An Iwasawa Conjecture for a Hyperbolic Threefold of a Finite Volume

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1 The motivation and a brief overview of our results

The Riemann's zeta function and its cousins have two main natures:

1. The geometric nature (the distribution of zeros);
   - The Riemann hypothesis.
2. The arithmetic nature (special values);
   - The class number formula (for Riemann or Dedekind zeta functions).
   - The Birch and Swinnerton-Dyer conjecture (for an L-function of an elliptic curve).

In the number theory, it has been considered two models of the zeta function.

1. The Hasse-Weil's congruent zeta function for a smooth projective variety over a finite field.
   - The geometric nature $\Rightarrow$ The Weil conjecture.
   - The arithmetic nature $\Rightarrow$ The Artin-Tate conjecture.

   A Point of the theory

   The Euler product

   $\Updownarrow$ The Grothendieck-Lefschetz trace formula

   An alternating product of the characteristic polynomials of the Frobenius on étale cohomologies
2. The p-adic zeta function.

- The geometric nature $\Rightarrow$?.
- The arithmetic nature $\Rightarrow$ The Iwasawa conjecture.

A Point of the theory

The p-adic zeta function

$\Updownarrow$ The Euler system

The Iwasawa polynomial

But the model over a finite field seems to be too special a little because

we do not have the Frobenius on a number field!.

So we want to propose an another model following Ruelle and Selberg. We will invesitigate the Ruelle L-function for a local system of rank one over a hyperbolic threefold of finite volume. Although there is no apparent "Frobenius", we will show it enjoys:

1. an analogue of the Riemann hypothesis.
2. an analogue of the Iwasawa conjecture.

A Point of our theory

The Ruelle L-function

$\Updownarrow$ The Selberg trace formula

The Alexander invariant

2 The Iwasawa power series of a cyclotomic $\mathbb{Z}_p$-extension

We will fix an odd prime p and will use the following notations.

Notations 2.1. *1.* $K_n = \mathbb{Q}(\zeta_{p^n})$*,* $\zeta_{p^n} = \exp(\frac{2\pi i}{p^n})$

2. $A_n = \mathcal{C}l(K_n)\{p\}$*: the p-primary part of the ideal class group of* K_n*,*

3. $\Gamma = \mathrm{Gal}(K_\infty/K_1)$*, where* $K_\infty = \lim_{n\to\infty} K_n$*.*

Here are some remarks.

1. The cyclotomic character χ_{cyc} yields an isomorphism:

$$\Gamma \overset{\chi_{cyc}}{\simeq} \mathbb{Z}_p.$$

2. By the action of $\mathrm{Gal}(K_1/\mathbb{Q}) \overset{\omega}{\simeq} \mathbb{F}_p^\times$, A_n is decomposed as

$$A_n = \oplus_{i=0}^{p-2} A_n^{\omega^i}$$

where

$$A_n^{\omega^i} = \{\alpha \in A_n \,|\, \gamma\alpha = \omega(\gamma)^i \alpha \,\mathrm{for}\, \gamma \in \mathrm{Gal}(K_1/\mathbb{Q})\}.$$

Definition 2.1. *For* $0 \le i \le p-2$*,* the Iwasawa module X_i *is defined to be*

$$X_i = \lim_{\leftarrow} A_n^{\omega^i}.$$

Here the limit is taken for the norm map.

We set

$$\Lambda_p = \mathbb{Z}_p[[\Gamma]],$$

which is isomorphic to a formal power series ring $\mathbb{Z}_p[[s]]$ in a non-canonical way. Then Iwasawa has shown:

X_i is a torsion Λ_p-module.

Let $\mathcal{L}_p^{alg,i} \in \Lambda_p$ be a generator of its characteristic ideal $\mathrm{Char}_{\Lambda_p} X_i$. It will be referred as *the Iwawasa power series.*

3 The Alexander invariant

Let X be a topological threefold which has a *sujective* homomorphism:

$$\pi_1(X) \xrightarrow{\pi} \mathbb{Z},$$

and

$$\pi_1(X) \xrightarrow{\rho} U(m)$$

a unitary representation. Here are some notations.

Notations 3.1. *1. X_∞ is the infinite cyclic covering of X which corresponds to* $\operatorname{Ker}\pi$.

2. $\Lambda = \mathbb{C}[\mathbb{Z}]$, which is isomorphic to $\mathbb{C}[t^{-1}, t]$ in a non-canonical way.

3. $\Lambda_\infty = \mathbb{C}[[s]]$, where $s = t - 1$.

In the following, we always assume:

Assumption:

$$\dim H_\cdot(X_\infty, \mathbb{C}), \quad \dim H_\cdot(X_\infty, \rho) < \infty.$$

Remark 3.1. *Under the assumption Milnor has shown:*

1.

$$H^i(X_\infty, \mathbb{C}) = H^i(X_\infty, \rho) = 0, \quad i \geq 3$$

and

$$H^2(X_\infty, \mathbb{C}) = \mathbb{C}.$$

2. (Milnor duality) For each $0 \leq i \leq 2$, the dimension of $H^i(X_\infty, \rho)$ is finite and there is a perfect pairing

$$H^i(X_\infty, \mathbb{C}) \times H^{2-i}(X_\infty, \mathbb{C}) \to H^2(X_\infty, \mathbb{C}) = \mathbb{C}$$

and

$$H^i(X_\infty, \rho) \times H^{2-i}(X_\infty, \rho) \to H^2(X_\infty, \mathbb{C}) = \mathbb{C}.$$

Thus the assumption implies

"$H^i(X_\infty,\, \rho)$ is a torsion Λ-module."

Let τ^* be the action of t on $H^i(X_\infty,\, \rho)$. We define *the twisted Alexander polynomial* $A_{\rho,i}$ to be the characteristic polynomial of τ^*:

$$A_{\rho,i}(t) = \det[t - \tau^* \mid H^i(X_\infty,\, \rho)].$$

Remark 3.2. *The characteristic ideal of $H^i(X_\infty,\, \rho)$ is generated by $A_{\rho,i}$:*

$$\mathrm{Char}_\Lambda(H^i(X_\infty,\, \rho)) = (A_{\rho,i}).$$

The Alexander invariant A_ρ is defined to be

$$A_\rho = \frac{A_{\rho,0} \cdot A_{\rho,2}}{A_{\rho,1}}.$$

Example 3.1. *(Milnor) Let $S^1 \xrightarrow{\kappa} S^3$ be a knot and X its complement. Then*

$$H_1(X,\, \mathbb{Z}) \simeq \mathbb{Z},$$

and we have an infnite cyclic covering

$$X_\infty \xrightarrow{\pi} X.$$

Moreover the dimension of $H.(X_\infty,\, \mathbb{C})$ are finite.

4 The Iwasawa main conjecture

The cyclotomic chracter χ_{cyc} induces a ring homorphism:

$$\Lambda_p = \mathbb{Z}_p[[\Gamma]] \stackrel{\chi_{cyc}}{\to} \mathbb{Z}_p,$$

by

$$\chi_{cyc}(\sum a_\gamma \gamma) = \sum a_\gamma \chi_{cyc}(\gamma).$$

For an integer $0 < i < p-1$, Kubota-Leopoldt, Iwasawa and Coleman have independently constructed an element $\mathcal{L}_p^{ana,i}$ (referred as *the p-adic ζ-function*) of Λ_p which satisfies

$$\chi_{cyc}^r(\mathcal{L}_p^{ana,i}) = (1-p^r)\zeta(-r)$$

for any

$$r \in \mathbb{N}, \quad r \equiv i \bmod p-1$$

Remark 4.1. *Special values of the Riemann zeta function at non-positive integers are given by*

$$\zeta(1-n) = -\frac{B_n}{n}, \quad n = 1, 2, 3, \cdots,$$

where B_n is the n-th Bernoulli number. In particular they are all rational numbers.

Now the Iwasawa main conjecture is

Theorem 4.1. *(Mazur-Wiles) For an odd i such that $0 < i < p-1$, we have*

$$(\mathcal{L}_p^{ana,i}) = (\mathcal{L}_p^{alg,i}),$$

as an ideal of Λ_p.

5 An Iwasawa conjecture for a compact hyperbolic threefold

Let X be a compact hyperbolic threefold which admits an infinite cyclic covering X_∞. Thus X is a quotient of $\mathbb{H}^3$ by a cocompact discrete subgroup Γ_g of $PSL_2(\mathbb{C})$.

Definition 5.1. *For a complex number z,* the Ruelle L-function *is defined to be*

$$R_\rho(z) = \prod_\gamma \det[1 - \rho(\gamma) e^{-zl(\gamma)}]^{-1}.$$

Here we have used the following conventions:

1. *Closed geodesics are identified with the hyperbolic conjugacy classes of Γ_g.*

2. *The index γ runs through prime closed geodesics.*

3. *$l(\gamma)$ is the length of γ.*

Remark 5.1. *The definition is still valid for a noncompact hyperbolic threefold of a finite volume.*

$R_\rho(z)$ is absolutely convergent for $\mathrm{Re}\, z >> 0$ and we can show that it is meromorphically continued on the whole plane.

In the following we assume:

Assumption

$$H^0(X_\infty, \rho) = 0.$$

Remark 5.2. *By the Milnor duality, this implies*

$$H^2(X_\infty, \rho) = 0.$$

We set

- $$\begin{aligned} \mathcal{L}_\rho(z) &= A_\rho(z+1)^{-1} \\ (&= \det[(z+1) - \tau^* \,|\, H^1(X_\infty, \rho)]\,). \end{aligned}$$

- $$h^1(\rho) = \dim H^1(X, \rho).$$

The next result is (a weak version of) a geometric analogue of the Iwasawa Main Conjecture.

Theorem 5.1. *We have*

$$\mathrm{ord}_{z=0} R_\rho(z) = 2h^1(\rho) \leq 2\mathrm{ord}_{z=0}\mathcal{L}_\rho(z),$$

and if the action of τ^* on $H^1(X_\infty, \rho)$ is semisimple, the identity holds. In particular if τ^* is semisimple,

$$(R_\rho(z)) = (\mathcal{L}_\rho(z)^2)$$

as an ideal of $\Lambda_\infty = \mathbb{C}[[z]]$.

Next we will compare their leading terms.

Theorem 5.2. *Supppose*

$$H^i(X, \rho) = 0$$

for each i. Then we have

$$|R_\rho(0)| = \delta_\rho |\mathcal{L}_\rho(0)|^2.$$

Here δ_ρ is a certain positive constant which can be computed explicitly.

If $H^1(X, \rho)$ does not vanish, we need an additional structure on X.

Suppose that X is homeomorphic to a mapping torus whose fiber is a compact Riemannian surface Σ:

$$X \xrightarrow{f} S^1, \quad f^{-1}(s) = \Sigma,$$

and that the surjection

$$\pi_1(X) \xrightarrow{\pi} \mathbb{Z}$$

is induced from f.

Remark 5.3. *In this case, X_∞ is a product of Σ and the real axis.*

Theorem 5.3. *(A limit formula) Suppose that $H^0(\Sigma, \rho)$ vanishes and that the action of τ^* on $H^1(X, \rho)$ is semisimple. Then*

$$\mathrm{ord}_{z=0} R_\rho(z) = 2\mathrm{ord}_{z=0} \mathcal{L}_\rho(z) = 2h^1(\rho),$$

and

$$\lim_{z \to 0} |z^{-h^1(\rho)} \mathcal{L}_\rho(z)|^2 = \lim_{z \to 0} |z^{-2h^1(\rho)} R_\rho(z)|.$$

Remark 5.4.

1. *Without semisimplicity of τ^*, we only have*

$$2h^1(\rho) = \mathrm{ord}_{z=0} R_\rho(z) \leq 2\mathrm{ord}_{z=0} \mathcal{L}_\rho(z).$$

2. *There is an example of a compact hyperbolic threefold which is a mapping torus. (due to W. Thurston.)*

6 An Iwasawa conjecture for a hyperbolic threefold of a finite volume

Let $X = \Gamma\backslash\mathbb{H}^3$ be a complete hyperbolic threefold of finite volume which has only one cusp. As before we assume that it admits an infinite cyclic covering X_∞.

Let ρ be a unitary character of Γ. We will treat our problem according to its behavior at the cusp.

Let Γ_∞ be the fundamental group at the cusp and $\rho|_{\Gamma_\infty}$ the restriction.

Theorem 6.1. *Let us put $h^i(\rho) = \dim H^i(X, \rho)$.*

1. *Suppose $\rho|_{\Gamma_\infty}$ is trivial. Then*
$$\mathrm{ord}_{z=0} R_\rho(z) = -2(2h^0(\rho) - h^1(\rho) + 1).$$
2. *Suppose $\rho|_{\Gamma_\infty}$ is nontrivial. Then*
$$\mathrm{ord}_{z=0} R_\rho(z) = 2h^1(\rho).$$

Suppose there is a surjective homomorphism from Γ to $\mathbb{Z}$ and let X_∞ be the corresponding infinite covering of X. Moreover suppose that all of the dimensions of $H.(X_\infty, \mathbb{C})$ and $H.(X_\infty, \rho)$ are finite. Let g be a generator of the infinite cyclic group.

Theorem 6.2.
1. *Suppose that $\rho|_{\Gamma_\infty}$ is trivial and that $H^0(X, \rho) = 0$. Then*
$$\mathrm{ord}_{z=0} R_\rho(z) \leq 2(1 + \mathrm{ord}_{z=0}\mathcal{L}_\rho(z)).$$
2. *Suppose $\rho|_{\Gamma_\infty}$ is nontrivial. Then*
$$\mathrm{ord}_{z=0} R_\rho(z) \leq 2\mathrm{ord}_{z=0}\mathcal{L}_\rho(z).$$

Moreover if the action of g on $H^1(X_\infty, \rho)$ is semisimple, they are equal.

Theorem 6.3. *Suppose that $\rho|_{\Gamma_\infty}$ is nontrivial and that $h^1(\rho)$ vanishes. Then*
$$R_\rho(0) = \tau_X(\rho)^2,$$
where $\tau_X(\rho)$ is the Reidemeister torsion of X and ρ. In particular this implies
$$|R_\rho(0)| = \delta_\rho |\mathcal{L}_\rho(0)|^2,$$
where δ_ρ is the positive constant in **Theorem 4.2**.

7 A philosophy of the proof

The proof of the theorems may be compared to one of **the Grothendieck's approach to the Weil conjecture.**

Let X be a proper smooth variety of dimension d over a finite field $\mathbb{F}_q$ and $\mathfrak{M}$ the set of its closed points. *The Hasse-Weil congruent zeta function* of X is defined to be

$$\zeta_X(t) = \prod_{P\in\mathfrak{M}} (1 - t^{\deg(P)})^{-1}.$$

Here $\deg(P)$ is the extension degree of the residue filed k_P of P over $\mathbb{F}_q$. Its logarithmic derivative is given as

$$\frac{d}{dt}\log\zeta_X(t) = \frac{1}{\zeta_X(t)}\frac{d}{dt}\zeta_X(t) = \sum_{n=1}^{\infty}|X(\mathbb{F}_{q^n})|t^{n-1},$$

where $|X(\mathbb{F}_{q^n})|$ is the number of $\mathbb{F}_{q^n}$ points of X.

Theorem 7.1. *(***The Grothendieck-Lefschetz Trace Formula***)* *Let ϕ be the action of q-th power Frobenius map on the cohomology group. Then*

$$|X(\mathbb{F}_{q^n})| = \sum_{i=0}^{2d}(-1)^i\mathrm{Tr}[(\phi^n \mid H^i_{et}(\overline{X},\, \mathbb{Q}_l)],$$

where $\overline{X}$ is the base extension of X to the algebraic closed field $\overline{\mathbb{F}}_q$.

By a simple computation,

$$\frac{d}{dt}\log\{\prod_i \det(1-\phi t \mid H^i_{et}(\overline{X},\, \mathbb{Q}_l))^{(-1)^{i+1}}\} = \sum_{n=0}^{\infty} t^{n-1}\sum_{i=0}^{2d}(-1)^i\mathrm{Tr}[(\phi^n \mid H^i_{et}(\overline{X},\, \mathbb{Q}_l)].$$

Thus we have

$$\zeta_X(t) = \prod_i \det(1 - \phi t \mid H^i_{et}(\overline{X},\, \mathbb{Q}_l))^{(-1)^{i+1}}.$$

We have changed this argument as

- $$\zeta_X(t) \Rightarrow \text{The Ruelle L-function}$$

- $$\text{Frobenius} \Rightarrow \text{the heat operator}$$

- $$\begin{aligned} & \prod_i \det(1 - \phi t \,|\, H^i_{et}(\overline{X},\, \mathbb{Q}_l))^{(-1)^{i+1}} \\ \Rightarrow \; & \text{the polynomial part of } \prod_i \det(1 - e^{-t\Delta})^{(-1)^{i+1}} \\ \doteqdot \; & \text{The Alexander invariant} \end{aligned}$$

- The Grothendieck-Lefschetz Trace Formula $\Rightarrow$ The Selberg Trace Formula.

Thus our theorem may be summerized by the following diagram:

$$\text{The Ruelle L-function} \overset{\text{The Selberg Trace formula}}{\Leftrightarrow} \text{The Alexander invariant},$$

which is quite similar to the solution of the Weil conjecture:

$$\text{The H-W congruent zeta-function} \overset{\text{The G-L Trace formula}}{\Leftrightarrow} \text{A rational function.}$$

8 An out line of the proof of the theorems

For simplicity we will assume the nontriviality of $\rho|_{\Gamma_\infty}$.

Notation

1. $\Omega^j_X(\rho)$: a vector bundle of j-forms on X twisted by ρ,
2. $L^2(X, \Omega^j_X(\rho))$: the space of its square integrable sections,
3. Δ : the positive Hodge Laplacian on $L^2(X, \Omega^j_X(\rho))$.

Here are some remarks.

1. The Hodge star operator induces an isomorphism of Hilbert spaces:
$$L^2(X, \Omega^j_X(\rho)) \simeq L^2(X, \Omega^{3-j}_X(\rho)), \quad j = 0, 1, \tag{1}$$
which commutes with Δ.
2. Since $\rho|_{\Gamma_\infty}$ is nontrivial we know that the spectrum of Δ consists of only eigenvalues.

Selberg trace formula

$$\mathrm{Tr}[e^{-t\Delta} \,|\, L^2(X, \Omega^j_X(\rho))] = \mathcal{I}_j(t) + \mathcal{H}_j(t) + \mathcal{U}_j(t),$$

where $\mathcal{I}_j(t)$, $\mathcal{H}_j(t)$ and $\mathcal{U}_j(t)$ are the identity, the hyperbolic and the unipotent term, respectively.

Notation

1.
$$\begin{aligned} \delta_0(t) &= \mathrm{Trace}[e^{-t\Delta} \,|\, L^2(X, \Omega^0_X(\rho))] \\ \delta_1(t) &= \mathrm{Trace}[e^{-t\Delta} \,|\, L^2(X, \Omega^1_X(\rho))] - \delta_0(t). \end{aligned}$$

2.
$$H_0(t) = \mathcal{H}_0(t), \quad H_1(t) = \mathcal{H}_1(t) - H_0(t),$$
$$I_0(t) = \mathcal{I}_0(t), \quad I_1(t) = \mathcal{I}_1(t) - I_0(t),$$
$$U_0(t) = \mathcal{U}_0(t), \quad U_1(t) = \mathcal{U}_1(t) - U_0(t).$$

In particular

$$\delta_0(t) = H_0(t) + I_0(t) + U_0(t), \quad \delta_1(t) = H_1(t) + I_1(t) + U_1(t).$$

We define the derivative of the Laplace transform of a function f on $\mathbb{R}$ to be

$$L'(f)(z) = 2z \int_0^\infty e^{-tz^2} f(t)dt,$$

if the RHS is defined.

1. **Step 1.** We will compute:

$$\frac{d}{dz} \log R_\rho(z) = L'(H_1)(z) - L'(e^t H_0)(z-1) - L'(e^t H_0)(z+1).$$

2. **Step 2.** We will show:

$$L'(I_1)(z) - L'(e^t I_0)(z-1) - L'(e^t I_0)(z+1) = 0$$

and

$$L'(U_1)(z) - L'(e^t U_0)(z-1) - L'(e^t U_0)(z+1)$$

is a polynomial.

3. **Step 3.** We will show

$$L'(\delta_1)(z) - L'(e^t \delta_0)(z-1) - L'(e^t \delta_0)(z+1)$$

is a meromorphic function on the whole plane with only simple poles whose residues are all integers. Moreover the Selberg trace formula, **Step 1** and **Step 2** imply

$$\mathrm{Res}_{z=0}\{\frac{d}{dz} \log R_\rho(z)\} = \mathrm{ord}_{z=0}[L'(\delta_1)(z) - L'(e^t\delta_0)(z-1) - L'(e^t\delta_0)(z+1)].$$

Thus $R_\rho(z)$ is meromorphically continued on the whole plane.

4. **Step 4.** Using the Hodge theory, we obtain

$$\mathrm{Res}_{z=0}[L'(\delta_1)(z) - L'(e^t\delta_0)(z-1) - L'(e^t\delta_0)(z+1)] = 2h^1(\rho),$$

which implies

$$\mathrm{ord}_{z=0} R_\rho(z) = 2h^1(\rho).$$

In the course of the proof, we will also obtain

Theorem 8.1. *(The Riemann hypothesis) The zeros and poles of $R_\rho(z)$ is, except for finitely many of them, are located on*

$$\{s \in \mathbb{C} \,|\, \mathrm{Re}\, s = -1,\, 0,\, 1\}.$$

Remark 8.1. *If $\rho|_{\Gamma_\infty}$ is trivial, there are another poles or zeros which derive from the scattering term. They are corresponding to the trivial zeros of the Riemann's zeta function.*