} + \boldsymbol{e}_j, k) - (\boldsymbol{x}, k) - (\boldsymbol{x} + \boldsymbol{e}_k, j)$$

where $\{i, j, k\}$ is taken in $\{\{1, 2, 3\}, \{2, 3, 1\}, \{3, 1, 2\}\}$. For each $\gamma = \sum_{\lambda \in \Lambda} m_\lambda \lambda \in \mathcal{G}$ let us define the boundary of γ by

$$\partial \gamma = \sum_{\lambda \in \Lambda} m_\lambda \partial \lambda,$$

and the all of the set $\partial\gamma$, $\gamma \in \mathcal{G}$ is denoted by $\mathcal{G}_1$. Then $\mathcal{G}_1$ is a $\boldsymbol{Z}$-module.

Now, let us define the boundary endomorphism θ associated with Θ if there exists the endomorphism θ which satisfies the following commutative relation:

$$\begin{array}{ccccc} & \mathcal{G} & \xrightarrow{\Theta} & \mathcal{G} & \\ \partial & \downarrow & & \downarrow & \partial \\ & \mathcal{G}_1 & \xrightarrow{\theta} & \mathcal{G}_1 & \end{array}$$

Theorem 5.1 (Ito-Sano) *Let us denote Pisot substitution σ by*

$$\sigma(i) = s_1^{(i)} \cdots s_{l(i)}^{(i)}, \quad i = 1, 2, 3,$$

and let us define θ by

$$\theta(\mathbf{o}, i) := \sum_{\substack{1 \leq t \leq 3 \\ 1 \leq u \leq 3}} \sum_{\substack{s_l{}^{(t)} = j \\ s_m^{(u)} = k}} \left(L_\sigma^{-1} \left(f \left(S_l^{(t)} \right) \right) + L_\sigma^{-1} \left(f \left(S_m^{(u)} \right) \right), t \wedge u \right)$$

where $\{i, j, k\}$ is taken in $\{\{1,2,3\}, \{2,3,1\}, \{3,1,2\}\}$ and $(\boldsymbol{x}, t \wedge u)$, $t, u \in \{1, 2, 3\}$ means

$$(\boldsymbol{x}, t \wedge u) = \begin{cases} (\boldsymbol{x}, s) & \textit{if} \quad (t, u) \in \{(2,3), (3,1), (1,2)\} \\ -(\boldsymbol{x}, s) & \textit{if} \quad (t, u) \in \{(3,2), (1,3), (2,1)\} \end{cases}$$

where $\{s, t, u\} = \{1, 2, 3\}$ and $S_l^{(t)}$ means the suffix of $s_l^{(t)}$ in $\sigma(t)$, that is,

$$\begin{aligned} \sigma(t) &= s_1{}^{(t)} \cdots s_l{}^{(t)} \cdots s_{l(t)}{}^{(t)}, \\ &= P_l{}^{(t)} s_l{}^{(t)} S_l{}^{(t)}. \end{aligned}$$

For $(\boldsymbol{x}, i)$ let us define

$$\begin{aligned} \theta(\boldsymbol{x}, i) &:= L_\sigma^{-1} \boldsymbol{x} + \theta(\mathbf{o}, i), \\ \theta(\sum_{\lambda \in \Lambda} n_\lambda \lambda) &:= \sum_{\lambda \in \Lambda} n_\lambda \theta(\lambda) \end{aligned}$$

where $\boldsymbol{y} + (\boldsymbol{x}, i) := (\boldsymbol{y} + \boldsymbol{x}, i)$. Then the map θ is the bounday endomorphism of Θ.

On Example 1, θ is given by the following manner:

$$\begin{aligned} \theta(\mathbf{o}, 1) &= (\mathbf{o}, 1 \wedge 2) = (\mathbf{o}, 3), \\ \theta(\mathbf{o}, 2) &= \left(L_\sigma{}^{-1}(\mathbf{e}_2), 2 \wedge 1 \right) + (\mathbf{o}, 2 \wedge 3) \\ &= - \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, 3 \right) + (\mathbf{o}, 1), \\ \theta(\mathbf{o}, 3) &= \left(L_\sigma^{-1}(e_3), 2 \wedge 1 \right) + (\mathbf{o}, 3 \wedge 1) \\ &= - \left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, 3 \right) + (\mathbf{o}, 2). \end{aligned}$$

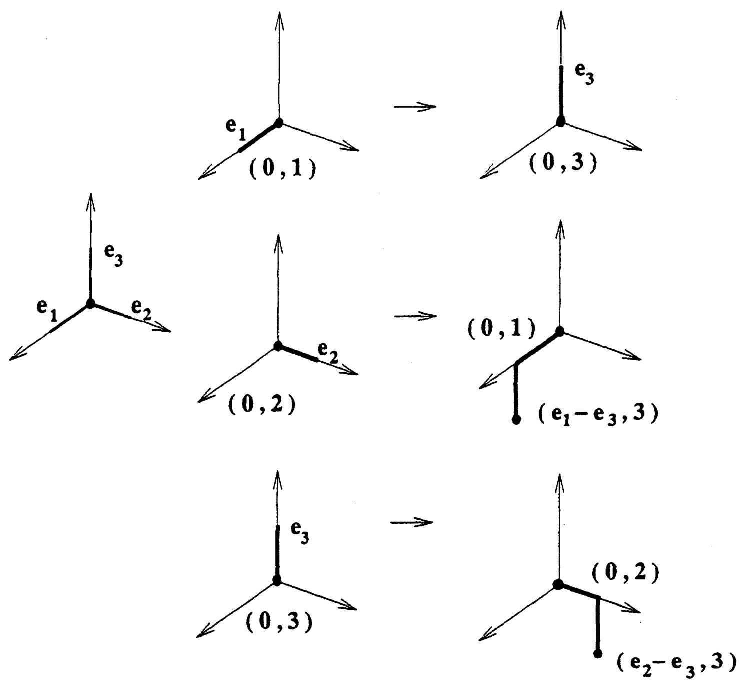

Figure 33: The figure of θ

Note. Using the boundary endomorphism θ, if θ satisfies some conditions with respect to the cancellation, then we can caluculate the Hausdorff dimention explicitely. (See [I-K], [I-O 93] and [I-O 91]).

6 Diophantine algorithm and substitutions

Let X be the domain given by $X = [0,1) \times [0,1)$ and let us define the transformation on T by

$$T(\alpha,\beta) := \begin{cases} \left(\dfrac{\beta}{\alpha}, \dfrac{1}{\alpha} - \left[\dfrac{1}{\alpha}\right]\right) & \text{if} \quad (\alpha,\beta) \in X_0 - \{(0,0)\} \\ \left(\dfrac{1}{\beta} - \left[\dfrac{1}{\beta}\right], \dfrac{\alpha}{\beta}\right) & \text{if} \quad (\alpha,\beta) \in X_1 \\ (0,0) & \text{if} \quad (\alpha,\beta) = (0,0) \end{cases}$$

where

$$\begin{aligned} X_0 &= \{(\alpha,\beta) \mid \alpha \geq \beta\}, \\ X_1 &= \{(\alpha,\beta) \mid \alpha < \beta\}. \end{aligned}$$

By using the integer value functions

$$\begin{aligned} a(\alpha,\beta) &:= \begin{cases} \left[\dfrac{1}{\alpha}\right] & \text{if} \quad (\alpha,\beta) \in X_0 \\ \left[\dfrac{1}{\beta}\right] & \text{if} \quad (\alpha,\beta) \in X_1 \end{cases}, \\ \varepsilon(\alpha,\beta) &:= \begin{cases} 0 & \text{if} \quad (\alpha,\beta) \in X_0 \\ 1 & \text{if} \quad (\alpha,\beta) \in X_1 \end{cases} \end{aligned}$$

on $X - \{(0,0)\}$, we define for each $(\alpha,\beta) \in X - \{(0,0)\}$ a sequence of digits ${}^t(a_n, \varepsilon_n)$ by

$${}^t(a_n,\varepsilon_n) := {}^t\left(a\left(T^{n-1}(\alpha,\beta)\right), \varepsilon\left(T^{n-1}(\alpha,\beta)\right)\right) \quad \text{if } T^{n-1}(\alpha,\beta) \neq (0,0).$$

The triple $(X, T, (a(\alpha,\beta), \varepsilon(\alpha,\beta)))$ is called Modified Jacobi-Perron algorithm. And we denote

$$(\alpha_n, \beta_n) := T^n(\alpha,\beta).$$

For the modified Jacobi-Perron algorithm, we introduce a transformation $(\overline{X}, \overline{T})$ called a natural extension of the modified Jacobi-Perron algorithm. Let $\overline{X} = X \times X$ and let us define the transformation $\overline{T}$ on $\overline{X}$ by

$$\overline{T}(\alpha,\beta,\gamma,\delta) = \begin{cases} \left(\dfrac{\beta}{\alpha}, \dfrac{1}{\alpha} - a_1, \dfrac{\delta}{a_1+\gamma}, \dfrac{1}{a_1+\gamma}\right) & \text{if} \quad (\alpha,\beta) \in X_0 - \{(0,0)\} \\ \left(\dfrac{1}{\beta} - a_1, \dfrac{\alpha}{\beta}, \dfrac{1}{a_1+\delta}, \dfrac{\gamma}{a_1+\delta}\right) & \text{if} \quad (\alpha,\beta) \in X_1 \\ (0,0,\gamma,\delta) & \text{if} \quad (\alpha,\beta) = (0,0) \end{cases}$$

Then we know that the transformation $\overline{T}$ is bijective from $(X - \{(0,0)\}) \times X$ to $X \times (X - \{(0,0)\})$.

We denote

$$(\alpha_n, \beta_n, \gamma_n, \delta_n) := \overline{T}^n(\alpha, \beta, \gamma, \delta).$$

Let us introduce the family of matrices as follows:

$$A_{\begin{pmatrix} a \\ 0 \end{pmatrix}} = \begin{bmatrix} a & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_{\begin{pmatrix} a \\ 1 \end{pmatrix}} = \begin{bmatrix} a & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

for each integral ${}^t(a, \varepsilon)$, $a \in \boldsymbol{N}$, $\varepsilon = \{0, 1\}$. Then we have the following formulas:

$$\begin{aligned}
\begin{pmatrix} 1 \\ \alpha_n \\ \beta_n \end{pmatrix} &= \frac{1}{\theta\theta_1 \cdots \theta_{n-1}} A^{-1}_{\begin{pmatrix} a_n \\ \varepsilon_n \end{pmatrix}} A^{-1}_{\begin{pmatrix} a_{n-1} \\ \varepsilon_{n-1} \end{pmatrix}} \cdots A^{-1}_{\begin{pmatrix} a_1 \\ \varepsilon_1 \end{pmatrix}} \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix}, \\
\begin{pmatrix} 1 \\ \gamma_n \\ \delta_n \end{pmatrix} &= \frac{1}{\eta\eta_1 \cdots \eta_{n-1}} {}^tA_{\begin{pmatrix} a_n \\ \varepsilon_n \end{pmatrix}} {}^tA_{\begin{pmatrix} a_{n-1} \\ \varepsilon_{n-1} \end{pmatrix}} \cdots {}^tA_{\begin{pmatrix} a_1 \\ \varepsilon_1 \end{pmatrix}} \begin{pmatrix} 1 \\ \gamma \\ \delta \end{pmatrix}
\end{aligned}$$

where

$$\begin{aligned}
\theta_k &= \max(\alpha_k, \beta_k), \\
\eta_k &= \begin{cases} a_k + \gamma_{k-1} & \text{if} \quad (\alpha_{k-1}, \beta_{k-1}) \in X_0 \\ a_k + \delta_{k-1} & \text{if} \quad (\alpha_{k-1}, \beta_{k-1}) \in X_1 \end{cases}
\end{aligned}$$

Let us introduce a transformation $\varphi_{\begin{pmatrix} a_n \\ \varepsilon_n \end{pmatrix}} : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ by

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = \varphi^{-1}_{\begin{pmatrix} a_n \\ \varepsilon_n \end{pmatrix}} \begin{pmatrix} x_{n-1} \\ y_{n-1} \\ z_{n-1} \end{pmatrix} := A^{-1}_{\begin{pmatrix} a_n \\ \varepsilon_n \end{pmatrix}} \begin{pmatrix} x_{n-1} \\ y_{n-1} \\ z_{n-1} \end{pmatrix}.$$

Then we see

$$\varphi^{-1}_{\begin{pmatrix} a_n \\ \varepsilon_n \end{pmatrix}} \mathcal{P}_{\gamma_{n-1}, \delta_{n-1}} = \mathcal{P}_{\gamma_n, \delta_n}.$$

Moreover the substitution $\Theta_{\begin{pmatrix} a \\ \varepsilon \end{pmatrix}}$ satisfies the following property:

$$\Theta_{\begin{pmatrix} a_n \\ \varepsilon_n \end{pmatrix}} \cdots \Theta_{\begin{pmatrix} a_2 \\ \varepsilon_2 \end{pmatrix}} \Theta_{\begin{pmatrix} a_1 \\ \varepsilon_1 \end{pmatrix}} (\mathcal{U}) \in \Gamma^+_{\gamma_n, \delta_n}.$$

Now let us consider the renormalization of $\pi^n \Theta_{\left(\begin{smallmatrix} a_n \\ \varepsilon_n \end{smallmatrix}\right)} \cdots \Theta_{\left(\begin{smallmatrix} a_2 \\ \varepsilon_2 \end{smallmatrix}\right)} \Theta_{\left(\begin{smallmatrix} a_1 \\ \varepsilon_1 \end{smallmatrix}\right)} (\mathcal{U})$, that is,

$$\lim_{n\to\infty} L^{-1}_{\left(\begin{smallmatrix} a_1 \\ \varepsilon_1 \end{smallmatrix}\right)} \cdots L^{-1}_{\left(\begin{smallmatrix} a_n \\ \varepsilon_n \end{smallmatrix}\right)} \pi_n \Theta_{\left(\begin{smallmatrix} a_n \\ \varepsilon_n \end{smallmatrix}\right)} \cdots \Theta_{\left(\begin{smallmatrix} a_1 \\ \varepsilon_1 \end{smallmatrix}\right)} (\mathcal{U})$$

where $\pi_n : \boldsymbol{R}^3 \to \mathcal{P}_{\gamma_n,\delta_n}$ be the projection along $\begin{bmatrix} 1 \\ \alpha_n \\ \beta_n \end{bmatrix}$.

Theorem 6.1 **([I-O 93])** *For almost everywhere $\gamma, \delta \in [0,1) \times [0,1)$ there exist the limit sets*

$$\begin{aligned} X^{(i)}_{\alpha,\beta,\gamma,\delta} &:= \lim_{n\to\infty} L^{-1}_{\left(\begin{smallmatrix} a_1 \\ \varepsilon_1 \end{smallmatrix}\right)} \cdots L^{-1}_{\left(\begin{smallmatrix} a_n \\ \varepsilon_n \end{smallmatrix}\right)} \pi_n \Theta_{\left(\begin{smallmatrix} a_n \\ \varepsilon_n \end{smallmatrix}\right)} \cdots \Theta_{\left(\begin{smallmatrix} a_1 \\ \varepsilon_1 \end{smallmatrix}\right)} (\mathbf{e}_i, i^*), \\ X^{(i)\prime}_{\alpha,\beta,\gamma,\delta} &:= \lim_{n\to\infty} L^{-1}_{\left(\begin{smallmatrix} a_1 \\ \varepsilon_1 \end{smallmatrix}\right)} \cdots L^{-1}_{\left(\begin{smallmatrix} a_n \\ \varepsilon_n \end{smallmatrix}\right)} \pi_n \Theta_{\left(\begin{smallmatrix} a_n \\ \varepsilon_n \end{smallmatrix}\right)} \cdots \Theta_{\left(\begin{smallmatrix} a_1 \\ \varepsilon_1 \end{smallmatrix}\right)} (\mathbf{o}, i^*), \end{aligned}$$

and satisfy the following property:

(1) $X_{\alpha,\beta,\gamma,\delta} = \bigcup_{i=1,2,3} X^{(i)}_{\alpha,\beta,\gamma,\delta}$ is the periodic tiling on $\mathcal{P}_{\gamma,\delta}$, that is,

$$\bigcup_{\boldsymbol{z}\in \boldsymbol{L}_0} (X_{\alpha,\beta,\gamma,\delta} + \boldsymbol{z}) = \mathcal{P}_{\gamma,\delta}$$

and

$$int.(X_{\alpha,\beta,\gamma,\delta} + \boldsymbol{z}) \bigcap int.(X_{\alpha,\beta,\gamma,\delta} + \boldsymbol{z}') = \phi \quad (\boldsymbol{z} \neq \boldsymbol{z}')$$

(2) the domain exchange transformation $W_{\alpha,\beta,\gamma,\delta} : X_{\alpha,\beta,\gamma,\delta} \to X_{\alpha,\beta,\gamma,\delta}$ such that

$$W_{\alpha,\beta,\gamma,\delta}(\boldsymbol{x}) = \boldsymbol{x} - \pi_0 \boldsymbol{e}_i \quad \textit{if} \quad \boldsymbol{x} \in X^{(i)}_{\alpha,\beta,\gamma,\delta}$$

is well-defined. Moreover, put

$$X_n := \bigcup_{i=1,2,3} X^{(i)}_{\alpha_n,\beta_n,\gamma_n,\delta_n} \subset \mathcal{P}_{\gamma_n,\delta_n}$$

and

$$\begin{aligned} \widehat{X}_n &:= L^{-1}_{\left(\begin{smallmatrix} a_1 \\ \varepsilon_1 \end{smallmatrix}\right)} \cdots L^{-1}_{\left(\begin{smallmatrix} a_n \\ \varepsilon_n \end{smallmatrix}\right)} (X_n) \subset X_{\alpha,\beta,\gamma,\delta}, \\ \widehat{X}_{n,i} &:= L^{-1}_{\left(\begin{smallmatrix} a_1 \\ \varepsilon_1 \end{smallmatrix}\right)} \cdots L^{-1}_{\left(\begin{smallmatrix} a_n \\ \varepsilon_n \end{smallmatrix}\right)} (X^{(i)}_{\alpha_n,\beta_n,\gamma_n,\delta_n}) \subset X_{\alpha,\beta,\gamma,\delta}, \end{aligned}$$

then

$$W^{k-1}_{\alpha,\beta,\gamma,\delta}\left(X^{(i)}_{\alpha_n,\beta_n,\gamma_n,\delta_n}\right) \subset X^{s_k^{(n,i)}}_{\alpha,\beta,\gamma,\delta},\quad 1 \le k \le q_n + p_n + r_n,$$

$$W^{q_n+p_n+r_n}_{\alpha,\beta,\gamma,\delta}\left(X^{(i)}_{\alpha_n,\beta_n,\gamma_n,\delta_n}\right) = X^{(i)'}_{\alpha,\beta,\gamma,\delta}$$

where $\sigma_{\left(\begin{smallmatrix} a_1 \\ \varepsilon_1 \end{smallmatrix}\right)} \cdots \sigma_{\left(\begin{smallmatrix} a_n \\ \varepsilon_n \end{smallmatrix}\right)}(i) = s_1^{(n,i)} \cdots s_{l(n,i)}^{(n,i)}$ *and*

$$\begin{bmatrix} q_n & * & * \\ p_n & * & * \\ r_n & * & * \end{bmatrix} := A_{\left(\begin{smallmatrix} a_1 \\ \varepsilon_1 \end{smallmatrix}\right)} A_{\left(\begin{smallmatrix} a_2 \\ \varepsilon_2 \end{smallmatrix}\right)} \cdots A_{\left(\begin{smallmatrix} a_n \\ \varepsilon_n \end{smallmatrix}\right)}$$

Note. In the 1-dimensional case, we consider the continued fraction algorithm

$$T\alpha = \frac{1}{\alpha} - \left[\frac{1}{\alpha}\right] \quad \text{on } [0,1]$$

and the continued fraction expansion:

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \cfrac{1}{\ddots + \cfrac{1}{a_n + T^n\alpha_n}}}}}$$

where $a_k := \left[\frac{1}{T^{k-1}\alpha}\right]$.

Instead of the plane $\mathcal{P}_{\gamma,\delta}$, we introduce the line l_γ,

$$l_\gamma := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \,\middle|\, \left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} 1 \\ \gamma \end{bmatrix} \right\rangle = 0 \right\}$$

and whose stepped surfaces $\mathcal{S}_\gamma$. We also introduce the natural extension of $\overline{T}$ by

$$\overline{T}(\alpha,\gamma) := \left(\frac{1}{\alpha} - a_1, \frac{1}{a_1+\gamma}\right).$$

Let us introduce a map $\varphi_a : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ by

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \varphi_a^{-1}\begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} := A_a^{-1}\begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}$$

where

$$A_a^{-1} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -a_1 \end{bmatrix}.$$

Then the map φ_a satisfies

$$\varphi_{a_n}^{-1} l_{\gamma_{n-1}} = l_{\gamma_n}$$

where

$$(\alpha_n, \gamma_n) := T^n(\alpha, \gamma).$$

Then we know

$$\Theta_{a_n}\left(\mathcal{S}_{\gamma_{n-1}}\right) = \mathcal{S}_{\gamma_n}$$

where Θ_a is given in Note of the section 3.

7 Applications

7.1 Quasi-periodic tiling related to the stepped surface

Let $\mathcal{P}_{\gamma,\delta}$, $0 < \gamma, \delta < 1$ be the plane given by

$$\mathcal{P}_{\gamma,\delta} = \left\{ \boldsymbol{x} \;\middle|\; \left\langle \boldsymbol{x}, \begin{bmatrix} 1 \\ \gamma \\ \delta \end{bmatrix} \right\rangle = 0 \right\},$$

and let $\mathcal{S}_{\gamma,\delta}$ be the stepped surface with respects to $\mathcal{P}_{\gamma,\delta}$.

Let $\pi : \boldsymbol{R}^3 \longrightarrow \mathcal{P}_{\gamma,\delta}$ along $\begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix}$, then we have a tiling $\pi\mathcal{S}_{\alpha,\beta}$ generated by three parallelograms $\pi(\mathbf{o}, i^*)$, $i = 1, 2, 3$ and whose translations.

Let us denote the above tiling by $\mathcal{T}_{\alpha,\beta}(= \pi\mathcal{S}_{\alpha,\beta})$. Let Γ_n be the family of patches which is generated by n parallelograms and simply conneted, that is,

$$\Gamma_n = \{\gamma \mid \gamma \prec \mathcal{T}_{\alpha,\beta},\ \#\gamma = n,\ \gamma \text{ is simply connected } \}.$$

Definition 7.1 *A tiling $\mathcal{T}$ of a plane is said to be* quasi-periodic *if for any $n > 0$ there exists $R > 0$ such that any configuration $\gamma \in \Gamma_n$ occurs somewhere in a neighbourhood of any point of the radius R.*

Theorem 7.2 *Let $(1, \gamma, \delta)$ be the linearly independent with respect to $\boldsymbol{Q}$, then the tiling $\mathcal{T}_{\gamma,\delta}$ is a quasi-periodic tiling.*

The essential idea is coming from the following fact: for each $(1, \gamma, \delta)$ there exists a sequence

$$\begin{pmatrix} a_1 & a_2 & \cdots & \cdots \\ \varepsilon_1 & \varepsilon_2 & \cdots & \cdots \end{pmatrix}$$

such that the stepped surface of $\mathcal{P}_{\gamma,\delta}$ is given by

$$\lim_{n\to\infty} \Theta_{\begin{pmatrix} a_1 \\ \varepsilon_1 \end{pmatrix}} \Theta_{\begin{pmatrix} a_2 \\ \varepsilon_2 \end{pmatrix}} \cdots \Theta_{\begin{pmatrix} a_n \\ \varepsilon_n \end{pmatrix}} (\mathcal{U}).$$

where the sequence is obtained by the modified Jacobi-Perron algorithm, and $\Theta_{\begin{pmatrix} a \\ \varepsilon \end{pmatrix}}$ is appeared in Example 2 (See [I-O 94] in detail).

7.2 Markov partition of group automorphisms on T^3

Let us consider the following special matrix $A = L_\sigma$ which is given by Pisot substitution, that is, which satisfies the assumption of Theorem 4.1. On the assumption, let us define the sets $\overline{X}_i, i = 1,2,3$ of $\boldsymbol{R}^3$ by

$$\overline{X}_i = \{(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x} \in X_i,\ \boldsymbol{y} = \lambda(e_i - \pi e_i),\ 0 \leq \lambda < 1\},$$

then the domain $\triangle$:

$$\triangle := \bigcup_{i=1,2,3} \overline{X}_i$$

is the 3-dimensional torus, that is,

(1) $\bigcup_{\boldsymbol{z} \in \boldsymbol{Z}^3} (\triangle + \boldsymbol{z}) = \boldsymbol{R}^3$

(2) $int.(\triangle + \boldsymbol{z}) \bigcap int.(\triangle + \boldsymbol{z}') = \phi$ if $\boldsymbol{z} \neq \boldsymbol{z}'$.

Theorem 7.3 *Let ξ be the partition of the 3-dimensional torus $\triangle(\simeq \boldsymbol{T}^3)$, that is,*

$$\xi = \left\{\overline{X}_i, i = 1,2,3\right\},$$

then the partition ξ be the Markov partition with structure matrix ${}^tL_\sigma$.

Note. The existence of Markov partitions of group automorphisms on $\boldsymbol{T}^n$ are discussed in [A-W] and [S]. Bowen claims that the boundary of Markov partition of 3-dimensional group automorphisms must not be smooth in [Bo]. Theorem 7.3 says that how we can construct the (not smooth) Markov partition (analogous discussion can be found in [Be]).

The following question is reasonable. For any element of $A \in SL(3, \boldsymbol{Z})$, does there exist the substituion σ and L_σ satisfying the assumption in Theorem 4.1? and is L_σ isomorphic to A? We only know with the private discussion between I and FURUKADO that for any A there exists $N > 0$ such that A^N which satisfies the assumption in Theorem 4.1.

7.3 Diophantine approximation

Let $\langle 1, \alpha, \beta \rangle$ be the integer basis of the cubic field $\boldsymbol{Q}(\lambda)$ given by

$$A \begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix}$$

for some $A \in SL(3, \boldsymbol{Z})$, and let us assume that λ be a complex Pisot number.

On the above setting, let us consider the limit set of the points

$$\left\{ \sqrt{q} \begin{pmatrix} q\alpha - p \\ q\beta - r \end{pmatrix} \;\middle|\; (q, p, r) \in \boldsymbol{Z}^3, q > 0 \right\}.$$

Theorem 7.4 *The limit set of above points consists of the family of ellipses.*

Therem 7.4 is found by the method of algebraic geometry in [A]. But by using the substitution, we can give another proof (See [F] in detail).

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