Conformal Field Theory and Operator Algebras

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Quantum field theory:

Study of Wightman fields

$\rightarrow$ Operator-valued distributions on spacetime with covariance with respect to spacetime symmetry group

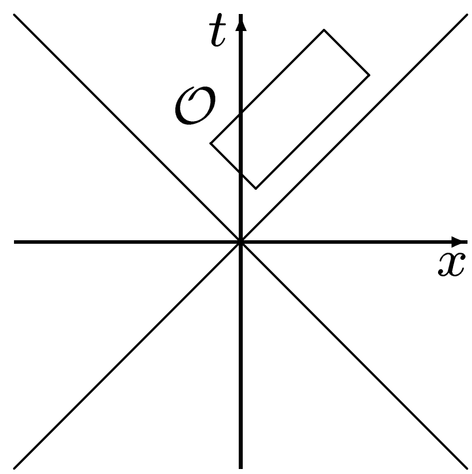
AQFT (Algebraic Quantum Field Theory) is an operator algebraic approach to quantum field theory using a family of operator algebras and it has a history of more than 40 years.

Full/chiral/boundary conformal field theories are studied in a unified framework in AQFT and we obtain classification results up to isomorphism for all of these for $c < 1$.

Our operator algebras (of bounded linear operators on a Hilbert space) are simple von Neumann algebras and they are called *factors*. The Jones theory of *subfactors* plays an important role here.

Full CFT in AQFT:

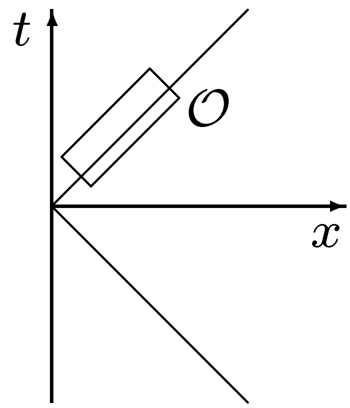
We consider rectangles $\mathcal{O}$ with edges parallel to $t = \pm x$ in $(1+1)$-dim Minkowski space as in the following picture.



Suppose we have operator-valued distributions Φ. Take test functions φ with supports in $\mathcal{O}$. We get many (unbounded) operators as $\langle \Phi, \varphi \rangle$. Fix $\mathcal{O}$ and consider the operator algebra $\mathcal{A}(\mathcal{O})$ of bounded linear operators generated by these operators. In this way, we get a family $\{\mathcal{A}(\mathcal{O})\}$ of operator algebras parameterized by space-time regions $\mathcal{O}$ (rectangles).

Boundary CFT:

We consider half-space $\{(x,t) \mid x > 0\}$ and only rectangles $\mathcal{O}$ contained in this half-space.

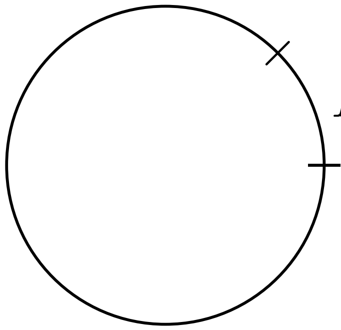


In this way, we have a similar family of operator algebras $\{\mathcal{A}(\mathcal{O})\}$.

The choice of the spacetime symmetry is not unique, and we can use the Poincaré symmetry, for example, but in CFT, we use *conformal symmetry*, (diffeomorphism covariance).

Full CFT *restricts* to two chiral theories on the light cones $\{x = \pm t\}$. In this way, we have a *chiral CFT* on the compactified S^1.

In chiral CFT, our "spacetime" is S^1 and a "spacetime region" is an interval I.



$I \mapsto \mathcal{A}(I) \subset B(H)$
von Neumann algebra
(factor)

We have a family $\{\mathcal{A}(I)\}$ of operator algebras on a Hilbert space H. These operator algebras are called *factors*, and $\{\mathcal{A}(I)\}$ is called a *net of factors*. In the usual situation, all the algebras $\mathcal{A}(I)$ are mutually isomorphic for all nets $\mathcal{A}$.

(1) $I \subset J \Rightarrow \mathcal{A}(I) \subset \mathcal{A}(J)$

(2) [locality] $I \cap J = \varnothing \Rightarrow [\mathcal{A}(I), \mathcal{A}(J)] = 0$

(3)[covariance] $u_g \mathcal{A}(I) u_g^* = \mathcal{A}(gI)$ for $g \in \mathrm{Diff}(S^1)$

(4) vacuum vector $\Omega \in H$ and positive energy

Comparison with a *vertex operator algebra*:

A vertex operator algebra (VOA) is an algebraic axiomatization of Wightman fields on S^1.

Both of one VOA and one net of factors should describe a chiral conformal field theory. So VOA's (with unitarity) and nets of factors should be in a bijective correspondence, at least under some "nice" conditions, but no general theorems have been known. (We have some candidates for the "nice" conditions.)

However, if we have one construction on one side, we can usually "translate" it to the other side, though it can be highly non-trivial from a technical viewpoint. Fundamental methods of constructions in the two approaches are listed:

VOA	net of factors
Kac-Moody/Virasoro algebras $\to$	A. Wassermann...
integral lattices $\to$	Dong-Xu
orbifold $\to$	Xu
coset $\to$	Xu
$\leftarrow$	Q-system (K-Longo)

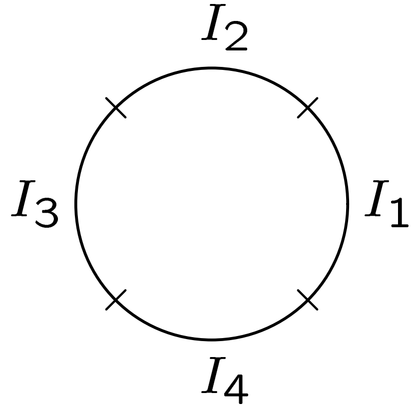
Important tool to study nets of factors is a representation theory. All $\mathcal{A}(I)$'s act on the initial Hilbert space H from the beginning, but we also consider their representations on another Hilbert space, that is, a family $\{\pi_I\}$ of representations $\pi_I : \mathcal{A}(I) \to B(K)$, where K is another Hilbert space, common for all I. A representation of a net of factors corresponds to a module over a VOA.

Each representation $\{\pi_I\}$ is in a bijective correspondence to a certain *endomorphism* λ of an infinite dimensional operator algebra, and we can restrict λ to a single factor $\mathcal{A}(I)$ for an arbitrarily fixed interval I. Then $\lambda(\mathcal{A}(I)) \subset \mathcal{A}(I)$ is a subfactor and we have its Jones index. Its square root is the *dimension* of the representation (Longo).

We can also compose endomorphisms and this composition gives a notion of *tensor products* (Doplicher-Haag-Roberts + Fredenhagen-Rehren-Schroer). We then get a *braided* tensor category.

In representation theory of VOA (and also of a quantum group), it happens that we have only finitely many irreducible representations. Such finiteness is often called *rationality*.

K-Longo-Müger (CMP 2001) gave an operator algebraic characterization of such rationality for a net $\{\mathcal{A}(I)\}$ of factors and it is called *complete rationality*.



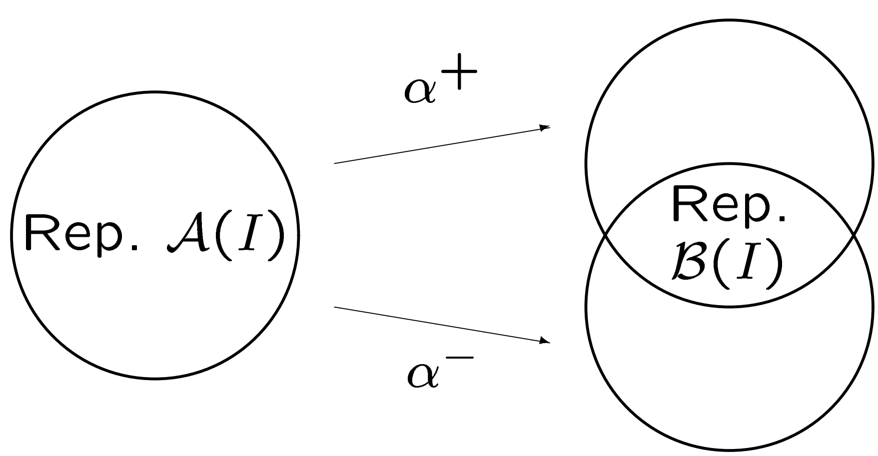
We split the circle into I_1, I_2, I_3, I_4 as above. Then complete rationality is given by the finiteness of the Jones index for a subfactor

$$\mathcal{A}(I_1) \vee \mathcal{A}(I_3) \subset (\mathcal{A}(I_2) \vee \mathcal{A}(I_4))'$$

where $'$ means the commutant. Then we automatically get a *modular* tensor category.

For an inclusion of nets of factors, $\mathcal{A}(I) \subset \mathcal{B}(I)$, we have an induction procedure analogous to the group representation. This procedure is called the α-induction and depends a choice of braiding, so we write α^+ and α^-.

[Longo-Rehren, Xu] + Ocneanu
$\rightarrow$ Böckenhauer-Evans-K (CMP 1999)



The intersection of the images of α^+ induction and α^- induction gives the true representation category of $\{\mathcal{B}(I)\}$. The others are called *soliton sectors*.

A modular tensor category produces a unitary representation π of $SL(2,\mathbb{Z})$ through its braiding, and its dimension is the number of irreducible objects. So a completely rational net of factors produces such a unitary representation. This is not irreducible in general, but is often *almost irreducible*.

Böckenhauer-Evans-K (CMP 1999) have shown that the matrix $(Z_{\lambda,\mu})$ defined by

$$Z_{\lambda,\mu} = \dim \mathrm{Hom}(\alpha^+_\lambda, \alpha^-_\mu)$$

is in the commutant of the representation π. (using Ocneanu's graphical calculus).

Such a matrix Z is called a *modular invariant*, and we have only finitely many such Z for a given π. For any completely rational net $\{\mathcal{A}(I)\}$, any extension $\{\mathcal{B}(I) \supset \mathcal{A}(I)\}$ produces such Z. Matrices Z are much easier to classify than extensions.

Classification of chiral CFT with $c < 1$:

For a net of factors, we can naturally define a *central charge* and it is known to take discrete values below 1. We have the *Virasoro net* $\{\mathrm{Vir}_c(I)\}$ for such c and it corresponds to the Virasoro VOA. Any net of factors $\{\mathcal{A}(I)\}$ with central charge $c < 1$ is an extension of the Virasoro net with the same central charge and it is automatically completely rational. So we can apply the above theory and we get the following complete classification list. (K-Longo, Ann. Math. 2004)

(1) Virasoro nets $\{\mathrm{Vir}_c(I)\}$ with $c < 1$

(2) Simple current extensions of the Virasoro nets with index 2

(3) Four exceptionals at $c = 21/22$, $25/26$, $144/145$, $154/155$

They are labeled with pairs of A-D_{2n}-$E_{6,8}$ Dynkin diagrams — *McKay correspondence*.

Three in (3) are identified with coset models, but the other does not seem to be related to any other known constructions. This is constructed with "extension by Q-system". A Q-system of Longo is a certain analogue of a Hopf algebra, and is essentially same as an "algebra in a tensor category".

From a viewpoint of tensor category, the above classification problem of extensions of a completely rational net of factors is the same as the following problem for VOA. (cf. Huang-Kirillov-Lepowsky)

Let V be a (rational) VOA and W_i be its irreducible modules. Classify VOA's arising from putting a VOA structure on $\bigoplus_i n_i W_i$ and using the same Virasoro element, where n_i is multiplicity and $W_0 = V$, $n_0 = 1$.

So the above classification theorem solves a classification problem of such extensions of the Virasoro VOA's with $c < 1$.

Classification of full CFT with $c < 1$:

Using the above results and more techniques, we can also completely classify full conformal field theories within AQFT framework for the case $c < 1$.

Full conformal field theories are given as certain nets of factors on 1+1-dimensional Minkowski space. Under natural symmetry and maximality conditions, those with $c < 1$ are completely labeled with the pairs of A-D-E Dynkin diagrams with the difference of their Coxeter numbers equal to 1. (K-Longo, CMP 2004). We now naturally have D_{2n+1}, E_7 as labels, unlike the chiral case.

The main difficulty in our work lies in proving uniqueness of the structure for each matrix in the Cappelli-Itzykson-Zuber list. This is done through 2-cohomology vanishing for certain tensor categories.

Classification of boundary CFT with $c < 1$:

Using the above results and further techniques, we can also completely classify boundary conformal field theories for the case $c < 1$.

Boundary conformal field theories are given as certain nets of factors on a $1+1$-dimensional Minkowski half-space. Under a natural maximality condition, these with $c < 1$ are completely labeled with the pairs of A-D-E Dynkin diagrams with distinguished vertices having the difference of their Coxeter numbers equal to 1. (K-Longo-Pennig-Rehren, to appear in CMP).

"Chiral fields" in boundary CFT should produce a net of factors on the boundary (which is compactified to S^1) as in the AQFT approach of Longo-Rehren. Then a general boundary CFT restricts to the boundary to produce a *non-local* extension of this *chiral* conformal field theory on the boundary.

Classification of $N = 1$ super CFT with $c < 3/2$:

We obtain a *super-local net* by replacing the commutator in the locality axiom with a supercommutator. The net now has a $\mathbb{Z}/2\mathbb{Z}$-grading, the even and odd parts.

The ordinary Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n},$$

where $m, n \in \mathbb{Z}$ and c is central.

For the Ramond/Neveu-Schwarz algebras, we have extra relations,

$$\begin{aligned} [L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r}, \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s}, \end{aligned}$$

where we have $r, s \in \mathbb{Z}$ for the Ramond algebra and $r, s \in \mathbb{Z} + 1/2$ for the Neveu-Schwarz algebra.

We now have discrete values for c up to 3/2. The values below 3/2 are again realized with coset construction, as in the case of the Virasoro nets with $c < 1$. Then we can classify their extensions of the "super-Virasoro nets" again with modular invariants and Q-system.

Now classification of modular invariants is more complicated due to existence of a "fixed point" of a symmetry in the fusion rules, but technique of Gannon-Walton with an extra care applies.

Then now the extensions are labeled with the pairs of A-D_{2n}-$E_{6,8}$ Dynkin diagrams with the difference of their Coxeter numbers equal to 2. Besides the super-Virasoro nets and their (easy) extensions of index 2, we have *six exceptionals* related to E_6 and E_8. (Carpi-K-Longo) (These are also understood as *mirror extensions* in the sense of Xu.)

Moonshine Conjecture (Conway-Norton 1979)

Mysterious relations between finite simple groups and modular functions (since McKay)

Monster: the largest among 26 sporadic finite simple groups whose order is about 8×10^{53}

Its non-trivial irreducible representation having the smallest dimension is 196883 dimensional.

The following function, called j-*function*, has been classically studied.

$$\begin{aligned} j(\tau) &= q^{-1} + 744 + 196884q + \\ &\qquad 21493760q^2 + 864299970q^3 + \cdots \end{aligned}$$

For $q = \exp(2\pi i\tau)$, $\operatorname{Im} \tau > 0$, we have modular invariance property, $j(\tau) = j\left(\dfrac{a\tau + b}{c\tau + d}\right)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, and this is the only function satisfying this property and starting with q^{-1}, up to freedom of the constant term.

McKay noticed $196884 = 196883+1$, and similar simple relations for other coefficients of the j-function and dimensions of irreducible representations of the Monster group turned out to be true. Then Conway-Norton formulated the Moonshine conjecture roughly as follows, which has been now proved by Borcherds in 1992.

(1) We have a "natural" infinite dimensional graded vector space $V = \bigoplus_{n=0}^{\infty} V_n$ with some algebraic structure having a Monster action preserving the grading and each V_n is finite dimensional.

(2) For any element g in the Monster, the power series $\sum_{n=0}^{\infty}(\mathrm{Tr}\ g|_{V_n})q^{n-1}$ is a special function called a *Hauptmodul* for some discrete subgroup of $SL(2,\mathbb{R})$. When g is the identity element, we obtain the j-function minus constant term 744.

Construction of Frenkel-Lepowsky-Meurman (1984) of the *Moonshine VOA* for part (1):

Leech lattice Λ: an exceptional lattice in dimension 24

$\Rightarrow$ lattice VOA V_Λ, which is “close” to our final object

We take a fixed point algebra under a natural action of $\mathbb{Z}/2\mathbb{Z}$, and then make a simple current extension of order 2. The resulting VOA is the Moonshine VOA $V^\natural$. (Twisted orbifold construction). The series $\sum_{n=0}^{\infty}(\dim V_n)q^{n-1}$ is indeed the j-function minus constant term 744.

Miyamoto's new construction (2004): realization of $V^\natural$ as an extension of a tensor power of the Virasoro VOA with $c = 1/2$, $L(1/2, 0)^{\otimes 48}$ (based on Dong-Mason-Zhu). This kind of extension of a Virasoro tensor power is called a *framed VOA*.

Operator algebraic counterpart:
K-Longo (Adv. Math. 2006)

We realize a Leech lattice net of factors on S^1 as an extension of $\mathrm{Vir}_{1/2}{}^{\otimes 48}$ using certain $\mathbb{Z}_4$-code. Then we can perform the twisted orbifold construction to get a net of factors, the *Moonshine net* $\mathcal{A}^\natural$. Theory of α-induction is used for getting various decompositions. We then get a Miyamoto-type description of this construction, as a counterpart of the framed VOA's. We then obtain the following properties.

(1) $c = 24$

(2) Representation theory is trivial

(3) The automorphism group is the Monster

(4) Hauptmodul property (as above)

Outline of the proof is as follows.

(1), (2) are easy.

Property (3) is the most difficult part. For the Virasoro VOA $L(1/2,0)$, the energy-momentum tensor is indeed a well-behaved Wightman field and smeared fields produce the Virasoro net $\mathrm{Vir}_{1/2}$. This nice property passes to the entire VOA and we can prove that the automorphism group as a VOA and the automorphism group as a net of factors are the same. Then (4) is now a trivial corollary of the Borcherds theorem.

Still, these examples are treated with various tricks case by case. We expect a bijective correspondence between VOA's and nets of factors on S^1 under some nice conditions. The C_2-finiteness condition of Zhu (with unitarity) for VOA and our complete rationality for nets of factors seem to be such "nice" conditions, but we do not know much yet.