CYCLE CLASSES ON THE MODULI OF K3 SURFACES

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ABSTRACT. This is a report on work in progress with Torsten Ekedahl on the cohomology classes of the cycles of the analogue of the Ekedahl-Oort stratification on the moduli of K3 surfaces in positive characteristic.

1. INTRODUCTION

Elliptic curves over an algebraically closed field k of characteristic $p > 0$ come in two sorts: ordinary and supersingular. One way to express the distinction for an elliptic curve E comes from looking at the formal group of E, i.e., the group law on a formal neighborhood of the origin $0 \in E(k)$: multiplication by p is given in terms of a local parameter t by

$$[p] \cdot t = a\, t^{p^h} + \text{higher order terms},$$

where $a \neq 0$ and h assumes the value 1 or 2. If $h = 1$ (resp. $h = 2$) we say that E is ordinary (resp. E is supersingular). There are only finitely many values of j for which the elliptic curve with j-invariant j is supersingular. Deuring's Mass Formula gives their number as

$$\sum_E \frac{1}{\#\mathrm{Aut}_k(E)} = \frac{p-1}{24},$$

where the summation is over all supersingular elliptic curves defined over k up to isomorphism and every elliptic curve is counted with a weight.

There is a generalization of this for principally polarized abelian varieties of given dimension g. The stratification defined by the distinction ordinary vs. supersingular is generalized into the Ekedahl-Oort stratification (cf. [7]) and the cycle classes of this E-O stratification were calculated in [5, 3]. The formulas for these classes can be seen as a generalization of Deuring's formula.

One would like to extend this to the moduli of K3 surfaces. By a K3 surface we mean a smooth projective surface X/k such that its canonical bundle K_X is trivial ($K_X \cong O_X$) and such that $H^1(X, O_X) = (0)$. For example, smooth quartic surfaces in $\mathbb{P}^3$ are examples of K3 surfaces. Both abelian varieties and K3 surfaces form a generalization of the notion of an elliptic curve.

One way to generalize the distinction ordinary vs. supersingular to K3 surfaces one might look at the formal Brauer group, i.e., one looks at the

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functor Φ

$$S \mapsto \ker H^2_{et}(X \times S, \mathbb{G}_m) \to H^2_{et}(X, \mathbb{G}_m)$$

on Artin rings of residue field k. As Artin and Mazur showed in [2] it defines a formal group, the formal Brauer group $\Phi = \Phi^2_X$. Its tangent space can be identified with $H^2(X, O_X)$ and for a K3 surface this has dimension 1. We thus get a 1-dimensional formal group. One knows that 1-dimensional formal groups are classified by their height: multiplication by p takes the form

$$[p] \cdot t = a\, t^{p^h} + \text{higher order terms,}$$

where $h \in \mathbb{Z}_{\geq 1}$ or $h = \infty$ in which case $\Phi \cong \hat{\mathbb{G}}_a$, the additive formal group. For $h < \infty$ we get a p-divisible formal group. A basic theorem of Artin-Mazur gives restrictions for this invariant.

Theorem 1.1. (Artin-Mazur) *If X is a K3 surface whose formal Brauer group has finite height $h < \infty$ then the rank ρ of the Néron-Severi group of X satisfies the inequality*

$$\rho \leq b_2 - 2h,$$

with $b_2 = 22$ the second Betti number.

In particular, if $\rho = 22$ then $h = \infty$; otherwise, $\rho \leq 20$ and $1 \leq h \leq 10$. The case $h = 1$ is the 'generic case' and $h = \infty$ is called the supersingular case. One should remark here that there are two notions of supersingularity for K3 surfaces: Artin's notion, meaning that $h = \infty$, and Shioda/Shafarevich's one, meaning that $\rho = 22$. It follows that supersingularity in the sense of Shioda and Shafarevich implies supersingularity in the sense of Artin and it is in fact conjectured that both notions coincide.

Let now M be a moduli space of polarized K3 surfaces over k of degree $2d$ with $2d$ prime to $p = \text{char}(k)$. Then one has a stratification on M with closed strata V_h loosely defined by

$$V_h = \{[X] \in M : h(X) \geq h\}.$$

In joint work with T. Katsura (see [6]) we determined the cycle classes of these loci. The formula is as follows:

Theorem 1.2. *The cycle class of the locus V_h of K3 surfaces of height $\geq h$ is given by*

$$[V_h] = (p-1)(p^2-1)\cdots(p^{h-1}-1)\,\lambda^h, \tag{1}$$

where $\lambda = c_1(\pi_*(\Omega^2_{\mathcal{X}/M})$ is the first Chern class of the Hodge bundle for the universal family $\pi : \mathcal{X} \to M$.

This can be seen as a generalization of Deuring's formula because we may write the latter as

$$[V_{\text{ss}}] = (p-1)\lambda,$$

with V_{ss} the supersingular locus on the j-line (the moduli space of elliptic curves) and λ the first Chern class of the Hodge bundle which is also the (stacky) line bundle whose sections are modular forms of weight 1. By

calculating the degree and interpreting everything correctly (stacks) we get Deurings's formula.

The dimensions of the strata occuring here range from 19 to 9. But the supersingular locus, which has dimension 9, allows a further stratification by the Artin invariant σ_0 with $1 \leq \sigma_0 \leq 10$. One way to define it is by looking at the discriminant of the intersection form on the Néron-Severi group. Alternatively, we can define it using crystalline cohomology or as

$$\sigma_0 = \dim \ker\{c_1 \otimes k : \mathrm{NS}(X) \otimes k \to H^1(X, Z_1)\},$$

where $Z_1 = \Omega^1_{X,\mathrm{closed}}$ is the sheaf of d-closed 1-forms.

But there is a unified way of defining a stratification on the moduli space that includes both the height strata and the Artin invariant strata. Our approach to the stratification in [4] follows the idea of [5, 3] to consider filtrations on the de Rham cohomology of a K3 surface.

2. Orthogonal Forms

In order to define the stratification on our moduli space M we need to discuss some notions related to an orthogonal vector space.

Let V be a finite-dimensional vector space, say of dimension n. We have to distinguish between n odd and n even.

For $n = 2m + 1$ odd we consider the Weyl group W_{B_m} of $SO(2m+1)$

$$\{\sigma \in S_{2m+1} : \sigma(i) + \sigma(2m + 2 - i) = 2m + 2\}$$

viewed as permutations of the set $\{1, 2, \ldots, 2m+1\}$. It contains $2m$ so-called final elements: these are elements that are reduced w.r.t. the root system obtained by deleting the first root. Explicitly, these are the elements

$$w_a = [2m + 2 - a, 1, 2, \ldots, 2m + 1, a],$$

where a permutation is given by its images of the elements $1, 2, \ldots, 2m + 1$. In particular there is a longest final element, also denoted by $w_\emptyset$,

$$w_\emptyset = [2m + 1, 2, 3, \ldots, 2m - 1, 2m, 1]$$

There is an ordering on the elements of W_{B_m}, the Bruhat ordering, We call the elements w with $w \leq w_\emptyset$ the admissible elements.

In case $n = 2m$ is even we consider the Weyl group W_{D_m} of $\mathrm{SO}(2m)$. We let

$$W'_{D_m} = W_{C_m} = \{\sigma \in S_{2m} : \sigma(i) + \sigma(2m + 1 - i) = 2m + 1\}$$

be the Weyl group of $\mathrm{O}(2m)$ and consider the subgroup

$$W_{D_m} = \{\sigma \in W_{C_m} : \#\{i \leq m : \sigma(i) > m + 1\} even\}$$

This is an index 2 subgroup of W'_{D_m}. Let $\mathrm{disc} : W_{C_m} \to \{\pm\}$ be the homomorphism with kernel W_{D_m}. Also here we have $2m$ final elements

$$w_a = [2m + 1 - a, 1, 2, \ldots, 2m + 1, a]$$

and we also have $2m$ twisted final elements $w'_a = w_a \cdot s'_m$ with s'_a the transposition $(m \, m + 1)$.

We now look at flags $(0) = V_0 \subset V_1 \subset \cdots \subset V_r$ on V which are isotropic, i.e., the quadratic form vanishes on V_r. We can complete such a flag by putting $V_{n-j} := V_j^\perp$ and we require that all dimensions s with $0 \le s \le n$ occur except possibly $n/2$. This hints at some subtleties for the even case. If $n = 2m$ then two flags are in the same or opposite family if

$$\dim(V_m^{(1)} \cap V_m^{(2)}) \equiv \begin{cases} m \, (\mathrm{mod}\, 2) \\ m - 1 \, (\mathrm{mod}\, 2) \end{cases}$$

For K3 surfaces in positive characteristic the choice of a complete isotropic flag on the cohomology will automatically gives us a second flag. The relative position of these two flags corresponds to an element in a Weyl group as the following proposition shows. The case n even displays some subtleties.

Proposition 2.1. *i) Let $n = 2m + 1$. Then the $SO(2m + 1)$-orbits of pairs of totally isotropic complete flags are in $1 - 1$ correspondence with elements of W_{B_m}. ii) $n = 2m$. Then the $SO(2m)$-orbits of pairs of totally isotropic comple te flags are in $1 - 1$ correspondence with elements of W'_{D_m}. If $(E., D.)$ corresponds to w then $\mathrm{disc}(w) = (-1)^d$ with $d = \dim(E_m \cap D_m)$.*

3. Filtrations on the cohomology of a K3 surface

Let X be a K3 surface and let N be a non-degenerate integral lattice. Suppose that $N \to \mathrm{NS}(X)$ is an isometric embedding. We say that this gives an N-polarization if N contains an ample line bundle. Its degree is by definition the absolute value of the discriminant of N.

We shall assume that this degree is prime to p.

By the primitive cohomology we mean the orthogonal complement of N on the de Rham cohomology $H = H^2_{\mathrm{dR}}(X)$. This has an extra structure, namely a Hodge filtration

$$(0) = U_{-1} \subset U_0 \subset U_1 \subset U_2 = H$$

with the property that it is self-dual: $U_0^\perp = U_1$. We then also have a conjugate filtration

$$(0) = U_{-1}^c \subset U_0^c \subset U_1^c \subset U_2^c = H$$

again self-dual. The dimensions of these steps are 0, 1, $n - 1$ and n. The inverse Cartier operator C^{-1} gives an isomorphism

$$C^{-1} : F^*(U_i/U_{i-1}) \cong U_{2-i}^c/U_{1-i}^c.$$

with $F : X \to X^{(p)}$ the relatice Frobenius. We now consider flags on the de Rham cohomology refining the conjugate filtration and use the inverse Cartier operator to transfer it to the Frobenius pull back of the Hodge filtration. This gives us two filtrations $E.$ and $D.$.

We call such a filtration stable if

$$D_j \cap E_{n-1} + E_1$$

is an element of the $E.$-filtration. We call a filtration canonical if every stable filtration is a refinement of it. We call it final if it is stable and complete.

Proposition 3.1. *A filtration $E.$ extending the Hodge filtration is final if under the correspondence of Prop. 2.1 it belongs to a final element or a twisted final element.*

The basic result is now that we can find final filtrations and that their type is related to the height and Artin invariant in the following way.

Theorem 3.2. *Let X be a polarized K3 surface. Let H be its primitive de Rham cohomology of dimension n and put $m = \lfloor n/2 \rfloor$. Then we have*

(1) *H has a canonical filtration. If k is separably closed then it has a final filtration. All final filtrations have the same type.*
(2) *If X has finite height h with $2h < n$ then H has final type w_h or w'_h.*
(3) *If X has finite height h with $2h = n$ then H has final type w'_m.*
(4) *If X is supersingular with Artin invariant $\sigma_0 < n/2$ then it has final type $w_{n-1-\sigma_0}$ or $w'_{n-1-\sigma_0}$.*
(5) *If X is supersingular with Artin invariant $\sigma_0 = n/2$ then it has final type w_{m-1}.*

4. Cycle Class Formulas in the Odd Orthogonal Case

We consider now the flag space $\mathcal{F} \to M_N$ over our moduli space of K3 surfaces. We restrict here to the odd orthogonal case referring for the other, more subtle case to [4]. The fibre parametrizes complete isotropic flags on the primitive de Rham cohomology of a K3 surface with polarization of type N. Using the Cartier operator one complete isotropic flag $E.$ on the primitive cohomology yields a second flag $D.$ and their relative position corresponds to an element w in our Weyl group, cf. 2.1. In $\mathcal{F}$ we can define loci $\mathcal{U}_w$ where the relative position of $E.$ and $D.$ corresponds to w

$$\mathcal{U}_w = \{(E.) \in \mathcal{F} : \dim(E_i \cap D_j) = \#\{1 \le \alpha \le i : w(\alpha) \le j\}\}.$$

These loci come with a scheme structure. We define the analogue of the E-O stratification on ithe moduli space M itself by setting for final w

$$\mathcal{V}_w = \phi(\mathcal{U}_w)$$

with its reduced structure.

Recall that an element $w \in W_{B_m}$ has a length $\ell(w)$ given by

$$\ell(w) = \#\{i < j \le n : w(i) > w(j)\} + \#\{i \le j : w(i) + w(j) > 2m + 2\}.$$

The strata on $\mathcal{F}$ are quite well-behave as the following proposition shows.

Proposition 4.1. (1) *The open stratum $\mathcal{U}_w$ is smooth of dimension $\ell(w)$.*
(2) *$\overline{\mathcal{U}}_w$ is reduced, Cohen-McCaulay and normal of dimension $\ell(w)$.*
(3) *The projection $\mathcal{U}_w \to \mathcal{V}_w$ is finite surjective and étale for final w.*

The proof of the proposition is analogous to the proof in [3] which employs the fact that our strata look up to order p neighborhoods as the Schubert cycles in flag spaces. We let $Y = \pi_*(\Omega^2_{\mathcal{X}/M})$ be the Hodge bundle and

we let $\lambda = c_1(Y)$ be its first Chern class (in the Chow group with rational coefficients). In view of the formulas for the case of abelian varieties one may expect that the cycle classes of these strata are polynomials in tautological classes with integral coefficients. Since the tautological ring is generated by λ we expect as cycle classes multiples of a power of λ. The problem is to find the coefficient.

Theorem 4.2. *Let $\mathcal{V}_{w_k}$ be the stratum on the moduli space M corresponding to the final element w_k for $1 \leq k \leq 2m$. For $k \leq m$ we have the following formulas for their cycle classes.*

(1)
$$[\overline{\mathcal{V}}_{w_k}] = (p-1)(p^2-1)\cdots(p^{k-1}-1)\,\lambda^{k-1}$$

(2)
$$[\overline{\mathcal{V}}_{w_{m+k}}] = \frac{1}{2}\frac{(p^{2k}-1)(p^{2k+2}-1)\cdots(p^{2m}-1)}{(p+1)(p^2+1)\cdots(p^{m-k+1}-1)}\,\lambda^{m+k-1}.$$

Remark 4.3.   (1) It is not clear from the formula in (2) that the expression has coefficients in $(1/2)\mathbb{Z}[p]$. But the formula in case 2) can also be written as

$$[\overline{V}_{w_{m+k}}] = \frac{1}{2}\Big(\prod_{j=1}^{m+k-1}(p^j-1)\Big)\begin{bmatrix}m\\m+1-k\end{bmatrix}_{p^2}\lambda^{m+k-1},$$

where $[n,i]_q$ is the usual q-binomial coefficient. This shows that the expression is up to the factor $1/2$ a polynomial in $\mathbb{Z}[p]$ times a power of λ.

(2) The formula in 1) coincides with the formula given in the joint work with Katsura, except for the factor $1/2$ for the supersingular locus corresponding to w_{m+1}. As explained in [6] the supersingular locus came there with a multiplicity 2, whereas it is reduced in the present approach.

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